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Editors

Infinite Dimensional Dynamical Systems

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Infinite Dimensional Dynamical Systems

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in the Mathematical Sciences

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Preface

In the typically colorful fall of 2008, a conference on Infinite Dimensional Dynamical Systems was held at York University, Toronto from September 24–28. Among the 80 participants from all over the world, 48 invited speakers presented their work covering a wide range of topics of infinite-dimensional dynamical systems generated by parabolic partial differential equations, hyperbolic partial differential equations, solitary equations, lattice differential equations, delay differential equations, and stochastic differential equations. This conference was also dedicated to Professor George Sell from University of Minnesota on the occasion of his 70th birthday.

Sell obtained his PhD degree from the University of Michigan in 1962. He joined the faculty of the School of Mathematics at the University of Minnesota in 1964, following 2 years as a Benjamin Pierce Instructor at Harvard University. At Minnesota, he was the cofounder of the Institute for Mathematics and Its Applications (IMA) and the founding director of the Army High Performance Computing Research Center. He is the founding editor of the Journal of Dynamics and Differential Equations and has been serving on the editorial boards of several other professional journals. Sell is a world-renowned leader in the field of dynamics of differential equations. An author of nine books and over 100 research articles, he has been an originator and pioneer in many important areas in this field, including nonautonomous dynamical systems, skew-product flows, invariant manifolds theory, infinite-dimensional dynamical systems, approximation dynamics, and fluid flows. He is a recipient of many honors and awards including an honorary doctorate degree from the University of St. Petersburg and an invited address at the 1982 International Congress of Mathematicians.

Infinite-dimensional dynamical systems are generated by evolutionary equations describing the evolution in time of systems whose status must be depicted in infinite-dimensional phase spaces. Studying the long-term behaviors of such systems is important to our understanding of their spatiotemporal pattern formation and global continuation, and has been among the major sources of motivation and applications of new development in nonlinear analysis and other mathematical theories. Theories of infinite-dimensional dynamical systems have also found more and more important applications in the physical, chemical, and life sciences.

Each of the 48 invited speakers gave a 40-minute lecture. The lectures are collected and published in this special volume of the Fields Institute Communications series, dedicated to George Sell.

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Contents

Persistence of Periodic Orbits for Perturbed Dissipative Dynamical Systems

Jack K. Hale and Geneviève Raugel

This paper is dedicated to Professor George Sell on the occasion of his seventieth birthday.

Abstract This paper is devoted to the study of the persistence of periodic solutions under perturbations in dynamical systems generated by evolutionary equations, which are not smoothing in finite time, but only asymptotically smoothing. Assuming that the periodic solution of the unperturbed system is non-degenerate, we want to prove the existence and uniqueness of a periodic solution for the perturbed equation in the neighbourhood of the unperturbed solution (with a period near the period of the periodic solution of the unperturbed problem). We review some methods of proofs, used in the case of systems of ordinary differential equations, and discuss their extensions to the infinite-dimensional case.

Mathematics Subject Classification (2010): Primary 35B10, 35B25, 37L50, 37L05; Secondary 35Q30, 35Q35

1 Introduction

Many evolutionary partial differential equations, arising in physics, chemistry or biology, … depend on various parameters, which are often only approximately known. Moreover, for computational reasons, a given evolutionary equation is often

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replaced by a discretized or approximate equation. So the question of persistence of periodic solutions under perturbation is very important. The persistence of non-degenerate periodic orbits of finite-dimensional systems of ordinary differential equations (ODE's) under autonomous perturbations is well-known. Many methods of proofs, such as the Poincaré map method, the Lyapunov-Schmidt method, integral equations methods or local coordinates systems method, can be used in order to prove this persistence result.

For ODE's, this problem is much simpler than the one to be discussed in infinite dimensions. In spite of this, it is instructive to briefly recall this case since it involves some of the underlying ideas for the general infinite dimensional case. Thus, let us consider the autonomous ODE in $\mathbb{R}^n$,

$$x_t = g_0(x), \tag{1}$$

where $g_0 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and x_t denotes the derivative $\frac{dx}{dt}$.

We suppose that there is a periodic orbit $\Gamma_0 = \{p_0(t), t \in [0, \omega_0)\}$ of (1) described by the periodic solution $p_0(t)$ of (1) of minimal period $\omega_0 > 0$. The linear variational equation about p_0 is given by

$$y_t = A_0(t)y, \quad A_0(t) = \frac{dg_0}{dx}(p_0(t)) \tag{2}$$

The matrix function $A_0(t)$ is ω_0-periodic in t. The Floquet multipliers of (2) are the eigenvalues of the mapping $\Pi_0(\omega_0, 0) : x_0 \in \mathbb{R}^n \mapsto \Pi_0(\omega_0, 0)x_0 = x(\omega_0, x_0)$, where $x(t, x_0)$ is the solution of (2) with initial value x_0 at $t = 0$.
The function p_{0t} is an ω_0-periodic solution of (2) and therefore, 1 is a Floquet multiplier of (2). The orbit Γ_0 (or the periodic solution $p_0(t)$ of period ω_0) is *nondegenerate* if 1 is a (algebraically) simple eigenvalue of $\Pi_0(\omega_0, 0)$.

To simplify the presentation, we consider the perturbed ODE

$$x_t = g_\varepsilon(x), \tag{3}$$

where $\varepsilon \in \mathbb{R}^k$ is a parameter, $g_\varepsilon(x) = g_0(x) + h(\varepsilon, x)$, $h(0, x) = 0$ for all $x \in \mathbb{R}^n$ and $h(\cdot, \cdot) \in C^1(\mathbb{R}^k \times \mathbb{R}^n, \mathbb{R}^n)$. To simplify the discussion, we assume, without loss of generality, that the solutions of (3) are defined on $[0, +\infty)$, for all initial data in $\mathbb{R}^n$ and for any ε small enough.

The following existence and uniqueness result of a periodic solution of the perturbed equation (3), for ε small enough, is well-known, if the orbit Γ_0 is nondegenerate. It is expected that the period will not be equal to ω_0, but should depend on ε and be close to ω_0.

Theorem 1.1. *If $\Gamma_0 = \{p_0(t), t \in [0, \omega_0)\}$ is a nondegenerate periodic orbit of (1) of (minimal) period ω_0, then there exist $\varepsilon_0 > 0$, $\delta_0 > 0$, and a neighbourhood $\mathcal{N}_0$ of Γ_0 such that, for $|\varepsilon| < \varepsilon_0$, there is a unique periodic orbit $\Gamma_\varepsilon = \{p_\varepsilon(t) \,|\, t \in [0, \omega_\varepsilon)\}$ of (3), which remains in $\mathcal{N}_0$ and has minimal period ω_ε with $|\omega_\varepsilon - \omega_0| \leq \delta_0$. Moreover, $\omega_\varepsilon \to \omega_0$, and, for any $T > 0$, $p_\varepsilon \to p_0$ in $C^0([0, T], \mathbb{R}^n)$ as $\varepsilon \to 0$.*

A classical method in ODE's for discussing the continuation problem and for proving this theorem is the *Poincaré transversal map*. We quickly recall it. For $\varepsilon \geq 0$, let $T_\varepsilon(t)x_0$ be the solution of (3) with $T_\varepsilon(0)x_0 = x_0$. Let Σ_0 be a transversal at $p_0(0)$ to Γ_0. For any $y \in \Sigma_0$ in a sufficiently small neighborhood of $p_0(0)$, there exists a first time $\tau(\varepsilon, y) > 0$, close to ω_0 for which $T_\varepsilon(\tau(\varepsilon, y))y \in \Sigma_0$, with $\omega_0 = \tau(0, p_0(0))$. By the implicit function theorem, $\tau(\varepsilon, y)$ is a well defined function, which belongs to $C^1((-\varepsilon_0, \varepsilon_0) \times B_{\Sigma_0}(p_0(0), r), \mathbb{R})$, where ε_0 and r are small positive numbers and $B_{\Sigma_0}(p_0(0), r)$ is the ball in Σ_0 of center $p_0(0)$ and radius r. We now define the Poincaré map $P_\varepsilon : y \in B_{\Sigma_0}(p_0(0), r) \mapsto P_\varepsilon(y) = T_\varepsilon(\tau(\varepsilon, y))y \in \Sigma_0$. The function $P_\varepsilon(y)$ is continuous in ε and $y \in B_{\Sigma_0}(p_0(0), r)$ and is continuously differentiable in y as a consequence of the continuous differentiability of $T_\varepsilon(t)y$ in t and y and of the one of $\tau(\varepsilon, y)$ in y. Of course, $P_0(p_0(0)) = p_0(0)$. It also is easy to observe that the nondegeneracy hypothesis implies that 1 is not an eigenvalue of $D_y(P_0)(p_0(0))$ and thus $D_y(P_0)(p_0(0)) - Id$ has a bounded inverse as a map from the tangent space of Σ_0 at $p_0(0)$ into itself. For $\varepsilon > 0$ small enough, the implicit function theorem implies that P_ε has a unique fixed point $p_\varepsilon(0)$ in a neighborhood of $p_0(0)$. Furthermore, $p_\varepsilon(0) \to p_0(0)$ as $\varepsilon \to 0$ and $\tau(\varepsilon, p_\varepsilon(0)) \to \omega_0$ as $\varepsilon \to 0$. We set $\omega_\varepsilon = \tau(\varepsilon, p_\varepsilon(0))$. The function $p_\varepsilon(t) = T_\varepsilon(t)p_\varepsilon(0)$, $t \in [0, \omega_\varepsilon)$, defines the periodic solution of the equation with minimal period ω_ε. This proves Theorem 1.1.

Remark. As we have seen, in this method of Poincaré sections, we associate a first return time to the initial data on the transversal section (this gives us the period of the periodic solution) and then look for a unique fixed point, which will give rise to the periodic solution. Since we are looking for a unique fixed point of the map $T_\varepsilon(\tau(\varepsilon, y))y$, this method requires at least a Lipschitz-continuous dependence in time of $T_\varepsilon(t)$.

This Lipschitz-continuity in time t is satisfied for retarded functional differential equations (RFDE's) with finite delay, when the time t is strictly larger than the delay. Thus, the Poincaré method can be adapted to RFDE's (which are, of course, infinite-dimensional systems) in the case where the period ω_0 is strictly larger than the delay (see [15, Chapter 10, Remark 4.2]).

We now turn to the infinite-dimensional case. For $\varepsilon \geq 0$, we consider a family of non-linear continuous semigroups (also called dynamical systems) $T_\varepsilon(t)$ defined on a given Banach space X. We assume that the limiting system $T_0(t)$ has a nontrivial periodic orbit $\Gamma_0 = \{p_0(s) \,|\, s \in [0, \omega_0)\}$, where $p_0 : \mathbb{R} \mapsto X$ is a periodic function of minimal period ω_0 such that $T_0(t)p_0(s) = p_0(s+t)$, for all $t \geq 0$ and all $s \in \mathbb{R}$. To simplify the discussion, we also assume that the family of semigroups $T_\varepsilon(t)$ is of class C^1, that is, $T_\varepsilon(t)u_0$ is continuously differentiable with respect to $u_0 \in X$.

In order to properly define the notion of *nondegenerate periodic solution*, we need to assume in addition that the function $p_0(t) : t \in [0, \omega_0) \mapsto p_0(t) \in X$ is of class C^1. Let $\Pi_0(t, 0) = D_u(T_0(t)p_0(0))$ be the evolution operator or process defined by the linearized operator around $p_0(t)$. Under the above regularity hypothesis on $p_0(t)$, we remark that $p_{0t}(0)$ is an eigenfunction of $\Pi_0(\omega_0, 0)$ associated to the eigenvalue 1 and we may say that the periodic solution $p_0(t)$ of period ω_0 is nondegenerate if 1 is a (algebraically) simple isolated eigenvalue of $\Pi_0(\omega_0, 0)$. Often, we will then say that Γ_0 is a non-degenerate periodic orbit (of period ω_0). Notice however that the periodic solution $p_0(t)$, regarded as a periodic function of period $n\omega_0$, for an integer $n \geq 2$, could be degenerate. We may now wonder if Theorem 1.1 can be extended to this general case, under some additional hypotheses. More precisely, we consider the following question:

Continuation Problem *If Γ_0 is a nondegenerate periodic orbit of $T_0(t)$, described by the periodic solution $p_0(t)$ of minimal period ω_0, determine methods for proving (under additional appropriate hypotheses) the existence and uniqueness of a periodic orbit $\Gamma_\varepsilon = \{p_\varepsilon(t) \,|\, t \in [0, \omega_\varepsilon)\}$ of $T_\varepsilon(t)$, where $p_\varepsilon(t)$ is a periodic solution of $T_\varepsilon(t)$ of minimal period ω_ε and where $(\Gamma_\varepsilon, \omega_\varepsilon)$ is "close" to (Γ_0, ω_0) if ε is small enough.*

For some classes of evolutionary equations, it is not difficult to verify that the periodic function $p_0(t) : t \in [0, \omega_0) \mapsto p_0(t) \in X$ is of class C^1. This is true for RFDE's with finite delay, since their solutions become smoother with increasing time. It is also true for parabolic equations, due to the fact that their solutions are smoother at positive time. As already noticed in the above remark, the method of Poincaré sections can be applied to RFDE's when the period ω_0 is strictly larger than the delay and thus Theorem 1.1 holds in this case (see [15]). The smoothing property in positive time also allows to apply the method of Poincaré sections to the parabolic equations and thus Theorem 1.1 is easily extended to this case (see [17, Theorem 8.3.2]).

For many types of evolutionary equations, including certain neutral functional differential equations (NFDE's) with finite delay, RFDE's with infinite delay and damped hyperbolic PDE's on bounded domains, the trajectories lying in compact invariant sets are smoother in time and, in particular, the periodic solutions are of class C^1 in time. These evolutionary equations belong to the class of *asymptotically smooth* or *asymptotically compact* dynamical systems, which are not smoothing in finite time, but are smoothing in infinite time. The class of *asymptotically smooth* dynamical systems (see [7, 8]) has been introduced in the early 1970s. The equivalent concept of *asymptotic compactness* was introduced later by Ladyshenskaya (see [26]). We recall that a dynamical system $T_0(t)$, $t \geq 0$, on a Banach space X is asymptotically smooth if, for any nonempty bounded set $B \subset X$ for which $T_0(t)B \subset B$, for $t \geq \tau \geq 0$, there exists a nonempty compact set $J = J(B)$ in the closure of B such that $\lim_{t\to\infty} \mathrm{dist}_X(T_0(t)B, J) = 0$, where dist_X is the Hausdorff semi-distance. A dynamical system $T_0(t)$ is asymptotically compact, if, for any bounded set $B \subset X$ such that $\cup_{t\geq\tau} T_0(t)B$ is bounded for some $\tau \geq 0$, every set of the form $\{T_0(t_n)z_n\}$, with $z_n \in B$ and $t_n \to_{n\to+\infty} +\infty$, $t_n \geq \tau$, is relatively compact. One easily sees that both concepts are equivalent (see [29]).

For most of the asymptotically smooth dynamical systems, we can define the notion of nondegenerate periodic solution (or periodic orbit). Let $T_0(t)$ be such an asymptotically smooth dynamical system and $p_0(t) \in C^1(\mathbb{R}, X)$ be a periodic solution (with Γ_0 being the corresponding periodic orbit). As in the proof of Theorem 1.1, one can define a transversal Σ_0 at $p_0(0)$ to Γ_0 and also, for any $v \in \Sigma_0$, the first return time $\tau(0,v)$ and the corresponding Poincaré map $P_0(v) = T_0(\tau(0,v))v$. This Poincaré map is continuous. But, unfortunately, for asymptotically smooth systems, which are not smoothing in finite time, this Poincaré map $P_0(v)$ is, in general, not even Hölder-continuous in $v \in X$ and thus we cannot directly apply the Poincaré method for generalizing Theorem 1.1. In [13], we have introduced a *modified* Poincaré method and proved that, under additional appropriate hypotheses, the above Continuation Problem has a positive answer, that is, Theorem 1.1 can be extended to a large class of asymptotically smooth dynamical systems. In [13], this modified Poincaré method is applied to perturbations (even non-regular perturbations) of semilinear systems [13, Sect. 3] and also of strongly nonlinear problems [13, Theorem 4.1 of Sect. 4]. This method is especially efficient in the case of strongly nonlinear systems. Since the proofs and hypotheses of the modified Poincaré method look a priori rather involved, one may wonder if one can generalize other known "easier" methods of proofs of Theorem 1.1 to the case of asymptotically smooth systems and thereby obtain weaker hypotheses or simpler methods of proofs. For perturbations of strongly nonlinear dynamical systems, there does not seem to exist general simpler methods of proofs, besides the modified Poincaré and the change of local coordinates methods. In this paper, we will not study the change of local coordinates method. This method, which has already been used by Urabe [30] and by Hale [6] in the frame of ODE's, has been extended to parabolic equations by Henry [17] and to equations of retarded type and simple neutral type by Hale and Weedermann [16]. In [14], we generalize it to the class of evolutionary equations, which are asymptotically regular, but not smoothing in finite time.

Since semilinear evolutionary equations are easier to handle, one expects that there are other classical methods (used for proving Theorem 1.1 in the case of ODE's), which can be generalized to the infinite-dimensional case. The main purpose of this paper is to consider such methods and to compare them with the modified Poincaré method. The typical semilinear evolutionary PDE is written in the following abstract form

$$u_t = B_0 u + f_0(u), \quad u(0) = u_0 \in X, \tag{4}$$

where B_0 is the infinitesimal generator of a linear C^0-semigroup on a Banach space X and the nonlinearity $f_0 \in C^1(X,X)$. We will consider *non-regular* perturbations of the Eq. (4) of the form

$$u_t = B_\varepsilon u + f_\varepsilon(u), \quad u(0) = u_0 \in X, \tag{5}$$

where, for $\varepsilon > 0$, B_ε is the infinitesimal generator of a linear C^0-semigroup on X and $f_\varepsilon \in C^1(X,X)$. For sake of simplicity, we suppose that f_ε converges to f_0 in $C^1(M,X)$ for any bounded closed subset M of X. We assume that the perturbation is *non-regular* in the sense that the semigroup $e^{B_\varepsilon t}$ does not converge to the semigroup

$e^{B_0 t}$ in $L(X,X)$, but converges in the following much weaker sense, for any $T > 0$, for any $0 \le t \le T$,

$$\|e^{B_\varepsilon t}v - e^{B_0 t}v\|_X \le C_0(T)\varepsilon^\beta \|v\|_Z, \ \forall v \in Z, \tag{6}$$

where $C_0(T)$ and β are two positive constants, and Z is a Banach subspace of X, the inclusion of Z in X being continuous (and also compact if $Z \neq X$). We may assume without restriction of generality that $D(B_\varepsilon) \subset Z$, for $\varepsilon > 0$. As already explained and as shown in [12], the invariant sets (and therefore the periodic orbits) of asymptotically smooth dynamical systems are more regular and thus, without loss of generality, we may assume that the periodic solutions are actually bounded in the space Z. In most of the cases, these invariant sets are also bounded in the domain $D(B_\varepsilon)$ of B_ε. As in the modified Poincaré method in [13], these "good regularity properties" will play a crucial role in all our proofs below.

The plan of the paper is as follows. In the next sections, we analyze two classical methods of proofs of Theorem 1.1 and discuss their extensions to the infinite-dimensional case. In Sect. 2, we study a method based on the variation of constants formula. We prove that this method admits a nice extension to the case of the semilinear equations (4) and (5) (see Theorem 2.9). In Paragraph 2.2, we compare it with the modified Poincaré method of [13]. Paragraph 2.3 is devoted to examples of applications of Theorem 2.9. In Sect. 3, we consider a simple method using the Fredholm alternative and the Lyapunov-Schmidt procedure. We will see that the generalization of this method leads to more involved technical problems than the one of Sect. 2.

Note: These last years, Jack Hale and I had been working on the problem of persistence of periodic orbits under perturbations in the case of dissipative evolutionary equations. This paper is based on various preliminary drafts and discussions with Jack Hale. During his sickness in 2009, Jack Hale strongly wished that we write a paper on this subject in honor of the seventieth birthday of George Sell. We had several discussions about the contents of this paper in the weeks preceding his passing away on December 9, 2009.

2 An Integral Equation Method

The integral method described below is based on the so-called variation of constants or Duhamel formula and goes back to many earlier works. For example, it has already been used by Malkin [27] and by Hale [5, p. 28] in the frame of non-autonomous equations. Later this method has been widely considered by several authors in the finite-dimensional case (see [24, 25], for example). In the last decade, this method has been developed in the infinite-dimensional frame for autonomous equations, when the non linear term in the equation has some compactness properties (see [2, 4, 18–22], for example). In [19, 20, 22], Johnson, Kamenskii and

Nistri have used it, together with a topological degree argument, in order to prove the persistence of periodic solutions of damped hyperbolic equations in thin product domains.

Before entering into the details of this method, we quickly describe it in the frame of a simple non autonomous equation in $\mathbb{R}^n$, as it was done in [5]. This method starts with an application of the variation of constants (or Duhamel) formula.

For $x \in \mathbb{R}^n$, consider the ODE

$$x_t = Bx + f(t), \tag{7}$$

where $f(t+\omega_0) = f(t)$ and $e^{B\omega_0} - I$ is nonsingular; that is, the zero solution of the equation

$$x_t = Bx \tag{8}$$

is nondegenerate in the class of ω_0-periodic functions.

The variation of constants formula represents any solution $x(t)$ of (7) as

$$x(t) = e^{Bt}x(0) + \int_0^t e^{B(t-s)} f(s)ds. \tag{9}$$

From the nondegeneracy hypothesis, it follows that (7) has a unique ω_0-periodic solution $x^*(t)$ with initial value $x^*(0)$ given by

$$x^*(0) = (I - e^{B\omega_0})^{-1} \int_0^{\omega_0} e^{B(\omega_0 - s)} f(s)ds \tag{10}$$

and the ω_0-periodic solution $x^*(t)$ is given by

$$\begin{aligned} x^*(t) &= e^{Bt}(I - e^{B\omega_0})^{-1} \int_0^{\omega_0} e^{B(\omega_0 - s)} f(s)ds + \int_0^t e^{B(t-s)} f(s)ds \\ &= e^{Bt}(e^{-B\omega_0} - I)^{-1} \int_t^{t+\omega_0} e^{-Bs} f(s)ds \\ &= \int_t^{t+\omega_0} e^{Bt}(e^{-B\omega_0} - I)^{-1} e^{-Bs} f(s)ds. \end{aligned} \tag{11}$$

Notice that, with this formula, it is very easy to prove the existence of periodic solutions under perturbations of (8), with terms which are periodic in t of period ω_0.

Remark 2.1. We point out that, in the formula (11), the initial data disappeared. If we go back to the nonlinear equation (5) and use the formulas corresponding to (11), we see that the ω_ε-periodic solutions $p_\varepsilon(t)$ of (5) have to satisfy the equality

$$p_\varepsilon(t) = \int_t^{t+\omega_\varepsilon} e^{B_\varepsilon t}(e^{-B_\varepsilon \omega_\varepsilon} - I)^{-1} e^{-B_\varepsilon s} f_\varepsilon(p_\varepsilon(s))ds, \tag{12}$$

where ω_ε is close to ω_0. Equality (12) shows that, in order to find the periodic solution p_ε, we are reduced to seek for a fixed point of a map defined on the space of ω_ε-periodic continuous solutions from $\mathbb{R}$ into X. This strategy can be really interesting in the infinite-dimensional case. Indeed, if the nonlinear term f_ε maps the Banach space X into X or into a smaller "more regular" space Y, we can hope to use this regularity property in the proof of the existence of a fixed point of (12). For example, this is the case for the important class of systems of weakly or strongly damped wave equations.

In the next paragraph, we are going to extend the equality (11) to the infinite-dimensional case in the frame of semilinear "dissipative" evolutionary equations and explain how one can use the integral equation (11) for showing the persistence of periodic solutions under perturbations. It will quickly become clear that this integral method is not well adapted to strongly non-linear equations. To conclude this section, we will briefly give some applications.

2.1 An Abstract Result

Let X and Z ($Z \neq X$) be two Banach spaces such that $Z \subset X$ is embedded in X with a *continuous* and *compact* injection. Let $\varepsilon_0 > 0$ be a small number and, for $0 \leq \varepsilon \leq \varepsilon_0$, let B_ε be the generator of a C^0-semigroup $S_\varepsilon(t) \equiv e^{B_\varepsilon t}$ on X satisfying the decay condition

$$\sup(\|e^{B_\varepsilon t}\|_{L(Z,Z)}, \|e^{B_\varepsilon t}\|_{L(X,X)}) \leq C_0 e^{-\alpha t}, \tag{13}$$

where $\alpha > 0$ and $C_0 > 0$ are two constants. We also assume that $D(B_\varepsilon)$, for $\varepsilon \geq 0$, is *embedded* in Z with a continuous injection.

We next introduce a nonlinear map $f_\varepsilon : X \to X$, which belongs to $C^2(X,X)$, "uniformly with respect to ε". In particular, for any bounded subset $\mathcal{B}$ of X, there exists a positive constant $C(\mathcal{B})$ such that, for any $0 \leq \varepsilon \leq \varepsilon_0$, for any $v \in \mathcal{B}$,

$$\|f_\varepsilon(v)\|_X + \|Df_\varepsilon(v)\|_{L(X,X)} + \|D^2 f_\varepsilon(v)\|_{L(X\times X,X)} \leq C(\mathcal{B}). \tag{14}$$

In one of the examples of systems of damped wave equations given in Sect. 3.2 below, the Banach space X also depends on ε. In this section, for sake of simplicity, we assume that the Banach space X does not depend on ε. The generalization of the abstract result below to the case where the spaces X and Z also depend on ε is rather straightforward and is described in details in [9] (see also [1, 2]).

For $0 \leq \varepsilon \leq \varepsilon_0$, we consider the following semilinear equation

$$u_t = B_\varepsilon u + f_\varepsilon(u), \quad u(0) = u_0 \in X. \tag{15}$$

We denote by $T_\varepsilon(t)$ the local semiflow on X generated by the Eq. (15).

We next make the following assumptions:

- (H1) The semiflow $T_0(t)$ admits a periodic solution $p_0(t) \in C^1(\mathbb{R},X) \cap C^0(\mathbb{R},D(B_0))$ of period $\omega_0 > 0$, which is *non-degenerate*.

- (H2) The functions $p_{0t}(t)$ and $f_0(p_0(t))$ are continuous from $\mathbb{R}$ into Z and from $\mathbb{R}$ into $D(B_0)$ respectively.

By Hypotheses (H1) and (H2), there exists a positive constant K_0 such that, for $0 \le t \le \omega_0$,

$$\|p_0(t)\|_{D(B_0)} + \|f_0(p_0(t))\|_{D(B_0)} + \|p_{0t}(t)\|_Z \le K_0. \tag{16}$$

We denote $\Gamma_0 = \{p_0(s) \,|\, s \in [0, \omega_0)\}$ the periodic orbit generated by $p_0(s)$. Let $\Pi_0(t,s)$ be the evolution operator or process defined by the linearized equation around $p_0(s)$,

$$w_t = B_0 w + Df_0(p_0)w, \quad w(s) = w_0. \tag{17}$$

The equation (17) admits a unique solution $w(t) = \Pi_0(t,s)w_0$, which belongs to $C^0(X,X)$. We recall that $p_0(t)$ is non-degenerate if 1 is a (algebraically) simple isolated eigenvalue of the period map $\Pi_0(\omega_0, 0)$.

We further assume that

- (H3) For $0 \le \varepsilon \le \varepsilon_0$, $D_u f_\varepsilon(p_0(s))$ is a linear bounded mapping from X into Z, the norm of which is uniformly bounded, with respect to ε and to $0 \le s \le 2\omega_0$, that is,

$$\sup_{t\in[0,2\omega_0]} \|D_u f_\varepsilon(p_0(t))\|_{L(X,Z)} \le C_0. \tag{18}$$

  This assumption implies in particular that, for any $w_0 \in Z$, the unique solution $\Pi_0(t,s)w_0$ belongs to $C^0(Z,Z)$.
- (H4) There exists a positive constant β_0 such that, for $0 \le t \le 2\omega_0$, for any $w \in Z$,

$$\|e^{B_0 t} w - e^{B_\varepsilon t} w\|_X \le C_0 \varepsilon^{\beta_0} \|w\|_Z. \tag{19}$$

- (H5) There exists a positive constant β_1 such that, for $0 \le \varepsilon \le \varepsilon_0$ and for any $s \in \mathbb{R}$,

$$\|f_\varepsilon(p_0(s)) - f_0(p_0(s))\|_X + \|D_u f_\varepsilon(p_0(s)) - D_u f_0(p_0(s))\|_{L(X,X)} \le C_0 \varepsilon^{\beta_1}. \tag{20}$$

In the proof of the uniqueness of the periodic solution for the perturbed equation, one needs additional hypotheses. Here, we will use the following two additional hypotheses,

- (H6) There exist a neighbourhood $\mathcal{V}_0$ of Γ_0 and a positive constant δ_0 such that, if $\Gamma_\varepsilon = \{p_\varepsilon(t) | t \in [0, \omega_\varepsilon]\}$ is a periodic orbit of (15), contained in $\mathcal{V}_0$ and if $p_\varepsilon(t)$ is of period ω_ε with $|\omega_0 - \omega_\varepsilon| \le \delta_0$, then $p_\varepsilon(t)$ and $f_\varepsilon(p_\varepsilon(t))$ are continuous functions from $t \in \mathbb{R}$ into $D(B_\varepsilon)$ and

$$\sup_{t\in\mathbb{R}} \|f_\varepsilon(p_\varepsilon(t))\|_{D(B_\varepsilon)} + \sup_{t\in\mathbb{R}} \|p_\varepsilon(t)\|_{D(B_\varepsilon)} \le K_0. \tag{21}$$

- (H7) The domain $D(B_0^*)$ is dense in X^* (this is true in particular when X is a reflexive Banach space). Moreover, B_ε^{-1} converges to B_0^{-1} in $L(X,X)$, when ε goes to zero.

Under the above hypotheses, in Theorem 2.9 below, we will give an existence and uniqueness result of a periodic solution of the perturbed problem (5).

Remark 2.2. 1. The above hypotheses can be made weaker. For example, the bounds ε^{β_0} and ε^{β_1} in the hypotheses (H5) and (H6) can be replaced by any function $g(\varepsilon)$, which goes to 0, when ε goes to 0.
2. In [9], we give an extension of the main Theorem 2.9 below to the case of general dissipative dispersive equations.
3. In the case of regular perturbations, that is, in the case where the estimate (19) holds for any $w \in X$, we still need to introduce a Banach subspace Z of X, which is compactly embedded in X. Indeed, at several steps of the proof of Theorem 2.9 below, we use the compactness of the linear map $D_u f_\varepsilon(p_0(t))$.

In order to introduce a formula analogous to (11) in the infinite-dimensional case, we need the following result.

Lemma 2.3. *Under the decay assumption* (13)*, for any* $t_1 > 0$*, there exists a positive constant* $C_1 = C_1(t_1)$ *such that, for any* $0 \le \varepsilon \le \varepsilon_0$ *and any* $t \ge t_1$*,*

$$\sup(\|(I - e^{B_\varepsilon t})^{-1}\|_{L(Z,Z)}, \|(I - e^{B_\varepsilon t})^{-1}\|_{L(X,X)}) \le C_1. \tag{22}$$

Proof. Since the proofs are the same in the spaces X and Z, we give it only in the case of the space X. We consider a new norm on X given by

$$|u|_X = \sup_{s \ge 0} \|e^{B_\varepsilon s} u\|_X e^{\alpha s}.$$

The inequality (13) implies on the one hand that

$$\|u\|_X \le |u|_X \le C_0 \|u\|_X, \tag{23}$$

and, on the other hand, that

$$\begin{aligned} |e^{B_\varepsilon t} u|_X &= \sup_{s \ge 0} \|e^{B_\varepsilon (t+s)} u\|_X e^{\alpha s} \\ &= \sup_{s \ge 0} \|e^{B_\varepsilon (t+s)} u\|_X e^{\alpha (t+s)} e^{-\alpha t} \\ &\le e^{-\alpha t} |u|_X. \end{aligned}$$

In particular, for $t \ge t_1$,

$$|e^{B_\varepsilon t} u|_X \le e^{-\alpha t_1} |u|_X.$$

It follows that $(I - e^{B_\varepsilon t})$ is invertible for $t \geq t_1$, and that its inverse satisfies the estimate

$$|(I - e^{B_\varepsilon t})^{-1} u|_X \leq \frac{1}{1 - e^{-\alpha t_1}} |u|_X. \tag{24}$$

The inequalities (23) and (24) imply that

$$\|(I - e^{B_\varepsilon t})^{-1} u\|_X \leq \frac{C_0}{1 - e^{-\alpha t_1}} \|u\|_X,$$

which is the desired estimate □

Determining a periodic solution $p_\varepsilon(t)$ of Eq. (15) for $\varepsilon > 0$ leads, by the variation of constants formula (see (10)), to seek $p_\varepsilon(0) \in X$, close to $p_0(0)$, and ω_ε close to ω_0, such that,

$$\begin{aligned} p_\varepsilon(0) &= (I - e^{B_\varepsilon \omega_\varepsilon})^{-1} \int_0^{\omega_\varepsilon} e^{B_\varepsilon(\omega_\varepsilon - s)} f_\varepsilon(p_\varepsilon(s)) ds \\ &= (I - e^{B_\varepsilon \omega_\varepsilon})^{-1} \int_0^{\omega_\varepsilon} e^{B_\varepsilon(\omega_\varepsilon - s)} f_\varepsilon(T_\varepsilon(s) p_\varepsilon(0)) ds. \end{aligned} \tag{25}$$

We could look for a solution $(p_\varepsilon(0), \omega_\varepsilon)$ of (25) and apply a fixed point theorem to the map defined by the right hand side of (25). But this would require to estimate terms like $\|Df_0(p_0(s))(DT_\varepsilon(s)(p_0(s))\varphi - Df_0(p_0(s))(DT_0(s)(p_0(s))\varphi\|_X$, when φ belongs to X. Since in general we do not have good estimates for $\|e^{B_0 t}\varphi - e^{B_\varepsilon t}\varphi\|_X$, when φ belongs to X only, this strategy would require additional hypotheses on the nonlinearity f_0 and on the comparison of the semigroups $e^{B_\varepsilon t}$ and $e^{B_0 t}$. For this reason, we do not look for solutions of (25), but follow the strategy explained in (11), that is, we look for a continuous ω_ε-periodic solution $p_\varepsilon(t)$ of

$$\begin{aligned} p_\varepsilon(t) = e^{B_\varepsilon t}(I - e^{B_\varepsilon \omega_\varepsilon})^{-1} \int_0^{\omega_\varepsilon} e^{B_\varepsilon(\omega_\varepsilon - s)} f_\varepsilon(p_\varepsilon(s)) ds \\ + \int_0^t e^{B_\varepsilon(t-s)} f_\varepsilon(p_\varepsilon(s)) ds, \end{aligned} \tag{26}$$

where ω_ε is close to ω_0. As we have mentioned above, such a strategy has already been used by Gurova and Kamenskii in the frame of parabolic equations [4], by Johnson et al. [18–20] in the particular case of systems of damped wave equations in thin product domains and by Abdelhedi [1, 2] in the case of more general thin two-dimensional domains. In what follows, we shall improve and generalize their results. Instead of applying topological degree arguments to (26) as done in [4, 18–20], we will simply apply fixed point theorems and show the existence and uniqueness of periodic solutions for the perturbed problem.

For any $\omega > 0$, we introduce the space

$$C_\omega(X) = \{w \in C^0(\mathbb{R}, X) \,|\, w \text{ is } \omega\text{-periodic}\},$$

equipped with the norm $\|w\|_{C_\omega(X)} = \sup_{t\in[0,\omega]} \|w(t)\|_X$ (more generally, for any Banach space Y, we introduce the space $C_\omega(Y) = \{w \in C^0(\mathbb{R}, Y) \,|\, w \text{ is } \omega\text{-periodic}\}$).

We remark that $p_\varepsilon(t) \in C_{\omega_\varepsilon}(X)$ is a solution of (26) if and only if $p_\varepsilon(t)$ is a fixed point of the mapping

$$w \in C_{\omega_\varepsilon}(X) \mapsto e^{B_\varepsilon t}(I - e^{B_\varepsilon \omega_\varepsilon})^{-1} \int_0^{\omega_\varepsilon} e^{B_\varepsilon(\omega_\varepsilon - s)} f_\varepsilon(w(s))ds + \int_0^t e^{B_\varepsilon(t-s)} f_\varepsilon(w(s))ds.$$

However, looking for a fixed point of a mapping defined on a variable space $C_{\omega_\varepsilon}(X)$ is not really convenient. For this reason, in order to fix the period equal to ω_0, we perform the change of time variable $t \mapsto \frac{\omega}{\omega_0} t$. Clearly, u is an ω-periodic solution of (15) if and only if $u^*(t) = u(\frac{\omega}{\omega_0} t)$ is a ω_0-periodic solution of the equation

$$u_t^* = \frac{\omega}{\omega_0} B_\varepsilon u^* + \frac{\omega}{\omega_0} f_\varepsilon(u^*), \quad u^*(0) = u_0 \in X. \tag{27}$$

We denote $S_{\varepsilon,\omega}(t) = e^{\frac{\omega}{\omega_0} B_\varepsilon t}$ the C^0-semigroup generated by the operator $\frac{\omega}{\omega_0} B_\varepsilon$. Like [18] (see also [2, 19]), we introduce the linear mapping $J_\varepsilon(\omega) : w \in C_{\omega_0}(X) \mapsto J_\varepsilon(\omega)w$ given by

$$\begin{aligned}(J_\varepsilon(\omega)w)(t) = S_{\varepsilon,\omega}(t)(I - S_{\varepsilon,\omega}(\omega_0))^{-1} \int_0^{\omega_0} S_{\varepsilon,\omega}(\omega_0 - s)w(s)ds \\ + \int_0^t S_{\varepsilon,\omega}(t-s)w(s)ds.\end{aligned} \tag{28}$$

Let w be a given element in $C_{\omega_0}(X)$ (resp. $C_{\omega_0}(Z)$). The compactness of the set $\cup_{t\in[0,\omega_0]} w(t)$ in X (resp. Z), for any w in $C_{\omega_0}(X)$ (resp. $C_{\omega_0}(Z)$) together with the Mazur lemma and the fact that $S_{\varepsilon,\omega}(\cdot)$ is a C^0-semigroup on X (resp. Z) imply that $J_\varepsilon(\omega)w$ belongs to $C_{\omega_0}(X)$ (resp. $C_{\omega_0}(Z)$). Moreover, the map $J_\varepsilon(\omega) : w \in C_{\omega_0}(X) \mapsto J_\varepsilon(\omega)w \in C_{\omega_0}(X)$ is a continuous linear mapping from $C_{\omega_0}(X)$ (resp. from $C_{\omega_0}(Z)$) into itself. In addition, taking into account the equality

$$\begin{aligned}&(I - S_{0,\omega}(\omega_0))^{-1} - (I - S_{0,\omega_0}(\omega_0))^{-1} \\ &\quad = (I - S_{0,\omega}(\omega_0))^{-1}(S_{0,\omega}(\omega_0) - S_{0,\omega_0}(\omega_0))(I - S_{0,\omega_0}(\omega_0))^{-1},\end{aligned} \tag{29}$$

and arguing as previously, one shows that, for any $w \in C_{\omega_0}(X)$, the map $J_\varepsilon(\omega)w : \omega \in (0, +\infty) \mapsto J_\varepsilon(\omega)w \in C_{\omega_0}(X)$ is continuous (and uniformly continuous in ω, when ω belongs to a compact set).

With similar arguments, one also proves that, if $w \in C_{\omega_0}(D(B_\varepsilon))$, then $J_\varepsilon(\omega)w : \omega \in (0, +\infty) \mapsto J_\varepsilon(\omega)w \in C_{\omega_0}(X)$ is of class C^1. For any $\varepsilon \geq 0$, for any $w \in C_{\omega_0}(D(B_\varepsilon))$, the derivative is given by

$$\left(\frac{dJ_\varepsilon}{d\omega}w\right)(t) = \frac{t}{\omega_0}B_\varepsilon(J_\varepsilon(\omega)W)(t) - \frac{B_\varepsilon}{\omega_0}(J_\varepsilon(\omega)W)(t)$$
$$+ B_\varepsilon S_{\varepsilon,\omega}(t)(I - S_{\varepsilon,\omega}(\omega_0))^{-2}\int_0^{\omega_0} S_{\varepsilon,\omega}(\omega_0 - s)w(s)ds, \tag{30}$$

where the function W is defined by $W(s) = sw(s)$, for any s.

We finally introduce the following non-linear map $F_\varepsilon(\omega) : (\omega, w) \in (0,+\infty) \times C_{\omega_0}(X) \mapsto F_\varepsilon(\omega, w) \in C_{\omega_0}(X)$:

$$F_\varepsilon(\omega, w) = J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0}f_\varepsilon(w)\right). \tag{31}$$

Obviously, for any $\omega \in (0,+\infty)$, the map $w \in C_{\omega_0}(X) \mapsto F_\varepsilon(\omega, w) \in C_{\omega_0}(X)$ is of class C^2. Moreover, as, by Hypothesis (H2), $f_0(p_0([0,\omega_0]))$ is a compact subset of $D(B_0)$, the map $\omega \in (0,+\infty) \mapsto F_0(\omega, p_0) \in C_{\omega_0}(X)$ is of class C^1 (see (30)).

We emphasize that, due to the variation of constants formula, u^* is a fixed point of $F_\varepsilon(\omega,\cdot)$ if and only if u^* is an ω_0-periodic solution of (27). In particular, we remark that

$$F_0(\omega_0, p_0) = p_0. \tag{32}$$

Before applying a fixed point theorem to the map $F_\varepsilon(\omega,\cdot)$, we state two elementary auxiliary lemmas.

Lemma 2.4. *1. An element $w \in C_{\omega_0}(X)$ is a fixed point of $D_uF_0(\omega_0, p_0)$ if and only if w is a ω_0-periodic solution of the linear system (17), that is, if and only if $w(0)$ is an eigenvector of the period map $\Pi_0(\omega_0, 0)$, associated with the eigenvalue 1. If 1 is a (algebraically) simple eigenvalue of $\Pi_0(\omega_0, 0)$, then 1 is a (algebraically) simple eigenvalue of $D_uF_0(\omega_0, p_0)$. Thus, under the hypotheses (H1) and (H3), 1 is a (algebraically) simple isolated eigenvalue of $D_wF_0(\omega_0, p_0)$.*

2. Under the hypotheses (H1) and (H3), $D_wF_0(\omega_0, p_0) - I$ is a Fredholm operator of index 0 from $C_{\omega_0}(X)$ (resp. $C_{\omega_0}(Z)$) into itself. The operator $D_wF_0(\omega_0, p_0)$ can be extended to a compact continuous linear mapping from $L^2((0,\omega_0), X)$ into $C_{\omega_0}(X)$. Moreover, $D_wF_0(\omega_0, p_0) - I$ is a Fredholm operator of index 0 from $L^2((0,\omega_0), X)$ into itself.

3. The property that 1 is an eigenvalue of $\Pi_0(\omega_0, 0)$ of algebraic multiplicity one, together with the regularity assumptions on p_0, imply that the equation

$$(D_uF_0(\omega_0, p_0) - I)w = D_\omega F_0(\omega_0, p_0) \tag{33}$$

does not admit any solution w in $C_{\omega_0}(X)$.

Proof. 1. From the definition (31), it follows that

$$(D_uF_0(\omega, p_0)w)(t) = \Big(S_{0,\omega}(t)(I - S_{0,\omega}(\omega_0))^{-1}\int_0^{\omega_0} S_{0,\omega}(\omega_0 - s)D_uf_0(p_0(s))w(s)ds$$
$$+ \int_0^t S_{0,\omega}(t-s)D_uf_0(p_0(s))w(s)ds\Big)\frac{\omega}{\omega_0}, \tag{34}$$

and, in particular,

$$(D_uF_0(\omega_0,p_0)w)(t) = S_0(t)(I-S_0(\omega_0))^{-1}\int_0^{\omega_0} S_0(\omega_0-s)D_uf_0(p_0(s))w(s)ds + \int_0^t S_0(t-s)D_uf_0(p_0(s))w(s)ds. \quad (35)$$

Hence, if $w \in C_{\omega_0}(X)$ is a fixed point of $D_uF_0(\omega_0,p_0)$, then,

$$w(\omega_0) = (I-S_0(\omega_0))^{-1}\int_0^{\omega_0} S_0(\omega_0-s)D_uf_0(p_0(s))w(s)ds,$$

and thus

$$w(\omega_0) = S_0(\omega_0)w(0) + \int_0^{\omega_0} S_0(\omega_0-s)D_uf_0(p_0(s))w(s)ds,$$

which means that $w(\omega_0) = w(0)$ is an eigenvector of the period map $\Pi_0(\omega_0,0)$, associated with the eigenvalue 1. Conversely, one at once shows that, if $w_0 \in X$ is an eigenvector of $\Pi_0(\omega_0,0)$, associated with the eigenvalue 1, then $\Pi_0(t,s)w_0$ is an ω_0-periodic continuous function and is a fixed point of $D_uF_0(\omega_0,p_0)$. In particular, $p_{0t}(t)$ is a fixed point of $D_uF_0(\omega_0,p_0)$.

We next show by contradiction that, if 1 is a (algebraically) simple eigenvalue of $\Pi_0(\omega_0,0)$, then 1 is also a (algebraically) simple eigenvalue of $D_uF_0(\omega_0,p_0)$. If it is not true, there exists a continuous function $w \in C_{\omega_0}(X)$, $w \neq p_{0t}$ such that

$$(D_uF_0(\omega_0,p_0)-I)w = p_{0t}, \quad (36)$$

which implies by (35) that

$$w(t) = \Pi_0(t,0)w(0) - p_{0t}(t)$$

and by the periodicity of the function $w(t)$ that

$$w(0) = \Pi_0(\omega_0,0)w(0) - p_{0t}(0).$$

The last equality implies that 1 is an eigenvalue of $\Pi_0(\omega_0,0)$ of algebraic multiplicity strictly larger than one.

Below, we show that, under Hypothesis (H3), $D_uF_0(\omega_0,p_0)$ is a compact map from $C_{\omega_0}(X)$ into itself. Thus, under the hypotheses (H1) and (H3), 1 is a (algebraically) simple isolated eigenvalue of $D_uF_0(\omega_0,p_0)$.

2. Let B be a fixed bounded subset of $C_{\omega_0}(X)$ and let us denote $\mathcal{B}$ the image $D_uF_0(\omega_0,p_0)B = \cup_{w\in B}D_uF_0(\omega_0,p_0)w$ of B by $D_uF_0(\omega_0,p_0)$. Since $D_uf_0(p_0(\cdot))$ is a linear bounded mapping from X into Z and that the injection of Z into X is compact, $\cup_{t\in[0,\omega_0]}\mathcal{B}(t) = \cup_{w\in B}D_uF_0(\omega_0,p_0)w([0,\omega_0])$ is a bounded subset of Z

and hence relatively compact in X. Thus, for any positive number δ_0, there exists η_1, $0 < \eta_1 < \delta_0$, such that, for any $0 \le \tau \le \eta_1$, and any $u_0 \in \cup_{t\in[0,\omega_0]}\mathcal{B}(t)$,

$$\|S_0(\tau)u_0 - u_0\|_X \le \delta_0.$$

Let $u = D_uF_0(\omega_0, p_0)w$, where $w \in B$ and $0 \le \tau \le \eta_1$, then

$$u(t+\tau) = S_0(\tau)u(t) + \int_0^\tau S_0(\tau - s)D_uf_0(p_0(t+s))w(t+s)ds.$$

Since B is a bounded set in $C_{\omega_0}(X)$, the above properties and (13) imply that, for $t \in [0, \omega_0)$ and, for example, for $0 \le \tau \le \inf(\eta_1, \omega_0 - t)$,

$$\begin{aligned} \|u(t+\tau) - u(t)\|_X &\le \delta_0 + C_0\tau \sup_{s\in[0,\omega_0]} \|D_uf_0(p_0(s))\|_{L(X,Z)} \sup_{s\in[0,\omega_0]} \|w(s)\|_X \\ &\le (1 + C_B)\delta_0, \end{aligned} \tag{37}$$

where C_B is a positive constant depending on B. The same inequality holds for $\tau \in (-\inf(t, \eta_1), 0)$. For any $\eta_0 > 0$, choose $\delta_0 > 0$ so that $(1 + C_B)\delta_0 \le \eta_0$. Then, for any $t \in [0, \omega_0]$, we have $\|u(t+\tau) - u(t)\|_X \le \eta_0$ for $|\tau| \le \eta_1$. This implies that $\mathcal{B} \equiv \cup_{w\in B}D_uF_0(\omega_0, p_0)w$ is a set of uniformly equicontinuous functions. By the Ascoli theorem, it follows that $\mathcal{B}$ is relatively compact in $C_{\omega_0}(X)$. Hence, $D_uF_0(\omega_0, p_0)$ is a compact mapping from $C_{\omega_0}(X)$ into itself and $I - D_uF_0(\omega_0, p_0)$ is a Fredholm operator of index 0 from $C_{\omega_0}(X)$ into itself (and also from $C_{\omega_0}(Z)$ into itself).

Using the classical properties of C^0-semigroups (see [28], for example) and the fact that $D_uf_0(p_0(s))$ is a bounded linear map from X into Z, we readily show that, if w belongs to $L^2((0, \omega_0), X)$, then $D_uF_0(\omega_0, p_0)w$ is in $C^0((0, \omega_0), Z)$. Moreover, $D_uF_0(\omega_0, p_0)w(\omega_0) = D_uF_0(\omega_0, p_0)w(0)$, which implies that $D_uF_0(\omega_0, p_0)w$ belongs to the space $C_{\omega_0}(X)$. If B_1 is a bounded set in $L^2((0, \omega_0), X)$, then $\cup_{w\in B_1}D_uF_0(\omega_0, p_0)w([0, \omega_0])$ is a bounded subset of Z and thus relatively compact in X. And one shows as above that $\cup_{w\in B_1}D_uF_0(\omega_0, p_0)w$ is a family of uniformly equicontinuous functions. The only change in the proof occurs in the estimate (37), which is replaced by

$$\begin{aligned} \|u(t+\tau) - u(t)\|_X &\le \delta_0 + C_0 \sup_{s\in[0,\omega_0]} \|D_uf_0(p_0(s))\|_{L(X,Z)} \tau^{1/2} \left(\int_0^\tau \|w(s)\|_X^2\right)^{1/2} \\ &\le (1 + C_B)\delta_0^{1/2}, \end{aligned} \tag{38}$$

where we have used the Cauchy–Schwarz inequality. Applying again the Ascoli theorem implies as above that $\cup_{w\in B_1}D_uF_0(\omega_0, p_0)w$ is relatively compact in $C_{\omega_0}(X)$. Hence, $D_uF_0(\omega_0, p_0)$ is a compact mapping from $L^2((0, \omega_0), X)$ into itself and $I - D_uF_0(\omega_0, p_0)$ is a Fredholm operator of index 0 from $L^2((0, \omega_0), X)$ into itself.

3. We first remark that, if ψ is an ω_0-periodic solution of the equation

$$\psi_t = B_0\psi + Df_0(p_0)\psi - \frac{p_{0t}}{\omega_0}, \tag{39}$$

then $\psi + tp_{0t}/\omega_0$ is a solution of Eq. (17) and

$$\Pi_0(\omega_0,0)\psi(0) = \psi(0) + p_{0t}(0),$$

that is, $\psi(0)$ is a generalized eigenvector of $\Pi_0(\omega_0,0)$ corresponding to the eigenvalue 1. Conversely, if φ_0 is a generalized eigenvector of $\Pi_0(\omega_0,0)$ corresponding to the eigenvalue 1, then $\Pi_0(s,0)\varphi_0 = \varphi(s)$ satisfies the equality

$$\begin{aligned}\frac{d}{dt}\left(\varphi - \frac{t}{\omega_0}p_{0t}\right) &= B_0\varphi + D_u f_0(p_0)\varphi - \frac{t}{\omega_0}p_{0tt} - \frac{p_{0t}}{\omega_0}\\ &= B_0\left(\varphi - \frac{t}{\omega_0}p_{0t}\right) + Df_0(p_0)\left(\varphi - \frac{t}{\omega_0}p_{0t}\right) - \frac{p_{0t}}{\omega_0},\end{aligned}$$

and $\varphi(\omega_0) - p_{0t}(\omega_0) = \varphi_0$, that is, $\varphi - \frac{t}{\omega_0}p_{0t}$ is an ω_0-periodic solution of the Eq. (39).

Notice that the regularity hypotheses on p_0 imply that $D_\omega F_0(\omega_0,p_0)$ is well defined. In [4] (see also [19]), it has been proved that if Eq. (39) has no ω_0-periodic solution, then Eq. (33) has no ω_0-periodic solution. Thus, since 1 is a (algebraically) simple eigenvalue of $\Pi_0(\omega_0,0)$, Eq. (33) has no ω_0-periodic solution. The proof of the lemma is completed. □

The next lemma, which is a comparison lemma, will be used in the proof of the existence of a periodic orbit for the perturbed problem.

Lemma 2.5. *Let $0 < t_1 < \omega_0$ be a fixed constant. Under the decay property (13) and Hypothesis (H4), there exist positive constants $C_2 = C_2(t_1)$ and $\beta_2 = \min(\beta_0,\beta_1)$ such that the following estimate holds, for any $t_1 \le \omega \le 2\omega_0$, and any $w \in C_{\omega_0}(Z)$,*

$$\|J_\varepsilon(\omega)w - J_0(\omega)w\|_{C_{\omega_0}(X)} \le C_2\varepsilon^{\beta_2}\|w\|_{C_{\omega_0}(Z)}. \tag{40}$$

Moreover, one can choose the constant $C_2 = C_2(t_1)$ large enough so that, under the decay property (13) and the Hypotheses (H.2)–(H.5), we have, for any $t_1 \le \omega \le 2\omega_0$,

$$\|F_\varepsilon(\omega,p_0) - F_0(\omega,p_0)\|_{C_{\omega_0}(X)} \le C_2\varepsilon^{\beta_2}, \tag{41}$$

and, for any $w \in C_{\omega_0}(X)$,

$$\|D_uF_\varepsilon(\omega,p_0)w - D_uF_0(\omega,p_0)w\|_{C_{\omega_0}(X)} \le C_2\varepsilon^{\beta_2}\|w\|_{C_{\omega_0}(X)}. \tag{42}$$

Proof. 1. We begin by estimating the term $\|J_\varepsilon(\omega)w - J_0(\omega)w\|_{C_{\omega_0}(X)}$, when w belongs to $C_{\omega_0}(Z)$. Going back to the definition of J_ε, we write,

$$\begin{aligned}
&J_\varepsilon(\omega)w - J_0(\omega)w \\
&\quad = S_{\varepsilon,\omega}(t)(I - S_{\varepsilon,\omega}(\omega_0))^{-1} \int_0^{\omega_0} \big(S_{\varepsilon,\omega}(\omega_0 - s) - S_{0,\omega}(\omega_0 - s)\big) w(s)ds \\
&\qquad + \int_0^t \big(S_{\varepsilon,\omega}(t-s) - S_{0,\omega}(t-s)\big) w(s)ds \\
&\qquad + S_{\varepsilon,\omega}(t)\big[(I - S_{\varepsilon,\omega}(\omega_0))^{-1} - (I - S_{0,\omega}(\omega_0))^{-1}\big] \int_0^{\omega_0} S_{0,\omega}(\omega_0 - s) w(s)ds \\
&\qquad + (S_{\varepsilon,\omega}(t) - S_{0,\omega}(t))(I - S_{0,\omega}(\omega_0))^{-1} \int_0^{\omega_0} S_{0,\omega}(\omega_0 - s) w(s)ds \\
&\quad \equiv J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{43}$$

The decay hypothesis (13), the property (22) of Lemma 2.3 and Hypothesis (H4) imply that

$$\|J_1 + J_2 + J_4\|_{C_{\omega_0}(X)} \leq 2C_0\varepsilon^{\beta_0}\omega_0(C_0C_1 + 1)\|w\|_{C_{\omega_0}(Z)}. \tag{44}$$

Remarking that

$$\begin{aligned}
&(I - S_{\varepsilon,\omega}(\omega_0))^{-1} - (I - S_{0,\omega}(\omega_0))^{-1} \\
&\qquad = (I - S_{\varepsilon,\omega}(\omega_0))^{-1}(S_{\varepsilon,\omega}(\omega_0) - S_{0,\omega}(\omega_0))(I - S_{0,\omega}(\omega_0))^{-1},
\end{aligned}$$

we also obtain that

$$\|J_3\|_{C_{\omega_0}(X)} \leq C_0^3 C_1^2 \omega_0 \varepsilon^{\beta_0} \|w\|_{C_{\omega_0}(Z)}. \tag{45}$$

From the equality (43) and the estimates (44) and (45), we deduce the inequality (40), for $t_1 \leq \omega \leq 2\omega_0$.

2. In order to estimate the term $\|F_\varepsilon(\omega, p_0) - F_0(\omega, p_0)\|_{C_{\omega_0}(X)}$, we remark that

$$\begin{aligned}
F_\varepsilon(\omega, p_0) - F_0(\omega, p_0) &= J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} f_\varepsilon(p_0)\right) - J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} f_0(p_0)\right) \\
&\quad + J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} f_0(p_0)\right) - J_0(\omega)\left(\frac{\omega}{\omega_0} f_0(p_0)\right).
\end{aligned} \tag{46}$$

As above, the decay hypothesis (13) and the property (22) of Lemma 2.3 imply that, for $t_1 \leq \omega \leq 2\omega_0$,

$$\sup(\|J_\varepsilon(\omega)\|_{L(X,X)}, \|J_\varepsilon(\omega)\|_{L(Z,Z)}) \leq C_0\omega_0(C_0C_1 + 2). \tag{47}$$

From the above estimate and from Hypothesis (H5), we deduce that, for $t_1 \leq \omega \leq 2\omega_0$,

$$\left\| J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} f_\varepsilon(p_0)\right) - J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} f_0(p_0)\right)\right\|_{C_{\omega_0}(X)} \leq 2C_0^2 \omega_0 (C_0 C_1 + 2)\varepsilon^{\beta_1}. \tag{48}$$

It remains to estimate the last term in the right hand side of the equality (46). From the equality (43) and the estimates (44) and (45) (or from the estimate (40)), we immediately derive that, for $t_1 \leq \omega \leq 2\omega_0$,

$$\left\| J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} f_0(p_0)\right) - J_0(\omega)\left(\frac{\omega}{\omega_0} f_0(p_0)\right)\right\|_{C_{\omega_0}(X)}$$
$$\leq C_0 \varepsilon^{\beta_0} \omega_0 (2C_0C_1 + 2 + C_0^2 C_1^2) \|f_0(p_0)\|_{C_{\omega_0}(Z)}. \tag{49}$$

The equality (46) and the estimates (48) and (49) imply the inequality (41) in the lemma.

3. We finally prove the estimate (42). Arguing as in (48), by using the regularity Hypothesis (H3) and (H5) together with the properties (22) and (47), we show that, for any $t_1 \leq \omega \leq 2\omega_0$, for any $w \in C_{\omega_0}(X)$,

$$\left\| J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} D_u f_\varepsilon(p_0) w\right) - J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} D_u f_0(p_0) w\right)\right\|_{C_{\omega_0}(X)}$$
$$\leq 2C_0^2 \omega_0 (2 + C_0 C_1)\varepsilon^{\beta_1} \|w\|_{C_{\omega_0}(X)}. \tag{50}$$

Since by Hypothesis (H3), $Df_0(p_0)$ is a bounded linear operator from X into Z, we can follow the lines of the proof of the estimate (40) (or directly use this inequality) to obtain that, for any $t_1 \leq \omega \leq 2\omega_0$ and for any $w \in C_{\omega_0}(X)$,

$$\left\| J_\varepsilon(\omega)\left(\frac{\omega}{\omega_0} D_u f_0(p_0) w\right) - J_0(\omega)\left(\frac{\omega}{\omega_0} D_u f_0(p_0) w\right)\right\|_{C_{\omega_0}(X)}$$
$$\leq 2C_0 \varepsilon^{\beta_0} \omega_0 (2C_0C_1 + 2 + C_0^2 C_1^2) \|D_u f_0(p_0) w\|_{C_{\omega_0}(Z)}$$
$$\leq C_2 \varepsilon^{\beta_0} \|w\|_{C_{\omega_0}(Z)}. \tag{51}$$

The estimates (50) and (51) imply the inequality (42) in the lemma. □

Remark 2.6. As a direct consequence of the inequality (40) of Lemma 2.5 and of the proof of the identity (30), by choosing a larger constant C_2 if necessary, we obtain the following estimate, for any $0 \leq \varepsilon \leq \omega_0$ and for any $w \in C_{\omega_0}(Z)$,

$$\left\| \frac{dJ_\varepsilon}{d\omega} B_\varepsilon^{-1} w - \frac{dJ_0}{d\omega} B_0^{-1} w \right\|_{C_{\omega_0}(X)} \leq C_2 \varepsilon^{\beta_2} \|w\|_{C_{\omega_0}(Z)}. \tag{52}$$

We shall use this estimate in the proof of the uniqueness of the periodic solution of $T_\varepsilon(t)$.

Lemma 2.4 implies that there exists an element $v^* \in L^2((0,\omega_0),X^*)$ such that the continuous linear form $\ell_0(\cdot) = \int_0^{\omega_0}\langle \cdot, v^*\rangle$ on $L^2((0,\omega_0),X)$ satisfies the following properties

$$\ell_0(p_{0t}) = 1, \quad \ell_0((I - D_wF_0(\omega_0,p_0))u) = 0, \quad \forall u \in L^2((0,\omega_0),X). \tag{53}$$

Thus, we can decompose the spaces $L^2((0,\omega_0),X)$ and $C_{\omega_0}(X)$ (resp. $C_{\omega_0}(Z)$) as follows

$$L^2((0,\omega_0),X) = \langle p_{0t}\rangle + L^2_0((0,\omega_0),X),$$

and

$$C_{\omega_0}(X) = \langle p_{0t}\rangle + \mathcal{X}, \quad \text{resp. } C_{\omega_0}(Z) = \langle p_{0t}\rangle + \mathcal{Z},$$

where $\langle p_{0t}\rangle$ denotes the one-dimensional space generated by p_{0t} and $L^2_0((0,\omega_0),X) = \{w \in L^2((0,\omega_0),X) \,|\, \ell_0(w) = 0\}$, $\mathcal{X} = \{w \in C_{\omega_0}(X) \,|\, \ell_0(w) = 0\}$ (resp. $\mathcal{Z} = \{w \in C_{\omega_0}(Z) \,|\, \ell_0(w) = 0\}$). We notice that $(I - D_wF_0(\omega_0,p_0))_{|L^2_0((0,\omega_0),X)}$, $(I - D_wF_0(\omega_0,p_0))_{|\mathcal{X}}$ and $(I - D_wF_0(\omega_0,p_0))_{|\mathcal{Z}}$ are isomorphisms.

For $\varepsilon > 0$ small and ω close to ω_0, we want to find a ("unique") ω_0-periodic solution $p_0 + \varphi$ of Eq. (27), or equivalently, a (unique) fixed point of $F_\varepsilon(\omega,\cdot)$, close to p_0. Let r and δ be small positive numbers. We introduce the following mapping $L_\varepsilon(\omega,\varphi) : (\omega,\varphi) \in (\omega_0 - \delta, \omega_0 + \delta) \times B_{\mathcal{X}}(0,r) \mapsto L_\varepsilon(\omega,\varphi) \in \mathcal{X}$,

$$L_\varepsilon(\omega,\varphi) = F_\varepsilon(\omega, p_0 + \varphi) - (p_0 + \varphi) - \ell_0(F_\varepsilon(\omega, p_0 + \varphi) - (p_0 + \varphi))p_{0t}.$$

The equation

$$F_\varepsilon(\omega, p_0 + \varphi) = p_0 + \varphi$$

is equivalent to the system

$$\begin{aligned} L_\varepsilon(\omega,\varphi) &= 0, \\ \ell_0(F_\varepsilon(\omega, p_0 + \varphi) - (p_0 + \varphi)) &= 0. \end{aligned} \tag{54}$$

Let us next introduce the map $\mathcal{L}_\varepsilon(\omega,\varphi) \in (\omega_0 - \delta, \omega_0 + \delta) \times B_{\mathcal{X}}(0,r) \mapsto \mathcal{L}_\varepsilon(\omega,\varphi) \in \mathcal{X}$, given by

$$\mathcal{L}_\varepsilon(\omega,\varphi) = \varphi - (D_wF_0(\omega_0,p_0) - I)^{-1}L_\varepsilon(\omega,\varphi). \tag{55}$$

Since φ is a zero of the map $L_\varepsilon(\omega,\cdot)$ if and only if φ is a fixed point of $\mathcal{L}_\varepsilon(\omega,\cdot)$, we are reduced to prove the following two properties:

Property 1: For ω close to ω_0, for $r > 0$ and $\varepsilon > 0$ small enough, find a unique fixed point $\varphi(\omega,\varepsilon) \in B_{\mathcal{X}}(0,r)$ of the map $\mathcal{L}_\varepsilon(\omega,\varphi)$.

Property 2: For $\varepsilon > 0$ small enough, find a unique solution ω_ε (close to ω_0) of $\ell_0(F_\varepsilon(\omega_\varepsilon, p_0 + \varphi(\omega_\varepsilon, \varepsilon)) - (p_0 + \varphi(\omega_\varepsilon, \varepsilon))) = 0$.

2.1.1 Step 1

In order to prove that, for ω close to ω_0, for $r > 0$ and $\varepsilon > 0$ small enough, the map $\mathcal{L}_\varepsilon(\omega, \varphi)$ has a unique fixed point $\varphi(\omega, \varepsilon) \in B_{\mathcal{X}}(0, r)$, we shall apply the strict contraction mapping theorem to $\mathcal{L}_\varepsilon(\omega, \cdot)$.

Theorem 2.7. *1. Under the hypotheses (H1)–(H5), there is a positive constant r_0, and, for $0 < r \leq r_0$, there exist $\varepsilon_0(r) > 0$ and $\delta_0(r) > 0$ such that, for $0 < r \leq r_0$, for $0 < \varepsilon \leq \varepsilon_0(r)$ and for $|\omega - \omega_0| \leq \delta_0(r)$, $\mathcal{L}_\varepsilon(\omega, \cdot)$ has a fixed point $\varphi(\varepsilon, \omega) \in B_{\mathcal{X}}(0, r)$. This fixed point is unique in $B_{\mathcal{X}}(0, r_0)$. The mapping $\varphi(\cdot, \varepsilon) : \omega \in [\omega_0 - \delta_0(r), \omega_0 + \delta_0(r)] \mapsto \varphi(\omega, \varepsilon) \in \mathcal{X}$ is a continuous map. Moreover, there exists a positive constant C_4 such that, for $0 < r \leq r_0$, for $0 < \varepsilon \leq \varepsilon_0(r)$ and for $|\omega - \omega_0| \leq \delta_0(r)$,*

$$\|\varphi(\varepsilon, \omega)\|_{\mathcal{X}} \leq C_4(\varepsilon^{\beta_2} + |\omega - \omega_0|). \tag{56}$$

2. Under the additional Hypothesis (H6), if $p_0 + \varphi(\varepsilon, \omega_i) \in \mathcal{X}$, $i = 1, 2$, are two periodic solutions of (15) *of periods ω_i with $|\omega_i - \omega_0| \leq \delta_0(r_0)$, $i = 1, 2$, and if $0 \leq \varepsilon \leq \varepsilon_0(r_0)$, then*

$$\|\varphi(\varepsilon, \omega_1) - \varphi(\varepsilon, \omega_2)\|_{\mathcal{X}} \leq C_4|\omega_1 - \omega_2|. \tag{57}$$

Proof. In this proof, c_1, c_2, ..., will denote positive constants independent of ε, ω, etc.

1. We first show that $\mathcal{L}_\varepsilon(\omega, \cdot)$ is a strict contraction, if the positive constants r, δ and ε are small enough. For $|\omega - \omega_0| \leq \delta$ and $\varphi_1 \in B_{\mathcal{X}}(0, r)$, $\varphi_2 \in B_{\mathcal{X}}(0, r)$, we can write

$$\begin{aligned}
&\mathcal{L}_\varepsilon(\omega, \varphi_1) - \mathcal{L}_\varepsilon(\omega, \varphi_2) \\
&\quad = (D_w F_0(\omega_0, p_0) - I)^{-1} \Big(D_w F_0(\omega_0, p_0)(\varphi_1 - \varphi_2) - F_\varepsilon(\omega, p_0 + \varphi_1) \\
&\qquad + F_\varepsilon(\omega, p_0 + \varphi_2) - \ell_0(F_\varepsilon(\omega, p_0 + \varphi_2) - F_\varepsilon(\omega, p_0 + \varphi_1) - (\varphi_2 - \varphi_1)) p_{0t} \Big),
\end{aligned}$$

or also, since $\ell_0((D_w F_0(\omega_0, p_0) - I)(\varphi_2 - \varphi_1))$ vanishes,

$$\begin{aligned}
&\|\mathcal{L}_\varepsilon(\omega, \varphi_1) - \mathcal{L}_\varepsilon(\omega, \varphi_2)\|_{\mathcal{X}} \\
&\quad \leq \Big\| \Big(\int_0^1 (D_w F_\varepsilon(\omega, p_0 + \varphi_1 + s(\varphi_2 - \varphi_1)) - D_w F_0(\omega_0, p_0))(\varphi_2 - \varphi_1)\, ds \\
&\qquad - \ell_0 \Big(\int_0^1 (D_w F_\varepsilon(\omega, p_0 + \varphi_1 + s(\varphi_2 - \varphi_1)) - D_w F_0(\omega_0, p_0))(\varphi_2 - \varphi_1) ds \Big) p_{0t} \Big) \Big\|_{\mathcal{X}} \\
&\qquad \times \|(D_w F_0(\omega_0, p_0) - I)^{-1}\|_{L(\mathcal{X}, \mathcal{X})}.
\end{aligned} \tag{58}$$

In order to obtain an upper-bound of the term in the right-hand side of the inequality (58), we first write

$$\|(D_wF_\varepsilon(\omega, p_0+\varphi_1+s(\varphi_2-\varphi_1))-D_wF_0(\omega_0,p_0))(\varphi_2-\varphi_1)\|_{\mathcal{X}} \leq M_1+M_2+M_3, \tag{59}$$

where

$$\begin{aligned} M_1 &= \|(D_wF_\varepsilon(\omega, p_0+\varphi_1+s(\varphi_2-\varphi_1))-D_wF_\varepsilon(\omega,p_0))(\varphi_2-\varphi_1)\|_{C_{\omega_0}(X)}, \\ M_2 &= \|(D_wF_\varepsilon(\omega,p_0)-D_wF_0(\omega,p_0))(\varphi_2-\varphi_1)\|_{C_{\omega_0}(X)}, \\ M_3 &= \|(D_wF_0(\omega,p_0)-D_wF_0(\omega_0,p_0))(\varphi_2-\varphi_1)\|_{C_{\omega_0}(X)}. \end{aligned} \tag{60}$$

Since f_ε is of class C^2 from X into X, uniformly with respect to ε (see the inequality (14)), there exists a positive constant C_3 such that,

$$M_1 \leq C_3 r\|\varphi_2-\varphi_1\|_{\mathcal{X}}. \tag{61}$$

The estimate (42) in Lemma 2.5 implies that

$$M_2 \leq C_2\varepsilon^{\beta_2}\|\varphi_2-\varphi_1\|_{\mathcal{X}}. \tag{62}$$

To estimate the term M_3, we first notice that

$$\begin{aligned} M_3 \leq\ & \frac{\omega-\omega_0}{\omega_0}\|J_0(\omega)D_uf_0(p_0)(\varphi_2-\varphi_1)\|_{C_{\omega_0}(X)} \\ & +\|(J_0(\omega)-J_0(\omega_0))D_uf_0(p_0)(\varphi_2-\varphi_1)\|_{C_{\omega_0}(X)} \\ \leq\ & \big(c_1|\omega-\omega_0|+\sup_{v\in B_1}\|(J_0(\omega)-J_0(\omega_0))D_uf_0(p_0)v\|_{C_{\omega_0}(X)}\big)\|\varphi_2-\varphi_1\|_{\mathcal{X}}, \end{aligned} \tag{63}$$

where B_1 is the ball of center 0 and radius 1 in $C_{\omega_0}(X)$.

Taking into account the expression of $J_0(\omega)-J_0(\omega_0)$ (see (43), for example, with $J_\varepsilon(\omega)$ and $S_{\varepsilon,\omega}$ replaced by $J_0(\omega_0)$ and by S_{0,ω_0} respectively) and the equality (29) and remarking that, by Hypothesis (H3), the sets

$$\cup_{s\in[0,\omega_0]}\cup_{w\in B_1} D_uf_0(p_0(s))w(s)$$

and

$$\cup_{w\in B_1}(I-S_{0,\omega_0}(\omega_0))^{-1}\int_0^{\omega_0} S_{0,\omega_0}(\omega_0-s)D_uf_0(p_0(s))w(s)ds$$

are bounded sets in Z and thus relatively compact sets in X, one readily shows that the mapping $\omega \mapsto J_0(\omega)D_u f_0(p_0)v \in C_{\omega_0}(X)$ is continuous in ω, uniformly with respect to $v \in B_1$. Hence, for any $\eta > 0$, there exists a positive constant $\delta_0(\eta) < \eta$ such that, for $|\omega - \omega_0| \leq \delta_0(\eta)$,

$$M_3 \leq c_2 \eta \|\varphi_2 - \varphi_1\|_{\mathcal{X}}. \tag{64}$$

The estimates (59), (61)–(64) imply that, for $|\omega - \omega_0| \leq \delta_0(\eta)$,

$$\begin{aligned}
&\|(D_w F_\varepsilon(\omega, p_0 + \varphi_1 + s(\varphi_2 - \varphi_1)) - D_w F_0(\omega_0, p_0))(\varphi_2 - \varphi_1)\|_{C_{\omega_0}(X)} \\
&\quad \leq (C_3 r + C_2 \varepsilon^{\beta_2} + c_2 \eta)\|\varphi_2 - \varphi_1\|_{\mathcal{X}}.
\end{aligned} \tag{65}$$

We next choose $r_0 > 0$, $\varepsilon_0 > 0$ and $\eta_0 > 0$ (and thus $\delta_0 \equiv \delta_0(\eta_0)$) such that,

$$\begin{aligned}
&\left(C_3 r_0 + C_2 \varepsilon_0^{\beta_2} + c_2 \eta_0\right)\left(1 + \|\ell_0\|_{L(C_{\omega_0}(X),\mathbb{R})}\|p_{0t}\|_X\right) \\
&\qquad \|(D_w F_0(\omega_0, p_0) - I)^{-1}_{|\mathcal{X}}\|_{L(\mathcal{X},\mathcal{X})} < \frac{1}{2}.
\end{aligned} \tag{66}$$

The inequality (58) and the estimates (65) and (66) imply that, for $0 < \varepsilon \leq \varepsilon_0$, $|\omega - \omega_0| \leq \delta_0$ and for φ_1, φ_2 in $B_{\mathcal{X}}(0, r_0)$,

$$\|\mathcal{L}_\varepsilon(\omega, \varphi_1) - \mathcal{L}_\varepsilon(\omega, \varphi_2)\|_{\mathcal{X}} \leq \frac{1}{2}\|\varphi_1 - \varphi_2\|_{\mathcal{X}}, \tag{67}$$

which means that $\mathcal{L}_\varepsilon$ is a strict contraction.

It remains to show that, for $0 < r \leq r_0$, there exist $\varepsilon_0(r) > 0$ and $\delta_0(r) > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0(r)$, for $|\omega - \omega_0| \leq \delta_0(r)$, $\mathcal{L}_\varepsilon(\omega, \cdot)$ maps the ball $B_{\mathcal{X}}(0, r)$ into itself.

Writing $\mathcal{L}_\varepsilon(\omega, \varphi) = (\mathcal{L}_\varepsilon(\omega, \varphi) - \mathcal{L}_\varepsilon(\omega, 0)) + \mathcal{L}_\varepsilon(\omega, 0)$, we deduce from (67) that, for $\varphi \in B_{\mathcal{X}}(0, r)$,

$$\|\mathcal{L}_\varepsilon(\omega, \varphi)\|_{\mathcal{X}} \leq \frac{1}{2} r + \|\mathcal{L}_\varepsilon(\omega, 0)\|_{\mathcal{X}}. \tag{68}$$

Since p_0 is a fixed point of $F_0(\omega_0, \cdot)$, we have the equality

$$\begin{aligned}
\mathcal{L}_\varepsilon(\omega, 0) = &(D_w F_0(\omega_0, p_0) - I)^{-1}\Big(F_\varepsilon(\omega, p_0) - F_0(\omega_0, p_0) \\
&\qquad - \ell_0(F_\varepsilon(\omega_0, p_0) - F_0(\omega_0, p_0))p_{0t}\Big).
\end{aligned} \tag{69}$$

As, by Hypothesis (H2), $f_0(p_0([0, \omega_0]))$ is a compact subset of $D(B_0)$, the map $\omega \in [t_1, \omega_0] \mapsto F_0(\omega, p_0) \in C_{\omega_0}(X)$ is of class C^1 and thus we obtain the following estimate

$$
\begin{aligned}
&\|F_0(\omega,p_0)-F_0(\omega_0,p_0)\|_{C_{\omega_0}(X)}\\
&\quad=\|(J_0(\omega)-J_0(\omega_0))\frac{\omega}{\omega_0}f_0(p_0)+\frac{\omega-\omega_0}{\omega_0}J_0(\omega_0)f_0(p_0)\|_{C_{\omega_0}(X)}\\
&\quad\le c_3|\omega-\omega_0|\Big(\sup_{s\in[0,\omega_0]}\|f_0(p_0(s))\|_{D(B_0)}+\sup_{s\in[0,\omega_0]}\|f_0(p_0(s))\|_X\Big).
\end{aligned}
\tag{70}
$$

The equality (69), the inequalities (70) and, the estimate (41) of Lemma 2.5 lead to the estimate

$$
\begin{aligned}
\|\mathcal{L}_\varepsilon(\omega,0)\|_{\mathcal{X}}&\le(C_2\varepsilon^{\beta_2}+2K_0c_3|\omega-\omega_0|)\\
&\qquad(1+\|\ell_0\|_{L(C_{\omega_0}(X),\mathbb{R})}\|p_{0t}\|_X)\|(D_wF_0(\omega_0,p_0)-I)^{-1}_{|\mathcal{X}}\|_{L(\mathcal{X})}
\end{aligned}
\tag{71}
$$

which, together with (68), implies that, for $|\omega-\omega_0|\le\delta_0(r)\le\delta_0$ and $0<\varepsilon\le\varepsilon_0(r)$,

$$
\begin{aligned}
\|\mathcal{L}_\varepsilon(\omega,\varphi)\|_{\mathcal{X}}\le\frac{r}{2}&+(C_2\varepsilon^{\beta_2}+2K_0c_3|\omega-\omega_0|)(1+\|\ell_0\|_{L(C_{\omega_0}(X),\mathbb{R})}\|p_{0t}\|_X)\\
&\times\|(D_wF_0(\omega_0,p_0)-I)^{-1}_{|\mathcal{X}}\|_{L(\mathcal{X})}\le r,
\end{aligned}
\tag{72}
$$

if $\varepsilon_0(r)>0$ and $\delta_0(r)>0$ are chosen so that

$$
\begin{aligned}
&(C_2\varepsilon_0(r)^{\beta_2}+2K_0c_3\delta_0(r))(1+\|\ell_0\|_{L(C_{\omega_0}(X),\mathbb{R})}\|p_{0t}\|_X)\\
&\quad\|(D_wF_0(\omega_0,p_0)-I)^{-1}_{|\mathcal{X}}\|_{L(\mathcal{X})}<\frac{1}{2}r.
\end{aligned}
$$

Hence, we have proved that, for any $0<r\le r_0$, there exist $\varepsilon_0(r)>0$ and $\delta_0(r)>0$ such that, for $0<\varepsilon\le\varepsilon_0(r)$, for $|\omega-\omega_0|\le\delta_0(r)$, $\mathcal{L}_\varepsilon(\omega,\cdot)$ is a strict contraction from $B_{\mathcal{X}}(0,r)$ into itself. Therefore, by the strict contraction fixed point theorem, for $0<\varepsilon\le\varepsilon_0(r)$ and $|\omega-\omega_0|\le\delta_0(r)$, $\mathcal{L}_\varepsilon(\omega,\cdot)$ has a unique fixed point $\varphi(\varepsilon,\omega)$ in $B_{\mathcal{X}}(0,r)$. Thus the equation $L_\varepsilon(\omega,\cdot)=0$ has a unique solution $\varphi(\varepsilon,\omega)$ in $B_{\mathcal{X}}(0,r)$.

2. Since $\varphi(\varepsilon,\omega)$ is a fixed point of $\mathcal{L}_\varepsilon(\omega,\cdot)$, the contraction property (67) implies that

$$
\|\varphi(\varepsilon,\omega)\|_{\mathcal{X}}\le 2\|\mathcal{L}_\varepsilon(\omega,0)\|_{\mathcal{X}}.
\tag{73}
$$

From the inequalities (71) and (73), we deduce that, for $|\omega-\omega_0|\le\delta(\eta)$,

$$
\|\varphi(\varepsilon,\omega)\|_{\mathcal{X}}\le C_4(\varepsilon^{\beta_2}+|\omega-\omega_0|),
\tag{74}
$$

where C_4 is a positive constant.

We next show that the map $\varphi(\cdot,\varepsilon):\omega\in[\omega_0-\delta_0(r),\omega_0+\delta_0(r)]\mapsto\varphi(\omega,\varepsilon)\in\mathcal{X}$ is continuous. Let ω^* belong to $[\omega_0-\delta_0(r),\omega_0+\delta_0(r)]$ and let $|\tau|$ be small enough. We denote by $\varphi(\varepsilon,\omega^*)$ and $\varphi(\varepsilon,\omega^*+\tau)$ the fixed point of $\mathcal{L}_\varepsilon(\omega^*,\cdot)$ and $\mathcal{L}_\varepsilon(\omega^*+\tau,\cdot)$ respectively. The contraction property (67) implies that

$$\|\varphi(\varepsilon,\omega^*)-\varphi(\varepsilon,\omega^*+\tau)\|_{\mathcal{X}}\leq 2\|\mathcal{L}_\varepsilon(\omega^*,\varphi(\varepsilon,\omega^*))-\mathcal{L}_\varepsilon(\omega^*+\tau,\varphi(\varepsilon,\omega^*))\|_{\mathcal{X}}. \tag{75}$$

Coming back to the definitions (55) and (31) of $\mathcal{L}_\varepsilon$ and F_ε and using the inequality (67), we obtain the estimates

$$\begin{aligned}
&\|\mathcal{L}_\varepsilon(\omega^*,\varphi(\varepsilon,\omega^*))-\mathcal{L}_\varepsilon(\omega^*+\tau,\varphi(\varepsilon,\omega^*))\|_{\mathcal{X}}\\
&\quad\leq c_4\|L_\varepsilon(\omega^*,\varphi(\varepsilon,\omega^*))-L_\varepsilon(\omega^*+\tau,\varphi(\varepsilon,\omega^*))\|_{C_{\omega_0}(X)}.\\
&\quad\leq c_5\left\|\left(\frac{\omega^*}{\omega_0}J_\varepsilon(\omega^*)-\frac{\omega^*+\tau}{\omega_0}J_\varepsilon(\omega^*+\tau)\right)\quad f_\varepsilon\left(p_0+\varphi(\varepsilon,\omega^*)\right)\right\|_{C_{\omega_0}(X)}.
\end{aligned} \tag{76}$$

Since, by Hypothesis (H3), $\cup_{s\in[0,\omega_0]}f_\varepsilon(p_0(s)+\varphi(\varepsilon,\omega^*)(s))$ is a compact subset of X and that $S_{\varepsilon,\omega}$ is a C^0-semigroup on X, the mapping $J_\varepsilon:\omega\in[t_1,2\omega_0]\mapsto J_\varepsilon(\omega)f_\varepsilon(p_0+\varphi(\varepsilon,\omega^*))\in C_{\omega_0}(X)$ is a (uniformly) continuous function of ω. And we deduce from the inequalities (75) and (76) that $\varphi(\varepsilon,\cdot)$ is a continuous map of ω at $\omega=\omega^*$.

It remains to show that the estimate (57) holds for the periodic solutions of (15), if Hypothesis (H6) holds. First, one remarks that one can choose r_0, $\varepsilon_0(r_0)$ and $\delta_0(r_0)\leq\delta_0$ small enough so that, if $p_0+\varphi(\varepsilon,\omega_\varepsilon)$ is a periodic solution of (15) of period ω_ε with $|\omega-\omega_\varepsilon|\leq\delta_0(r_0)$, then $\Gamma_\varepsilon=\{p_0+\varphi(\varepsilon,\omega_\varepsilon)(t)\}$ is contained in $\mathcal{V}_0$. Hence, by Hypothesis (H6), the set $\cup_{t\in[0,\omega_0]}f_\varepsilon(p_0(t)+\varphi(\varepsilon,\omega_\varepsilon)(t))$ is a compact subset of $D(B_\varepsilon)$. Therefore the map $\omega\in[t_1,+\infty)\mapsto J_\varepsilon(\omega)f_\varepsilon(p_0+\varphi(\varepsilon,\omega_\varepsilon))$ is of class C^1.

Let now $p_0+\varphi(\varepsilon,\omega_i)$, $i=1,2$, be two solutions of (15). The inequalities (75) and (76) imply that

$$\begin{aligned}
\|\varphi(\varepsilon,\omega_1)-\varphi(\varepsilon,\omega_2)\|_{\mathcal{X}}\leq c_6\Bigg(&\left|\frac{\omega_1-\omega_2}{\omega_0}\right|\|J_\varepsilon(\omega_1)f_\varepsilon(p_0+\varphi(\varepsilon,\omega_1))\|_{C_{\omega_0}(X)}\\
&+\left|\frac{\omega_2}{\omega_0}\right|\|\left(J_\varepsilon(\omega_1)-J_\varepsilon(\omega_2)\right)f_\varepsilon(p_0+\varphi(\varepsilon,\omega_1))\|_{C_{\omega_0}(X)}\Bigg).
\end{aligned}$$

Since the map $\omega\in[t_1,+\infty)\mapsto J_\varepsilon(\omega)f_\varepsilon(p_0+\varphi(\varepsilon,\omega_1))\in C_{\omega_0}(X)$ is of class C^1, by taking into account the equality (30) together with the estimates (21) and (22), we deduce from the above inequality that,

$$
\begin{aligned}
&\|\varphi(\varepsilon,\omega_1)-\varphi(\varepsilon,\omega_2)\|_{\mathcal{X}} \\
&\quad\leq \left|\frac{\omega_1-\omega_2}{\omega_0}\right| \|J_\varepsilon(\omega_1)f_\varepsilon(p_0+\varphi(\varepsilon,\omega_1))\|_{C_{\omega_0}(X)} \\
&\qquad+\left|\frac{\omega_2(\omega_1-\omega_2)}{\omega_0}\right| \left\|\int_0^1 \frac{dJ_\varepsilon}{d\omega}(\omega_1+s(\omega_2-\omega_1))f_\varepsilon(p_0+\varphi(\varepsilon,\omega_1))ds\right\|_{C_{\omega_0}(X)} \\
&\quad\leq c_7|\omega_1-\omega_2|. \qquad (77)
\end{aligned}
$$

Hence, the theorem is proved. □

2.1.2 Step 2

It remains to show that, for $\varepsilon>0$ small enough, there exists a unique element ω_ε, close to ω_0, satisfying

$$\ell_0(F_\varepsilon(\omega_\varepsilon,p_0+\varphi(\varepsilon,\omega_\varepsilon)-(p_0+\varphi(\varepsilon,\omega_\varepsilon)))=0. \qquad (78)$$

To simplify the notations, we write $\omega=\omega_0+\tau$ and $\varphi_\varepsilon(\tau)=\varphi(\varepsilon,\omega_0+\tau)$. For $0\leq\varepsilon\leq\varepsilon_0$, looking for the zeros of the function

$$M_\varepsilon(\tau)=\ell_0(F_\varepsilon(\omega_0+\tau,p_0+\varphi_\varepsilon(\tau))-(p_0+\varphi_\varepsilon(\tau))),$$

is equivalent to looking for the fixed points of the function,

$$
\begin{aligned}
K_\varepsilon(\tau)&=\tau-\ell_0(D_\tau F_0(\omega_0,p_0))^{-1}\ell_0(F_\varepsilon(\omega_0+\tau,p_0+\varphi_\varepsilon(\tau))-(p_0+\varphi_\varepsilon(\tau))) \\
&=d_0\ell_0(F_\varepsilon(\omega_0+\tau,p_0+\varphi_\varepsilon(\tau))-(p_0+\varphi_\varepsilon(\tau))-\tau D_\tau F_0(\omega_0,p_0)), \qquad (79)
\end{aligned}
$$

where $d_0=-\ell_0(D_\tau F_0(\omega_0,p_0))^{-1}$. Since by Lemma 2.4, $\ell_0(D_\tau F_0(\omega_0,p_0))$ does not vanish, the above function is well-defined. According to Theorem 2.7, it is a continuous function with respect to τ.

We first show the following existence result. We recall that the positive constants r_0, $\delta_0(r)$ and $\varepsilon_0(r)$ have been defined in Theorem 2.7.

Lemma 2.8. *Under the Hypotheses (H1)–(H5), for any r, $0<r\leq r_0$, there exist positive constants $\varepsilon_1(r)\leq\varepsilon_0(r)$ and $\delta_1(r)\leq\delta_0(r)$ such that, for $0<\varepsilon\leq\varepsilon_1(r)$, the map K_ε has a fixed point τ_ε with $|\tau_\varepsilon|\leq\delta_1(r)$, and therefore $F_\varepsilon(\omega_0+\tau_\varepsilon,\cdot)$ has a fixed point $p_0+\varphi(\varepsilon,\omega_0+\tau_\varepsilon)\in B_{C_{\omega_0}(X)}(p_0,r)$.*

Moreover, there exists a positive constant C_5 such that

$$|\tau_\varepsilon|\leq C_5\varepsilon^{\beta_2}. \qquad (80)$$

Proof. Let $0 < r \leq r_0$ be fixed. To show the existence of a fixed point of K_ε, we shall apply the Leray fixed point theorem. It is thus sufficient to show that there exists $\delta_1(r) > 0$ so that K_ε maps the interval $[-\delta_1, \delta_1]$ into itself. To simplify the notation, we will write δ_1 instead of $\delta_1(r)$. We begin by estimating the term K_ε.

Since $\ell_0((D_u F_0(\omega_0, p_0) - I)\varphi_\varepsilon)$ vanishes, we can write

$$
\begin{aligned}
K_\varepsilon(\tau) &= d_0\ell_0\big(F_\varepsilon(\omega_0+\tau, p_0+\varphi_\varepsilon(\tau)) - F_\varepsilon(\omega_0+\tau, p_0) - D_u F_0(\omega_0, p_0)\varphi_\varepsilon(\tau) \\
&\quad + F_\varepsilon(\omega_0+\tau, p_0) - F_0(\omega_0+\tau, p_0) \\
&\quad + F_0(\omega_0+\tau, p_0) - F_0(\omega_0, p_0) - \tau D_\tau F_0(\omega_0, p_0)\big) \\
&= K_1 + K_2 + K_3 + K_4 + K_5,
\end{aligned} \tag{81}
$$

where

$$
\begin{aligned}
K_1 &= d_0\ell_0\Big(\int_0^1 \Big(D_u F_\varepsilon(\omega_0+\tau, p_0+s\varphi_\varepsilon) - D_u F_\varepsilon(\omega_0+\tau, p_0)\Big)\varphi_\varepsilon(\tau)ds\Big), \\
K_2 &= d_0\ell_0((D_u F_\varepsilon(\omega_0+\tau, p_0) - D_u F_0(\omega_0+\tau, p_0))\varphi_\varepsilon(\tau)), \\
K_3 &= d_0\ell_0((D_u F_0(\omega_0+\tau, p_0) - D_u F_0(\omega_0, p_0))\varphi_\varepsilon(\tau)), \\
K_4 &= d_0\ell_0(F_\varepsilon(\omega_0+\tau, p_0) - F_0(\omega_0+\tau, p_0)), \\
K_5 &= d_0\ell_0(F_0(\omega_0+\tau, p_0) - F_0(\omega_0, p_0) - \tau D_\tau F_0(\omega_0, p_0)).
\end{aligned} \tag{82}
$$

Since f_ε belongs to $C^2(X, X)$, we deduce from Theorem 2.7 that

$$
K_1 \leq c_1\|\varphi_\varepsilon(\tau)\|_{\mathcal{X}}^2 \leq c_1 C_4^2(\varepsilon^{\beta_2} + \tau)^2. \tag{83}
$$

Lemma 2.5 and Theorem 2.7 imply that

$$
K_2 + K_4 \leq c_2 C_2 \varepsilon^{\beta_2}\big(1 + \|\varphi_\varepsilon(\tau)\|_{\mathcal{X}}\big) \leq c_3\varepsilon^{\beta_2}\big(1 + C_4(\varepsilon^{\beta_2} + \tau)\big) \leq c_4\varepsilon^{\beta_2}. \tag{84}
$$

The estimate of K_3 is similar to the estimate of M_3 in the proof of Theorem 2.7 (see the inequalities (63) in particular). Arguing like there, we show that, for any $\eta > 0$, there exists $0 < \tau_0(\eta) \leq \eta$ such that, for $|\tau| \leq \tau_0(\eta)$,

$$
K_3 \leq c_5\eta\|\varphi_\varepsilon(\tau)\|_{\mathcal{X}},
$$

and thus, by Theorem 2.7, that

$$
K_3 \leq \eta c_5 C_4(\varepsilon^{\beta_2} + |\tau|). \tag{85}
$$

It remains to estimate the term K_5. We recall that the mapping $\omega \in [\tau_1, +\infty) \mapsto F_0(\omega, p_0) \in C_{\omega_0}(X)$ is of class C^1. By the Taylor formula, we can write,

$$
\begin{aligned}
|K_5| &= d_0\left(|\tau\int_0^1\big(D_\omega F_0(\omega_0+s\tau,p_0)-D_\omega F_0(\omega_0,p_0)\big)ds|\right)\\
&\le d_0\left(|\tau\int_0^1\big(D_\omega J_0(\omega_0+s\tau)-D_\omega J_0(\omega_0)\big)f_0(p_0)ds|\right.\\
&\qquad+|\frac{\tau^2}{\omega_0}\int_0^1\int_0^1 D_\omega J_0(\omega_0+s\sigma\tau)f_0(p_0)sdsd\sigma|\\
&\qquad\left.+|\frac{\tau^2}{\omega_0}\int_0^1 D_\omega J_0(\omega_0+s\tau)f_0(p_0)sds|\right).
\end{aligned}
\tag{86}
$$

Arguing as in the proof of the estimate (64) of M_3, we show that there exists a positive constant $\tau_1(\eta)\le\tau_0(\eta)$, such that, for $|\tau|\le\tau_1(\eta)$,

$$
|\tau\int_0^1\big(D_\omega J_0(\omega_0+s\tau)-D_\omega J_0(\omega_0)\big)f_0(p_0)ds|\le\eta|\tau|. \tag{87}
$$

From the estimates (86) and (87), one deduces that, for $|\tau|\le\tau_1(\eta)$,

$$
|K_5|\le c_6|\tau|^2+\eta|\tau|. \tag{88}
$$

The estimates (83)–(85) and (88) imply that, for $|\tau|\le\tau_1(\eta)$,

$$
|K_\varepsilon(\tau)|\le c_7(\varepsilon^{\beta_2}+|\tau|^2+\eta|\tau|). \tag{89}
$$

One first chooses positive constants η so that $c_7\eta\le 1/4$, then δ_1 so that $\delta_1\le\inf(1/4,\delta_0(r),\tau_1(\eta))$ and finally ε_1 so that $c_7\varepsilon_1^{\beta_2}\le\delta_1/2$ and $\varepsilon_1\le\varepsilon(r)$. With this choice, we deduce from (89) that, for $0<\varepsilon\le\varepsilon_1$, $K_\varepsilon(\tau)$ maps the interval $[-\delta_1,\delta_1]$ into itself and hence admits a fixed point $\tau_\varepsilon\in[0,\delta_1]$. Moreover, (89) implies that

$$
\tau_\varepsilon\le 2c_7\varepsilon^{\beta_2}.
$$

Then the lemma is proved. □

We next prove the uniqueness of the fixed point

$$
p_0+\varphi(\varepsilon,\omega_0+\tau_\varepsilon)\in B_{C_{\omega_0}(X)}(p_0,r_0)
$$

of the mapping $F_\varepsilon(\omega_0+\tau_\varepsilon,\cdot)$, under the additional Hypotheses (H6) and (H7), that is, the uniqueness of the fixed point $\tau_\varepsilon\in[-\delta_1(r_0),\delta_1(r_0)]$ of $K_\varepsilon(\cdot)$.

Theorem 2.9. *1. Under the Hypotheses (H1)–(H7), there exist positive constants $\varepsilon_2\le\varepsilon_1(r_0)$ and $\delta_2\le\delta_1(r_0)$, such that, for $0\le\varepsilon\le\varepsilon_2$, $K_\varepsilon(\cdot)$ has a unique fixed point $\tau_\varepsilon\in[-\delta_2,\delta_2]$.*

2. *Hence, there exists a neighbourhood* $\mathcal{V}_2 \subset \mathcal{V}_0$ *of the periodic orbit* Γ_0 *so that, for* $0 \le \varepsilon \le \varepsilon_2$*, the Eq.* (15) *has a periodic solution* $p_\varepsilon(t) = p_0(t) + \varphi_\varepsilon(\varepsilon, \omega_0 + \tau_\varepsilon)(t)$ *of minimal period* $\omega_\varepsilon \in [\omega_0 - \delta_2, \omega_0 + \delta_2]$ *such that* $\Gamma_\varepsilon = \{p_\varepsilon(t)\,|\,t \in [0, \omega_\varepsilon]\}$ *is the unique periodic orbit of* (15) *contained in* $\mathcal{V}_2$ *with minimal period* $\omega_\varepsilon \in [\omega_0 - \delta_2, \omega_0 + \delta_2]$ *. Moreover, we have*

$$|\tau_\varepsilon| + \|\varphi_\varepsilon\|_{\mathcal{X}} \le C_6 \varepsilon^{\beta_2}. \tag{90}$$

Proof. Due to Theorem 2.7 and Lemma 2.8, we only have to prove the uniqueness of the fixed point τ_ε of the map $K_\varepsilon(\cdot)$, when the additional Hypotheses (H6) and (H7) hold.

Assume that, for $\varepsilon > 0$ small enough, the map K_ε has two fixed points τ_1 and τ_2 with $\tau_i \le \delta_1$, $i = 1, 2$, where $\delta_1 = \delta_1(r_0)$ has been defined in Lemma 2.8. We will show that

$$|\tau_1 - \tau_2| = |K_\varepsilon(\tau_1) - K_\varepsilon(\tau_2)| < |\tau_1 - \tau_2|,$$

which implies that $\tau_1 = \tau_2$.

As before, in this proof, c_1, c_2, ..., will denote positive constants independent of ε, τ_1, τ_2, etc.

We recall that

$$K_\varepsilon(\tau) = d_0\langle F_\varepsilon(\omega_0 + \tau, p_0 + \varphi_\varepsilon(\tau)) - p_0 - D_u F_0(\omega_0, p_0)\varphi_\varepsilon(\tau) - \tau D_\tau F_0(\omega_0, p_0), v^*\rangle,$$

where $v^* \in L^2((0, \omega_0), X^*)$. Using Hypothesis (H7), it is easy to show that the space $L^2((0, \omega_0), D(B_0^*))$ is dense in $L^2((0, \omega_0), X^*)$. Therefore, for any $\eta_0 > 0$, there exists $\tilde{v} \in L^2((0, \omega_0), D(B_0^*))$ such that

$$\|v^* - \tilde{v}\|_{L^2((0,\omega_0),X^*)} \le \eta_0.$$

We now fix $\eta_0 > 0$ (we will specify later the condition satisfied by η_0) and introduce the mapping

$$\tilde{K}_\varepsilon(\tau) = d_0\langle F_\varepsilon(\omega_0 + \tau, p_0 + \varphi_\varepsilon(\tau)) - p_0 - D_u F_0(\omega_0, p_0)\varphi_\varepsilon(\tau) - \tau D_\tau F_0(\omega_0, p_0), \tilde{v}\rangle.$$

To simplify the notations, we set $\varphi_\varepsilon(\tau_i) = \varphi_\varepsilon^i$, $i = 1, 2$. Due to Hypothesis (H6), $F_\varepsilon(\omega, p_0 + \varphi_\varepsilon^1)$ is differentiable with respect to ω and we can write

$$\begin{aligned}
K_\varepsilon(\tau_1) \; - \; & K_\varepsilon(\tau_2) - (\tilde{K}_\varepsilon(\tau_1) - \tilde{K}_\varepsilon(\tau_2)) \\
&= d_0 \left\langle \int_0^1 D_\tau F_\varepsilon(\omega_0 + \tau_2 + s(\tau_1 - \tau_2), p_0 + \varphi_\varepsilon^1)(\tau_1 - \tau_2)ds, v^* - \tilde{v} \right\rangle \\
&\quad + d_0 \left\langle \int_0^1 D_u F_\varepsilon(\omega_0 + \tau_2, p_0 + \varphi_\varepsilon^2 + s(\varphi_\varepsilon^1 - \varphi_\varepsilon^2))(\varphi_\varepsilon^1 - \varphi_\varepsilon^2)ds, v^* - \tilde{v} \right\rangle \\
&\quad + d_0 \langle D_u F_0(\omega_0, p_0)(\varphi_\varepsilon^2 - \varphi_\varepsilon^1), v^* - \tilde{v}\rangle
\end{aligned}$$

$$
\begin{aligned}
&+d_0\langle(\tau_2-\tau_1)D_\tau F_0(\omega_0,p_0),v^*-\tilde{v}\rangle\\
&=K_1^*+K_2^*+K_3^*+K_4^*. \qquad (91)
\end{aligned}
$$

Due to Hypothesis (H6), the property (22) and the equalities (30) and (31), we have,

$$
\begin{aligned}
|K_1^*| \le c_1 d_0|\tau_1-\tau_2|\eta_0 \int_0^1 \big(&\|D_\tau J_\varepsilon(\omega_0+\tau_2+s(\tau_1-\tau_2))f_\varepsilon(p_0+\varphi_\varepsilon^1)\|_{C_{\omega_0}(X)}\\
&+\|J_\varepsilon(\omega_0+\tau_2+s(\tau_1-\tau_2))f_\varepsilon(p_0+\varphi_\varepsilon^1)\|_{C_{\omega_0}(X)}\big)ds\\
\le c_2 d_0|\tau_1-\tau_2|\eta_0 K_0. \qquad (92)
\end{aligned}
$$

Using the fact that F_ε is differentiable with respect to u, uniformly with respect to ε (see the hypothesis (14)) and applying the inequality (57) of Theorem 2.7, we obtain the estimate

$$
|K_2^*| \le c_3 d_0 C_4|\tau_1-\tau_2|\eta_0. \qquad (93)
$$

In the same way, we have,

$$
|K_3^*| \le c_4 d_0 C_4|\tau_1-\tau_2|\eta_0. \qquad (94)
$$

We have already seen that, by Hypothesis (H2), $F_0(\cdot,p_0)$ is differentiable with respect to ω. Thus we can write,

$$
|K_4^*| \le c_5 d_0|\tau_1-\tau_2|\eta_0 K_0. \qquad (95)
$$

Choosing $\eta_0>0$ such that $\eta_0\le 1$ and,

$$
d_0\big((c_2+c_5)K_0+(c_3+c_4)C_4\big)\eta_0 < 1/4,
$$

we deduce from the equality (91) and the estimates (92)–(95) that

$$
|K_\varepsilon(\tau_1)-K_\varepsilon(\tau_2)-(\tilde{K}_\varepsilon(\tau_1)-\tilde{K}_\varepsilon(\tau_2))| \le \frac{1}{4}|\tau_1-\tau_2|. \qquad (96)
$$

We next estimate the term $|\tilde{K}_\varepsilon(\tau_1)-\tilde{K}_\varepsilon(\tau_2)|$. This term can be decomposed as follows:

$$
\tilde{K}_\varepsilon(\tau_1)-\tilde{K}_\varepsilon(\tau_2)=\tilde{K}_1+\tilde{K}_2, \qquad (97)
$$

where

$$
\begin{aligned}
\tilde{K}_1&=d_0\langle F_\varepsilon(\omega_0+\tau_1,p_0+\varphi_\varepsilon^1)-F_\varepsilon(\omega_0+\tau_2,p_0+\varphi_\varepsilon^1)-(\tau_1-\tau_2)D_\tau F_0(\omega_0,p_0),\tilde{v}\rangle\\
\tilde{K}_2&=d_0\langle F_\varepsilon(\omega_0+\tau_2,p_0+\varphi_\varepsilon^1)-F_\varepsilon(\omega_0+\tau_2,p_0+\varphi_\varepsilon^2)-D_uF_0(\omega_0,p_0)(\varphi_\varepsilon^1-\varphi_\varepsilon^1),\tilde{v}\rangle.
\end{aligned}
$$

Since $F_\varepsilon(\omega_0+\tau, p_0+\varphi_\varepsilon^1)$ is differentiable with respect to τ, by using the integral Taylor formula, we get,

$$\begin{aligned}\tilde{K}_1 &= d_0\left\langle \int_0^1 (D_\tau F_\varepsilon(\omega_0+\tau_s, p_0+\varphi_\varepsilon^1) - D_\tau F_0(\omega_0,p_0))(\tau_1-\tau_2)ds, \tilde{v}\right\rangle \\ &\equiv \tilde{K}_1^1 + \tilde{K}_1^2, \end{aligned} \tag{98}$$

where $\tau_s = \tau_2 + s(\tau_1-\tau_2)$ and where

$$\tilde{K}_1^1 = d_0\left\langle \int_0^1 (D_\tau F_\varepsilon(\omega_0+\tau_s, p_0+\varphi_\varepsilon^1) - D_\tau F_\varepsilon(\omega_0, p_0+\varphi_\varepsilon^1))(\tau_1-\tau_2)ds, \tilde{v}\right\rangle$$

$$\tilde{K}_1^2 = d_0\langle (D_\tau F_\varepsilon(\omega_0, p_0+\varphi_\varepsilon^1) - D_\tau F_0(\omega_0,p_0))(\tau_1-\tau_2), \tilde{v}\rangle.$$

A short computation shows that

$$\begin{aligned}\tilde{K}_1^1 = d_0\Big\langle \int_0^1 \Big(\big(D_\tau J_\varepsilon(\omega_0+\tau_s)\tau_s + (J_\varepsilon(\omega_0+\tau_s) - J_\varepsilon(\omega_0))\big)\frac{f_\varepsilon(p_0+\varphi_\varepsilon^1)}{\omega_0} \\ +(D_\tau J_\varepsilon(\omega_0+\tau_s) - D_\tau J_\varepsilon(\omega_0)) f_\varepsilon(p_0+\varphi_\varepsilon^1)\Big)(\tau_1-\tau_2)ds, \tilde{v}\Big\rangle. \end{aligned} \tag{99}$$

Since, by Hypothesis (H6), $f_\varepsilon(p_0+\varphi_\varepsilon^1)(t)$ is uniformly bounded in $C^0([0,\omega_0], D(B_\varepsilon))$, we obtain the following bound,

$$\begin{aligned} d_0\Big\langle \int_0^1 \Big(D_\tau J_\varepsilon(\omega_0+\tau_s)\tau_s + (J_\varepsilon(\omega_0+\tau_s) - J_\varepsilon(\omega_0))\Big)\frac{f_\varepsilon(p_0+\varphi_\varepsilon^1)}{\omega_0}(\tau_1-\tau_2)ds, \tilde{v}\Big\rangle \\ \leq d_0\left\langle \int_0^1 D_\tau J_\varepsilon(\omega_0+\tau_s)\frac{\tau_s}{\omega_0} f_\varepsilon(p_0+\varphi_\varepsilon^1)(\tau_1-\tau_2)ds, \tilde{v}\right\rangle \\ + d_0\left\langle \int_0^1\int_0^1 D_\tau J_\varepsilon(\omega_0+\sigma\tau_s)\frac{\tau_s}{\omega_0} f_\varepsilon(p_0+\varphi_\varepsilon^1)(\tau_1-\tau_2)dsd\sigma, \tilde{v}\right\rangle \\ \leq c_6 d_0 K_0 \sup(|\tau_1|,|\tau_2|)|\tau_1-\tau_2|. \end{aligned} \tag{100}$$

We remark that

$$\begin{aligned} d_0\left\langle \int_0^1 (D_\tau J_\varepsilon(\omega_0+\tau_s) - D_\tau J_\varepsilon(\omega_0)) f_\varepsilon(p_0+\varphi_\varepsilon^1)(\tau_1-\tau_2)ds, \tilde{v}\right\rangle \\ = d_0\langle (B_0^{-1} - B_\varepsilon^{-1})\int_0^1 (D_\tau J_\varepsilon(\omega_0+\tau_s) - D_\tau J_\varepsilon(\omega_0)) f_\varepsilon(p_0+\varphi_\varepsilon^1)(\tau_1-\tau_2)ds, B_0^*\tilde{v}\rangle \\ + d_0\langle \int_0^1 (D_\tau J_\varepsilon(\omega_0+\tau_s) - D_\tau J_\varepsilon(\omega_0)) B_\varepsilon^{-1} f_\varepsilon(p_0+\varphi_\varepsilon^1)(\tau_1-\tau_2)ds, B_0^*\tilde{v}\rangle. \end{aligned} \tag{101}$$

The first term in the right-hand side of the above equality is simply estimated as follows

$$d_0 \left\langle (B_0^{-1} - B_\varepsilon^{-1}) \int_0^1 (D_\tau J_\varepsilon(\omega_0 + \tau_s) - D_\tau J_\varepsilon(\omega_0)) f_\varepsilon(p_0 + \varphi_\varepsilon^1)(\tau_1 - \tau_2) ds, B_0^* \tilde{v} \right\rangle$$
$$\leq c_7 d_0 \|B_0^{-1} - B_\varepsilon^{-1}\|_{L(X,X)} |\tau_1 - \tau_2| \, \|B_0^* \tilde{v}\|_{L^2((0,\omega_0),X^*)}. \tag{102}$$

Since $f_\varepsilon(p_0 + \varphi_\varepsilon^1)(t)$ is uniformly bounded in $C^0([0,\omega_0], D(B_\varepsilon))$, the identity (30) shows that $D_\tau J_\varepsilon(\omega_0 + \tau) B_\varepsilon^{-1} f_\varepsilon(p_0 + \varphi_\varepsilon^1)$ is differentiable with respect to τ and that we have

$$d_0 \left\langle \int_0^1 (D_\tau J_\varepsilon(\omega_0 + \tau_s) - D_\tau J_\varepsilon(\omega_0)) B_\varepsilon^{-1} f_\varepsilon(p_0 + \varphi_\varepsilon^1)(\tau_1 - \tau_2) ds, B_0^* \tilde{v} \right\rangle$$
$$= d_0 \left\langle \int_0^1 \int_0^1 (D_{\tau\tau} J_\varepsilon(\omega_0 + \sigma \tau_s) B_\varepsilon^{-1} f_\varepsilon(p_0 + \varphi_\varepsilon^1) \tau_s (\tau_1 - \tau_2) d\sigma ds, B_0^* \tilde{v} \right\rangle$$
$$\leq c_8 d_0 K_0 \sup(|\tau_1|, |\tau_2|) |\tau_1 - \tau_2| \, \|B_0^* \tilde{v}\|_{L^2((0,\omega_0),X^*)}. \tag{103}$$

We next estimate the term $\tilde{K}_1^2$. This term can be decomposed as

$$\tilde{K}_1^2 = d_0 \langle (B_0^{-1} - B_\varepsilon^{-1}) D_\tau F_\varepsilon(\omega_0, p_0 + \varphi_\varepsilon^1)(\tau_1 - \tau_2), B_0^* \tilde{v} \rangle$$
$$+ d_0 \langle B_\varepsilon^{-1} (D_\tau F_\varepsilon(\omega_0, p_0 + \varphi_\varepsilon^1) - D_\tau F_\varepsilon(\omega_0, p_0))(\tau_1 - \tau_2), B_0^* \tilde{v} \rangle$$
$$+ d_0 \langle (B_\varepsilon^{-1} D_\tau F_\varepsilon(\omega_0, p_0) - B_0^{-1} D_\tau F_0(\omega_0, p_0))(\tau_1 - \tau_2), B_0^* \tilde{v} \rangle. \tag{104}$$

As in (102), the first term in the right hand side of (104) is simply estimated as follows

$$d_0 \langle (B_0^{-1} - B_\varepsilon^{-1}) D_\tau F_\varepsilon(\omega_0, p_0 + \varphi_\varepsilon^1)(\tau_1 - \tau_2), B_0^* \tilde{v} \rangle$$
$$\leq c_9 d_0 K_0 \|B_0^{-1} - B_\varepsilon^{-1}\|_{L(X,X)} |\tau_1 - \tau_2| \, \|B_0^* \tilde{v}\|_{L^2((0,\omega_0),X^*)}. \tag{105}$$

The second term in the right hand side of (104) can be written as

$$d_0 \langle B_\varepsilon^{-1} (D_\tau F_\varepsilon(\omega_0, p_0 + \varphi_\varepsilon^1) - D_\tau F_\varepsilon(\omega_0, p_0))(\tau_1 - \tau_2), B_0^* \tilde{v} \rangle$$
$$= d_0 (\tau_1 - \tau_2) \left\langle \int_0^1 (D_\tau J_\varepsilon(\omega_0) + \frac{1}{\omega_0} J_\varepsilon(\omega_0)) B_\varepsilon^{-1} D_u f_\varepsilon(p_0 + s\varphi_\varepsilon^1) \varphi_\varepsilon^1 ds, B_0^* \tilde{v} \right\rangle,$$

and hence, by using the equality (30) and the estimate (56) of Theorem 2.7, we obtain,

$$d_0 \, \langle B_\varepsilon^{-1} (D_\tau F_\varepsilon(\omega_0, p_0 + \varphi_\varepsilon^1) - D_\tau F_\varepsilon(\omega_0, p_0))(\tau_1 - \tau_2), B_0^* \tilde{v} \rangle$$
$$\leq c_{10} d_0 C_4 (\varepsilon^{\beta_2} + |\tau_1|) |\tau_1 - \tau_2| \, \|B_0^* \tilde{v}\|_{L^2((0,\omega_0),X^*)}. \tag{106}$$

The last term in the right hand side of (104) writes as follows

$$\begin{aligned} d_0\langle(B_\varepsilon^{-1}D_\tau F_\varepsilon(\omega_0,p_0)-B_0^{-1}D_\tau F_0(\omega_0,p_0))(\tau_1-\tau_2),B_0^*\tilde v\rangle \\ = d_0\langle D_\tau J_\varepsilon(\omega_0)B_\varepsilon^{-1}(f_\varepsilon(p_0)-f_0(p_0))(\tau_1-\tau_2),B_0^*\tilde v\rangle \\ +d_0\langle(D_\tau J_\varepsilon(\omega_0)B_\varepsilon^{-1}-D_\tau J_0(\omega_0)B_0^{-1})f_0(p_0)(\tau_1-\tau_2),B_0^*\tilde v\rangle \\ +d_0\langle\frac{1}{\omega_0}(B_\varepsilon^{-1}-B_0^{-1})F_\varepsilon(\omega_0,p_0)(\tau_1-\tau_2),B_0^*\tilde v\rangle \\ +d_0\left\langle\frac{1}{\omega_0}B_0^{-1}(F_\varepsilon(\omega_0,p_0)-F_0(\omega_0,p_0))(\tau_1-\tau_2),B_0^*\tilde v\right\rangle. \end{aligned} \tag{107}$$

Taking into account the Hypotheses (H2), (H5) and (H7), the estimate (52) of Remark 2.6 and the inequality (41) of Lemma 2.5, we deduce from the above equality that

$$\begin{aligned} d_0\langle(B_\varepsilon^{-1}D_\tau F_\varepsilon(\omega_0,p_0)-B_0^{-1}D_\tau F_0(\omega_0,p_0))(\tau_1-\tau_2),B_0^*\tilde v\rangle \\ \le c_{11}d_0\big(\varepsilon^{\beta_2}+\|B_0^{-1}-B_\varepsilon^{-1}\|_{L(X,X)}\big)|\tau_1-\tau_2|\,\|B_0^*\tilde v\|_{L^2((0,\omega_0),X^*)}. \end{aligned} \tag{108}$$

From the equalities and inequalities (98)–(108), we infer that

$$\tilde K_1\le c_{12}d_0|\tau_1-\tau_2|\big(\varepsilon^{\beta_2}+|\tau_1|+|\tau_2|+\|B_0^{-1}-B_\varepsilon^{-1}\|_{L(X,X)}\big)\big(1+\|B_0^*\tilde v\|_{L^2((0,\omega_0),X^*)}\big). \tag{109}$$

The estimate of the term $\tilde K_2$ is shorter. We first write $\tilde K_2$ as a sum of the following terms:

$$\begin{aligned} \tilde K_2 &= d_0\left\langle\int_0^1(D_uF_\varepsilon(\omega_0+\tau_2,p_0+\varphi_{\varepsilon,s})-D_uF_\varepsilon(\omega_0+\tau_2,p_0))(\varphi_\varepsilon^1-\varphi_\varepsilon^2)ds,\tilde v\right\rangle \\ &\quad +d_0\langle(D_uF_\varepsilon(\omega_0+\tau_2,p_0)-D_uF_0(\omega_0+\tau_2,p_0))(\varphi_\varepsilon^1-\varphi_\varepsilon^2),\tilde v\rangle \\ &\quad +d_0\langle(D_uF_0(\omega_0+\tau_2,p_0)-D_uF_0(\omega_0,p_0))(\varphi_\varepsilon^1-\varphi_\varepsilon^2),\tilde v\rangle \\ &\equiv \tilde K_2^1+\tilde K_2^2+\tilde K_2^3, \end{aligned} \tag{110}$$

where $\varphi_{\varepsilon,s}=\varphi_\varepsilon^2+s(\varphi_\varepsilon^1-\varphi_\varepsilon^2)$.

Since, by (14), $D_{uu}^2f_\varepsilon(u)\in L(X\times X,X)$ is uniformly bounded (with respect to ε) on the bounded sets of X, using the estimates (56) and (57) of Theorem 2.7, we prove that

$$\begin{aligned} \tilde K_2^1 &= d_0\left\langle\int_0^1\int_0^1 D_{uu}^2F_\varepsilon(\omega_0+\tau_2,p_0+\sigma\varphi_{\varepsilon,s})\varphi_{\varepsilon,s}(\varphi_\varepsilon^1-\varphi_\varepsilon^2)d\sigma ds,\tilde v\right\rangle \\ &\le c_{13}d_0\|\varphi_{\varepsilon,s}\|_{\mathcal X}\,\|\varphi_\varepsilon^1-\varphi_\varepsilon^2\|_{\mathcal X} \\ &\le c_{14}d_0(\varepsilon^{\beta_2}+|\tau_1|+|\tau_2|)|\tau_1-\tau_2|. \end{aligned} \tag{111}$$

We directly deduce from Lemma 2.5 and Theorem 2.7 that

$$\tilde{K}_2^2 \leq c_{15} d_0 \varepsilon^{\beta_2} |\tau_1 - \tau_2|. \tag{112}$$

Finally, using the fact that $B_0^{-1} D_u f_0(p_0)(\varphi_\varepsilon^1 - \varphi_\varepsilon^2)$ belongs to $C^0([0,\omega_0], D(B_0))$ and applying Theorem 2.7, we obtain that

$$\begin{aligned}
\tilde{K}_2^3 &= d_0 \langle B_0^{-1}(J_0(\omega_0+\tau_2) - J_0(\omega_0) + J_0(\omega_0+\tau_2)\tau_2\omega_0^{-1}) D_u f_0(p_0)(\varphi_\varepsilon^1 - \varphi_\varepsilon^2), B_0^* \tilde{v}\rangle \\
&= d_0 \left\langle \int_0^1 D_\tau J_0(\omega_0 + s\tau_2) B_0^{-1} D_u f_0(p_0) \tau_2 (\varphi_\varepsilon^1 - \varphi_\varepsilon^2) ds, B_0^* \tilde{v} \right\rangle \\
&\quad + d_0 \langle \tau_2 \omega_0^{-1} J_0(\omega_0+\tau_2) B_0^{-1} D_u f_0(p_0)(\varphi_\varepsilon^1 - \varphi_\varepsilon^2), B_0^* \tilde{v}\rangle \\
&\leq c_{16} d_0 |\tau_2| |\tau_2 - \tau_1| \, \|B_0^* \tilde{v}\|_{L^2((0,\omega_0),X^*)}.
\end{aligned} \tag{113}$$

The inequalities (109), (111)–(113) imply that there exists a positive constant c_{17} such that

$$|\tilde{K}_\varepsilon(\tau_1) - \tilde{K}_\varepsilon(\tau_2)| \leq c_{17} |\tau_1 - \tau_2| \big(\varepsilon^{\beta_2} + |\tau_1| + |\tau_2| + \|B_0^{-1} - B_\varepsilon^{-1}\|_{L(X,X)}\big)(1 + C_{\eta_0}), \tag{114}$$

where $\|B_0^* \tilde{v}\|_{L^2((0,\omega_0),X^*)} \leq C_{\eta_0}$. It remains to choose $\varepsilon_2 > 0$ and $\delta_2 > 0$ small enough so that, for any $0 < \varepsilon \leq \varepsilon_2$,

$$c_{17}\big(\varepsilon_2^{\beta_2} + 2\delta_2 + \|B_0^{-1} - B_\varepsilon^{-1}\|_{L(X,X)}\big)(1 + C_{\eta_0}) \leq \frac{1}{4}. \tag{115}$$

Finally, we deduce from the inequalities (96), (114) and from the condition (115) that

$$|\tau_1 - \tau_2| \leq \frac{1}{2} |\tau_1 - \tau_2|,$$

which implies that $\tau_1 = \tau_2$. The theorem is proved. □

2.2 Comparison with the Modified Poincaré Method

In the introduction, we have proved the persistence of periodic solutions for perturbed ode's by applying the Poincaré method (see the proof of Theorem 1.1). In [13], we have extended this method to the perturbations in the class of evolutionary partial differential equations, which are not smoothing in finite time and obtained the "Modified Poincaré method". We have seen in [13] that the Poincaré method does not only apply to semilinear partial differential equations, but also to fully non-linear equations. Also we can apply it to perturbations of dynamical systems, which are not necessarily generated by an evolutionary equation. All these features make that the modified Poincaré method is in general a more powerful and better method than the integral equation method of the previous section.

Here we want to compare the hypotheses of the modified Poincaré method with the ones of the integral equation method in the frame of the semilinear equations (4) and (5). For sake of simplicity, we assume that $f_0 = f_\varepsilon \equiv f$ in (4) and (5). The two methods are actually quite different. Indeed, the integral method consists of looking for a fixed point of a mapping defined on the space of continuous functions from the Banach space X into itself, which are periodic of a fixed period ω_0. As we shall see below, the modified Poincaré method consists in looking for a fixed point of a mapping defined on a subspace of the phase space X. However, the required hypotheses are not so different. We shall see that, in the general context of the Eqs. (4) and (5), the hypotheses required in the integral method are somehow weaker than those in the modified Poincaré method.

Whereas we compare both methods in the particular case of (4) and (5), we shall briefly recall the modified Poincaré method. In [13], like here, we have introduced two Banach spaces X and Z such that $Z \subset X$ is embedded in X with a continuous and compact injection and we have assumed that $D(B_0)$ is embedded in Z with a continuous injection. In the integral method, we had also to assume that $D(B_\varepsilon)$, for $\varepsilon > 0$, is embedded in Z with a continuous injection. Notice that, in [13], we needed to assume that $D(B_0)$ is included in Z with a *compact* injection, which is usually satisfied in the applications. In addition, we had to suppose that Z is a reflexive Banach space. In both methods, we required that the exponential decay property of $e^{B_\varepsilon t}$ and $e^{B_0 t}$ holds (see (13)). Of course, in both methods, we introduced a non-degenerate periodic solution $p_0(t) \in C^1(\mathbb{R}, X) \cap C^0(\mathbb{R}, D(B_0))$ of minimal period $\omega_0 > 0$. In [13], we needed to assume that moreover $p_0(t) \in C^{1,1}(\mathbb{R}, X)$. Here, we only assumed that the nonlinearity f belongs to $C^2(X, X)$, that the functions p_{0t}, $f(p_0(t))$ are continuous from $\mathbb{R}$ into Z and from $\mathbb{R}$ into $D(B_0)$ respectively (see Hypothesis (H2) in Sect. 2.1), and that $D_u f(p_0(t))$ is a bounded linear mapping from X into Z, the norm $\|D_u f(p_0(t))\|_{L(X,Z)}$ being uniformly bounded with respect to t (see the Hypotheses (H2) and (H3) of Sect. 2.1). In [13, Sect. 3.1], we needed to assume that the nonlinearity f belongs to $C^2(X, Z)$, which implied in particular that $p_{0t}(t)$ is bounded in Z, for any t. In both methods, we assumed that the perturbations are non-regular in the sense that we only know that, for $0 \le t \le 2\omega_0$,

$$\|e^{B_0 t} w - e^{B_\varepsilon t} w\|_X \le C_0 \varepsilon^{\beta_0} \|w\|_Z,$$

for any $w \in Z$ (see Hypothesis (H4) above). In [13], besides this assumption, we also needed to suppose that, for $0 \le t \le 2\omega_0$,

$$\|e^{B_0 t} w - e^{B_\varepsilon t} w\|_Z \le C_0 \varepsilon^{\beta_0} \|w\|_Y,$$

where $D(B_0) \subset Y \subset Z$ (see Assumption (A2) in [13, Sect. 3]).

Under the above recalled hypotheses (see the conditions (13), (14) and the Hypotheses (H1)–(H5)), the integral method (see Theorem 2.7 and Lemma 2.8) implies that the perturbed equation (5) admits a periodic solution $p_\varepsilon(t)$ close to

$p_0(t)$ with period ω_ε close to ω_0. For proving the same property with the modified Poincaré method, we need to make some more assumptions.

Before continuing the comparison of the hypotheses of both methods, we describe the modified Poincaré method. In the introduction, we have seen that the classical Poincaré method consists in introducing a transversal Σ_0 at $p_0(0)$ to Γ_0 and the Poincaré map P_ε associated to Σ_0, and also in looking for a fixed point of P_ε. In [13], we remarked that the classical Poincaré method could be interpreted in terms of a Lyapunov-Schmidt method. Indeed, notice that the non-degeneracy hypothesis implies that there exists a unique element q_0^* in the dual space X^* of X such that

$$(\Pi_0(\omega_0,0))^* q_0^* = q_0^*, \quad \langle \dot{p}_{0t}(0), q_0^* \rangle = 1, \tag{116}$$

where the period map $\Pi_0(\omega_0,0)$ has been defined in the introduction. If we define $X_0 = \{u \in X \,|\, \langle u, q_0^* \rangle = 0\}$, then the space X can be decomposed as follows

$$X = \mathrm{Vect}\,(\dot{p}_{0t}(0)) \oplus X_0,$$

and $X_0 =$ Range $(\Pi_0(\omega_0,0) - I)$. We remark that $p_0(0) + X_0$ is a transversal (set) to the periodic orbit $\Gamma_0 = \{p_0(s) | s \in [0,T_0]\}$ at the point $p_0(0)$.

Let P denote the projection onto the one-dimensional space generated by $p_{0t}(0)$ and $Q = I - P$. Looking for a time ω_ε close to ω_0 and an element $u_\varepsilon \in X_0$, close to 0, satisfying

$$T_\varepsilon(\omega_\varepsilon)(p_0(0) + u_\varepsilon) = p_0(0) + u_\varepsilon$$

is equivalent to solving the system

$$\begin{aligned} P\big(T_\varepsilon(\omega_\varepsilon)(p_0(0)+u_\varepsilon) - (p_0(0)+u_\varepsilon)\big) &= 0\,, \\ Q\big(T_\varepsilon(\omega_\varepsilon)(p_0(0)+u_\varepsilon) - (p_0(0)+u_\varepsilon)\big) &= 0. \end{aligned} \tag{117}$$

If we choose the above transversal $p_0(0) + X_0$ to Γ_0 at $p_0(0)$, the classical Poincaré method is nothing else as a Lyapunov-Schmidt method in which one first solves the equation in the P-component (leading to the determination of a function $\omega_\varepsilon(u)$) and then one solves the equation in the Q-component

$$Q\big(T_\varepsilon(\omega_\varepsilon(u_\varepsilon))(p_0(0)+u_\varepsilon) - (p_0(0)+u_\varepsilon)\big) = 0.$$

The regularity assumptions on f imply that, for any fixed t, $T_\varepsilon(t) : u_0 \in X \mapsto T_\varepsilon(t)u_0$ is a C^1-mapping. Therefore, one shows, by applying the implicit function theorem (or the strict contraction fixed point theorem), that, for any u in a small neighbourhood of 0 in X_0, there exists a unique time $\omega_\varepsilon(u)$ close to 0 such that

$$P\big(T_\varepsilon(\omega_\varepsilon(u))(p_0(0)+u) - (p_0(0)+u)\big) = 0.$$

Unfortunately, we did not make any assumption, implying that, for any $u \in X$, $T_\varepsilon(t)u : t \in \mathbb{R} \mapsto T_\varepsilon(t)u \in X$ is Lipschitz-continuous or even Hölder-continuous in t. Thus, we cannot apply the implicit function theorem or the strict contraction fixed point theorem for solving the Q-equation, that is, for finding a unique u_ε in X_0 such that the second equation in (117) holds. A strategy to overcome this difficulty consists in applying the Leray–Schauder fixed point theorem directly to the (117) problem. To this end, we need to introduce a compact convex set and a mapping from this set into itself, to which we can apply the Leray–Schauder fixed point theorem. We will next explain how to introduce such a set and such a mapping.

The assumption that Z is compactly embedded in X together with the exponential decay rate of $e^{B_0 t}$ actually implies that the radius of the essential spectrum of the period map $\Pi_0(\omega_0, 0)$ (defined by the linearized equation around $p_0(t)$, see (17)) is strictly less than 1. In [13], this property is essential [13, Hypothesis (H3) of the introduction]. It implies that there exists $0 < \rho_0 < 1$, such that, outside the ball of center 0 and radius ρ_0 in the complex plane, the spectrum $\sigma(\Pi_0(\omega_0, 0))$ consists only in a finite number of eigenvalues. In particular, besides the eigenvalue $\lambda_0 = 1$, $\Pi_0(\omega_0, 0)$ admits a finite number ℓ_0 of eigenvalues λ_i, $i = 1, \ldots, \ell_0$, counted with their multiplicities, such that

$$|\lambda_i| \geq 1.$$

Let P_0 (respectively P_1) denote the projection onto the generalized eigenspace generated by the generalized eigenfunctions associated with the $\ell_0 + 1$ (respectively ℓ_0) eigenvalues λ_i, $i = 0, 1, \ldots, \ell_0$ (respectively $i = 1, \ldots, \ell_0$). Moreover, without loss of generality, we may assume that there exists a positive number $k < 1$ such that, if $\varphi \in (I - P_0)X$, then

$$\|\Pi_0(\omega_0, 0)\varphi\|_X \leq k\|\varphi\|_X. \tag{118}$$

In [13], in order to find a periodic orbit $\Gamma_\varepsilon = \{p_\varepsilon(t) \,|\, t \in [0, \omega_\varepsilon)\}$ of (5), of minimal period close to ω_0 and belonging to a small neighbourhood of $\Gamma_0 = \{p_0(t) \,|\, t \in [0, \omega_0)\}$, we first constructed a convex compact set in which we applied the Leray–Schauder fixed point theorem. Since the perturbation is not regular and that the systems T_ε, $\varepsilon \geq 0$, are in general not even Hölder continuous in time, we work with convex sets involving at least two topologies. More precisely, assume that

$$\sup_{t \in [0, \omega_0]} \|p_0(t)\|_Z \leq R_0,$$

where $R_0 > 0$ is large enough. Without loss of generality, we may replace the decay hypothesis (13) by the stronger assumption, for any $t \geq 0$,

$$\|e^{B_\varepsilon t}\|_{L(Z,Z)} \leq e^{-\alpha t}. \tag{119}$$

If this is not the case, we can replace the norm of Z by an equivalent one, for which (119) holds.

For $R_1 > 0$ large enough ($R_1 \geq 2R_0$) and $r_1 > 0$ small enough, we introduce the following sets $\mathcal{W}(r_1, R_1)$, given by

$$\mathcal{W}(r_1,R_1) = \{\varphi \in (I-P_0)X \cap Z \mid \|\varphi\|_Z \le R_1, \quad \|\varphi\|_X \le r_1\}.$$

We equip $\mathcal{W}(r_1,R_1)$ with the metric associated with the norm of X. Since Z is a reflexive Banach space, the set $\mathcal{W}(r_1,R_1)$ is a closed complete metric space. For $r_1 > 0$, $R_1 > R_0$, $r_2 > 0$ and $\eta > 0$ chosen in an appropriate way, we are thus led to find $(\varphi,\psi) \in \mathcal{W}(r_1,R_1) \times B_{P_1X}(0,r_2)$ and $\omega_\varepsilon \in B_{\mathbb{R}}(\omega_0,\eta)$, for $\varepsilon > 0$ small enough, such that

$$T_\varepsilon(\omega_\varepsilon)(p_0^0 + \varphi + \psi) = p_0^0 + \varphi + \psi. \tag{120}$$

The equation (120) is equivalent to the system

$$\begin{aligned}(I-P_0)(T_\varepsilon(\omega_\varepsilon)(p_0^0+\varphi+\psi) - p_0^0) &= \varphi \\ P_1(T_\varepsilon(\omega_\varepsilon)(p_0^0+\varphi+\psi) - p_0^0) &= \psi \\ \langle (T_\varepsilon(\omega_\varepsilon)(p_0^0+\varphi+\psi) - p_0^0), q_0^* \rangle &= 0. \end{aligned} \tag{121}$$

Remarking that the solutions of the second equation in (121) coincide with those of the equation

$$(D_u(T_0(\omega_0)p_0^0) - I)^{-1} P_1 \left[T_\varepsilon(\omega_\varepsilon)(p_0^0+\varphi+\psi) - (p_0^0+\psi) \right] = 0, \tag{122}$$

leads to introduce the mapping $\mathcal{H}_\varepsilon(\varphi,\psi,\tau)$ from $\mathcal{W}(r_1,R_1) \times B_{P_1X}(0,r_2) \times B_{\mathbb{R}}(0,\eta)$ into $(I-P_0)Z \times P_1X \times \mathbb{R}$ defined by

$$\mathcal{H}_\varepsilon(\varphi,\psi,\tau) = (\mathcal{H}_{\varepsilon 1}(\varphi,\psi,\tau), \mathcal{H}_{\varepsilon 2}(\varphi,\psi,\tau), \mathcal{H}_{\varepsilon 3}(\varphi,\psi,\tau)), \tag{123}$$

where

$$\begin{aligned}\mathcal{H}_{\varepsilon 1}(\varphi,\psi,\tau) &= (I-P_0)(T_\varepsilon(\omega_0+\tau)(p_0(0)+\varphi+\psi) - p_0(0)) \\ \mathcal{H}_{\varepsilon 2}(\varphi,\psi,\tau) &= \psi - (D_u(T_0(\omega_0)p_0(0)) - I)^{-1} P_1 \\ &\qquad \left[T_\varepsilon(\omega_0+\tau)(p_0(0)+\varphi+\psi) - (p_0(0)+\psi) \right] \\ \mathcal{H}_{\varepsilon 3}(\varphi,\psi,\tau) &= -[\langle (T_\varepsilon(\omega_0+\tau)(p_0(0)+\varphi+\psi) - p_0(0)), q_0^* \rangle - \tau]. \end{aligned}$$

The contraction property (118) allows to show that the mapping $\mathcal{H}_{\varepsilon 1}$ maps $\mathcal{W}(r_1,R_1) \times B_{P_1X}(0,r_2) \times B_{\mathbb{R}}(0,\eta)$ into $\mathcal{W}(r_1,R_1)$, while the facts that the range of P_0 is finite-dimensional and is contained in Z allow to prove that $\mathcal{H}_{\varepsilon 2}$ maps $\mathcal{W}(r_1,R_1) \times B_{P_1X}(0,r_2) \times B_{\mathbb{R}}(0,\eta)$ into $B_{P_1X}(0,r_2)$. In order to show that the mapping $\mathcal{H}_\varepsilon(\varphi,\psi,\tau)$ admits a fixed point $(\varphi_\varepsilon,\psi_\varepsilon,\tau_\varepsilon) \in \mathcal{W}(r_1,R_1) \times B_{P_1X}(0,r_2) \times B_{\mathbb{R}}(0,\eta)$, besides the hypotheses recalled earlier, one needs additional properties. In [13, Sect. 3.1], we assumed that the map $t \to T_0(t)p_0(0) \in Z$ is Hölderian of order a_0, $0 < a_0 \le 1$ and, that the linear map $t \to e^{B_0 t} \in L(Z,X)$ as well as the linearized operator $t \to \Pi_0(t,0) \in L(Z,X)$ are Hölderian of order a, $0 < a \le 1$ (see [13, Assumption (A3) in Sect. 3.1]). We also supposed that the eigenvector q_0^* belongs to the domain $D(B_0^*)$ [13, Assumption (A4) in Sect. 3.1], but this assumption could be replaced by a much weaker one. If the Hölder exponent a satisfies the condition $a > 1/2$ [13, Assumption (A5)(a) in Sect. 3.1], we showed in [13, Sect. 3.3], by applying the Leray fixed point theorem, that the mapping $\mathcal{H}_\varepsilon(\varphi, \psi, \tau)$ admits a fixed point $(\varphi_\varepsilon, \psi_\varepsilon, \tau_\varepsilon) \in \mathcal{W}(r_1, R_1) \times B_{P_1X}(0, r_2) \times B_{\mathbb{R}}(0, \eta)$. If $a \leq 1/2$, we needed to introduce an additional hypothesis (for more details, see [13, Assumption (A5)(b) in Sect. 3.1]) and to replace the compact convex set $\mathcal{W}(r_1, R_1)$ by a slightly more complicated space, involving the completion of the space X for the norm $\|B_0^{-1} \cdot \|_X$.

To prove the "uniqueness" of the periodic solution $p_\varepsilon(t)$ near $p_0(t)$, with period ω_ε close to ω_0, one needs additional regularity hypotheses in the modified Poincaré method as well as in the integral method. In both methods, the uniqueness of the periodic solution $p_\varepsilon(t)$ is proved separately, with similar types of arguments. For the integral method in Sect. 2.1, we assumed in Hypothesis (H5) that, if $q_\varepsilon(t)$ is a periodic solution of (5), close to $p_0(t)$, and of minimal period near ω_0, then $q_\varepsilon(t)$ and $f(q_\varepsilon(t))$ are continuous functions from $t \in \mathbb{R}$ into $D(B_\varepsilon)$ and their norms in $D(B_\varepsilon)$ are uniformly bounded with respect to ε and to $t \in [0, 2\omega_0]$. In the case of the modified Poincaré method (see [13, Assumption (A6) in Sect. 3.1]), we assumed the same regularity properties for the periodic solutions $q_\varepsilon(t)$, but no supplementary hypothesis on $f(q_\varepsilon(t))$. In both methods, we made the natural hypothesis that B_ε^{-1} converges to B_0^{-1} in $L(X,X)$, when ε goes to zero (see the above Hypothesis (H7) and the Assumption (A8) in [13]). In the integral method, we also use the property (see Hypothesis (H7)) that the domain $D(B_0^*)$ is dense in X^*, which is closely related to the Assumption (A4) used in the uniqueness argument in [13]. In the uniqueness proof of the Poincaré method, we need in addition that the linear map $t \mapsto e^{B_\varepsilon t} \in L(Z,X)$ is Hölderian of order b, $0 < b \leq 1$, uniformly with respect to ε.

In summary, the integral and the modified Poincaré methods are completely different methods. The modified Poincaré method applies to more general problems that the integral method. In the case of the semilinear equations (4) and (5) (with $f_0 = f_\varepsilon = f$), the hypotheses that we impose in each method are rather similar. However, the integral method requires less hypotheses. For example, in the modified Poincaré method, we need to assume that f belongs to $C^2(X,Z)$ whereas, in the integral method, we only suppose that $f \in C^2(X,X)$, $f(p_0(\cdot)) \in C^1(\mathbb{R}, D(B_0))$ and $D_u f(p_0(t)) \in L(X,Z)$. Also, in the Poincaré method, we require additional Hölder properties, which implies that we have less "flexibility" in the choice of the space Z in the Poincaré method than in the integral one.

2.3 Some Examples of Applications

The examples that we give here have already been considered in [13], where we applied the modified Poincaré method to them. Here we show that we can also apply the above integral method and that this integral method is very natural and even requires less work than the Poincaré method.

2.3.1 Systems of Weakly Damped Wave Equations with Positive Damping

We first consider a simple example consisting of a system of damped wave equations with positive damping, defined on a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$ and supplemented with homogeneous Neumann boundary conditions.

More precisely, we introduce a nonlinear function $g = (g_1, g_2) : (x, y) \in \mathbb{R}^2 \mapsto g(x, y) \in \mathbb{R}^2$ of class C^2. In the case $n = 1$, we suppose that g is any C^2-function. If $n = 2$, we assume that g has a polynomial growth rate in (x, y). In the case where $n = 3$, we suppose that

$$|D^2 g(x,y)| \leq C(1 + |x|^{\alpha^*} + |y|^{\alpha^*}), \tag{124}$$

where $0 \leq \alpha^* \leq 1$, which implies that the Nemitskyi operator $g_i : (u, v) \in H^1(\Omega) \times H^1(\Omega) \mapsto H^s(\Omega)$, $i = 1, 2$, is Lipschitz-continuous on the bounded sets of $H^1(\Omega) \times H^1(\Omega)$, where $s \in (0, 1]$ if $n = 1$, $s \in (0, 1)$ if $n = 2$ and $s \in [0, \frac{1-\alpha^*}{2}]$ if $n = 3$. For later use, we choose $s_1 = 1$ in the case $n = 1$, $0 < s_1 < 1$ in the case $n = 2$ and $s_1 = \frac{1-\alpha^*}{2}$ in the case $n = 3$.

Let X be the Sobolev space $(H^1(\Omega) \times L^2(\Omega))^2$. Let γ_0 and δ be two fixed positive constants and $\gamma(x)$ be a continuous function in $\overline{\Omega}$ such that $\gamma_0 + \gamma(x) > 0$ in $\overline{\Omega}$. For $0 < \varepsilon \leq 1$, we consider the system of two damped wave equations defined on the domain Ω:

$$\begin{aligned} u_{tt} + (\gamma_0 + \varepsilon\gamma(x))u_t - \Delta u + \delta u &= (1+\varepsilon)g_1(u,v) \\ v_{tt} + (\gamma_0 + \varepsilon\gamma(x))v_t - \Delta v + \delta v &= (1+\varepsilon)g_2(u,v) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \ \text{ on } \partial\Omega \\ (u, u_t, v, v_t)(0,x) &= (u_0, u_1, v_0, v_1) \in X. \end{aligned} \tag{125}$$

System (125) can be written as the first order system

$$U_t = B_\varepsilon U(t) + f_\varepsilon(U) \ , \ U(0) = U_0 \in X, \tag{126}$$

where $U = (u, u_t, v, v_t) \in X$, and, for any $0 \leq s \leq s_1$,

$$f_\varepsilon(U) = (0, (1+\varepsilon)g_1(u,v), 0, (1+\varepsilon)g_2(u,v)) \in (H^{s+1}(\Omega) \times H^s(\Omega))^2$$

and

$$B_\varepsilon U = (u_t, \Delta u - \delta u - (\gamma_0 + \varepsilon\gamma(x))u_t, v_t, \Delta v - \delta v - (\gamma_0 + \varepsilon\gamma(x))v_t).$$

It is well-known that system (125) (or system (126)) generates a local dynamical system $T_\varepsilon(t) : X \to X$.

We introduce the operator $A = -\Delta + \delta I$ with domain $D(A) = \{u \in H^2(\Omega) \,|\, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$ and set $E^s = D(A^{s/2})$. We recall that E^s coincides with the classical Sobolev space $H^s(\Omega)$, for $0 \le s < 3/2$. Thus, if $X^s = (E^{s+1} \times E^s)^2$, X^s coincides with $(H^{s+1}(\Omega) \times H^s(\Omega))^2$ for $s < 1/2$. This is no longer the case for $s \ge 1/2$ since then boundary conditions appear in the definition of the spaces E^{s+1}. We also remark that the domain $D(B_\varepsilon)$ coincides with $(D(A) \times H^1(\Omega))^2$ for any $\varepsilon \ge 0$.

It is well-known that the operator B_ε is the generator of a linear C^0-semigroup $\exp B_\varepsilon t$ on $X = X^0$ and on X^1 and that there exist positive constants C_0 and α such that, for $0 \le s \le 1$,

$$\|e^{B_\varepsilon t}\|_{L(X^s,X^s)} \le C_0 e^{-\alpha t}. \tag{127}$$

Thus hypothesis (13) holds. For more details, we refer to [13]. Moreover, from [13, Inequality (3.158)] we deduce that, for $t \ge 0$,

$$\|e^{B_\varepsilon t} w - e^{B_0 t} w\|_X \le C_0 \varepsilon \|w\|_X, \tag{128}$$

which implies Hypothesis (H4).

We have to distinguish two different situations. In the cases $n = 1$, $n = 2$ and $n = 3$ with $\alpha^* < 1$, due to the compactness of the embedding of $H^{s_1}(\Omega)$ into $L^2(\Omega)$, the non-linear map $f_\varepsilon : X \to X$ is compact. Therefore, one easily proves that the local dynamical system $T_\varepsilon(t)$ is asymptotically smooth (see [8] or [29], for example). In the case $n = 3$ with $\alpha^* = 1$, f_ε is no longer a compact map from X into X and $T_\varepsilon(t)$ is no longer asymptotically smooth in general. However, under additional classical sign conditions (also called dissipative conditions), $T_\varepsilon(t)$ is asymptotically smooth (see [3] and also [8] or [29], for example).

Let $\mathcal{P}_0(t) = (p_0(t), p_{0t}(t), q_0(t), q_{0t}(t))$ be a non-degenerate periodic orbit of $T_0(t)$ of minimal period $\omega_0 > 0$ (such periodic orbits can exist, see [1, 2], for example). Hypothesis (H5) is obviously satisfied with $\beta_1 = 1$. The regularity of the invariant sets of the systems of damped wave equations (see [12]) implies that $\mathcal{P}_0(t)$ belongs to the space $C^0(\mathbb{R}, X^1)$ and that $\mathcal{P}_0(t) : t \in \mathbb{R} \mapsto \mathcal{P}_0(t) \in X$ is a C^2-function of t and thus Hypothesis (H1) holds. The proof of the regularity of $\mathcal{P}_0(t)$ can be given by a bootstrap argument in the cases $n = 1$, $n = 2$ or $n = 3$ with $\alpha^* < 1$. In the case $n = 3$, $\alpha^* = 1$, with additional dissipative conditions on f_ε, the proof is more involved (see [12] and also [3]).

We next introduce the space $Z = X^{s_2}$. In the cases $n = 1$, $n = 2$ or $n = 3$ with $\alpha^* < 1$, we choose $0 < s_2 < \inf(1/2, s_1)$ for sake of simplicity. In the case $n = 3$, $\alpha^* = 1$, we simply choose s_2 satisfying $0 < s_2 < 1/2$. The regularity results of [12] imply that $(p_{0t}(t), p_{0tt}(t), q_{0t}(t), q_{0tt}(t))$ is continuous from $\mathbb{R}$ into Z. Since $(p_0(t), q_0(t))$ belongs to $C^0(\mathbb{R}, (H^2(\Omega))^2)$, $g_i(p_0, q_0)$ belongs to $C^0(\mathbb{R}, H^1(\Omega))$ and thus Hypothesis (H2) is satisfied. In the cases $n = 1$, $n = 2$ or $n = 3$ with $\alpha^* < 1$, $D_U f_\varepsilon(U)$ is a linear continuous mapping from X into $(H^{s_1+1}(\Omega) \times H^{s_1}(\Omega))^2$, for any $U \in X$, and thus Hypothesis (H3) is automatically satisfied. In the case $n = 3$, $\alpha^* = 1$, since $(p_0(t), q_0(t))$ belongs to $C^0(\mathbb{R}, (H^2(\Omega))^2)$, we deduce from assumption (124) made on the second derivative of $g(\cdot)$ that $D_U f_\varepsilon(\mathcal{P}_0(t))$ is a linear continuous mapping from X into $(H^2(\Omega) \times H^1(\Omega))^2$ and thus Hypothesis (H3) is also satisfied. Again, as a direct consequence of the regularity properties of the periodic solutions of the system $T_\varepsilon(\cdot)$, Hypothesis (H6) is satisfied.

Finally, $X = X^*$ is a reflexive space and in [13, Sect. 3.5.1], we have also shown that B_ε^{-1} converges to B_0^{-1} in $L(X,X)$ when ε goes to zero. Thus Hypothesis (H7) is satisfied.

All the hypotheses of Theorem 2.9 being satisfied, we may apply it and conclude that there exist small positive constants ε_2 and η_2 and a small neighbourhood $\mathcal{V}_2$ of $\Gamma_0 \equiv \{\mathcal{P}_0(t) \,|\, t \in [0,\omega_0]\}$ such that, for $0 < \varepsilon \le \varepsilon_2$, the system (126) has a periodic solution $\mathcal{P}_\varepsilon(t) = \mathcal{P}_0(t) + \varphi_\varepsilon(\varepsilon, \omega_0 + \tau_\varepsilon)(t)$ of minimal period $\omega_\varepsilon \in [\omega_0 - \eta_2, \omega_0 + \eta_2]$ such that $\Gamma_\varepsilon = \{\mathcal{P}_\varepsilon(t) \,|\, t \in [0,\omega_\varepsilon]\}$ is the unique periodic orbit in $\mathcal{V}_2$ of period $\omega_\varepsilon \in [\omega_0 - \eta_2, \omega_0 + \eta_2]$. And the estimate (90) holds with $\beta_2 = 1$.

Remark 2.10. In [13, Sect. 3.5.1], we have shown that the system (125) satisfies the hypotheses of the modified Poincaré method, obtaining thereby the existence and uniqueness result of a periodic orbit of the perturbed system close to the one of the unperturbed system. Here in order to obtain the same result, we have applied Theorem 2.9. Comparing the length of the proofs in both cases, we easily see that the verification of the hypotheses of Theorem 2.9 is shorter (especially in the case $n = 3$, $\alpha^* = 1$).

Remark 2.11. The generalization of the above result to the case of homogeneous Dirichlet boundary conditions, that is, to the case where the boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

in (125), are replaced by

$$u = v = 0 \text{ on } \partial\Omega,$$

is straightforward. Notice that, in the case of Dirichlet boundary conditions, we may set $\delta = 0$ in (125). We proceed as above by introducing the first order system (126). But now the domain $D(A)$ of the operator $A = -\Delta_D$ with homogeneous Dirichlet boundary conditions, is $D(A) = \{u \in H^2(\Omega) \,|\, u = 0 \text{ on } \partial\Omega\}$. Here $E^s = D(A^{s/2})$ coincides with the Sobolev space $H^s(\Omega)$, for $0 \le s < 1/2$, with $H_0^s(\Omega)$, for $1/2 < s < 1$ and, with $H^s(\Omega) \cap H_0^1(\Omega)$, for $1 \le s < 5/2$. We also remark that the domain $D(B_\varepsilon)$ coincides with $D(A) \times H_0^1(\Omega)$ for any $\varepsilon \ge 0$. Since, in Hypotheses (H2) and (H6), we impose that $f_0(\mathcal{P}_0(t))$ and, for $\varepsilon > 0$, $f_\varepsilon(\mathcal{P}_\varepsilon(t))$ belong to $D(A) \times H_0^1(\Omega)$, for the periodic solutions of system (126), this implies at first glance that we need to assume that $g_1(0,0) = g_2(0,0) = 0$.

Actually, we do not need to make this restrictive assumption on the nonlinearity g. Indeed, if $g(0,0) \neq 0$, we first consider the solution $(u_\varepsilon, v_\varepsilon) \in (H_0^1(\Omega) \cap H^2(\Omega))^2$ of the elliptic system

$$\begin{aligned} -\Delta_D u_{0\varepsilon} &= (1+\varepsilon) g_1(0,0), \\ -\Delta_D v_{0\varepsilon} &= (1+\varepsilon) g_2(0,0). \end{aligned} \tag{129}$$

We then set $U_{0\varepsilon} = (u_{0\varepsilon}, 0, v_{0\varepsilon}, 0)$ and replace (126) by the modified system

$$W_t = B_\varepsilon W(t) + f_\varepsilon(W + U_{0\varepsilon}) - f_\varepsilon(0). \tag{130}$$

We remark that

$$\|U_{0\varepsilon} - U_{00}\|_{X^1} \leq C\varepsilon.$$

We then proceed with the modified system (130) as in the case of homogeneous Neumann boundary conditions.

2.3.2 Systems of Weakly Damped Wave Equations with Variable Non-negative Damping

We have previously described a simple example of a regular perturbation. If one replaces the positive constant damping γ_0 by a non-negative variable damping $\gamma_\varepsilon(x)$, we obtain an interesting non-regular perturbation of system (125). Here, as in [13], we quickly describe an example of variable non-negative "interior" damping $\gamma_\varepsilon(x)$ converging to damping on the boundary, when $\Omega \equiv I = (0,1)$ (for the study of this problem in general smooth domains, see [23]).

As in Sect. 2.3.1, let $X = (H^1(I) \times L^2(I))^2$ and let $g = (g_1, g_2)$ be a C^2-nonlinearity. For any $n \in \mathbf{N}$, we consider the system of partially damped wave equations on I,

$$\begin{aligned} u^n_{tt} + \gamma_n(x) u^n_t - u^n_{xx} + \delta u^n &= g_1(u^n, v^n) \\ v^n_{tt} + \gamma_n(x) v^n_t - v^n_{xx} + \delta v^n &= g_2(u^n, v^n) \\ (u^n_x(y,t), v^n_x(y,t)) &= 0, \text{ if } y = 0, 1 \\ (u^n, u^n_t, v^n, v^n_t)(0,x) &= (u_0, u_1, v_0, v_1) \in X. \end{aligned} \tag{131}$$

where

$$\gamma_n(x) = \begin{cases} n\gamma_0 & \text{if } a_n \leq x \leq a_n + \frac{1}{n}, \\ 0 & \text{elsewhere,} \end{cases}$$

and where $\gamma_0 \neq 1$ is a positive constant and $a_n \in [0,1)$, $n \geq 0$, is a sequence of real numbers tending to 0 when n tends to infinity and satisfying the condition $\sup_n n a_n < \infty$. System (131) defines a local dynamical system $T_n(t)$. When n tends to infinity, the limiting system is given by

$$\begin{aligned} u_{tt} - u_{xx} + \delta u &= g_1(u, v) \\ v_{tt} - v_{xx} + \delta v &= g_2(u, v) \\ (-u_x(0,t) + \gamma_0 u_t(0,t), -v_x(0,t) + \gamma_0 v_t(0,t)) &= (u_x(1,t), v_x(1,t)) = 0 \\ (u, u_t, v, v_t)(0,x) &= (u_0, u_1, v_0, v_1) \in X. \end{aligned} \tag{132}$$

This system generates a local dynamical system $T_\infty(t)$ on X and on $Z = H^{1+s}(I) \times H^s(I)$, $0 < s < 1/2$. The convergence of $T_n(t)$ towards $T_\infty(t)$ and the regularity properties have been studied by Joly [23]. Using his results and proceeding exactly as in Sect. 2.3.1, one easily checks that, if Γ_∞ is a non-degenerate periodic orbit of $T_\infty(t)$, all the hypotheses of Theorem 2.9 are satisfied and that thus this theorem applies. Again, verifying the hypotheses of Theorem 2.9 is shorter than verifying the hypotheses of the modified Poincaré method of [13].

2.3.3 A System of Damped Wave Equations in a Thin Domain

We can also generalize the integral method to the case where the Banach space X is replaced by a Banach space X_ε depending on the perturbation parameter ε. In [9], we will state an extension of the abstract Theorem 2.9 to this case. In [13, Sect. 5.1], we gave the example of a system of damped wave equations in a thin domain and indicated that we could apply a generalized version of the modified Poincaré method. An easy generalization of the abstract Theorem 2.9 can also be applied. Here, we will only describe the system of damped wave equations in thin two-dimensional or three-dimensional domains and give the main properties that allow to apply a generalized form of Theorem 2.9 (for details, see [9]).

Before describing this problem, we recall that the problem of persistence of periodic orbits in thin product domains for damped wave equations has been first studied by Johnson et al. [19, 21], who used the operators J_ε together with topological degree arguments. Using the same integral form as Johnson, Kamenskii and Nistri and the fixed point theorem of strict contraction, Abdelhedi [1,2] showed the persistence of periodic orbits for a system of damped wave equations in more general thin two-dimensional domains. However, she could not apply her method to the case of thin three-dimensional domains.

Let Ω be a smooth bounded domain in $\Omega \subset \mathbb{R}^n$, $n = 1,2$. As in Sect. 2.3.1, we introduce a C^2-nonlinear function $g = (g_1, g_2) : (x,y) \in \mathbb{R}^2 \mapsto g(x,y) \in \mathbb{R}^2$. If $n = 1$, we suppose that g has a polynomial growth rate in (x,y). If $n = 2$, we assume that g satisfies the condition (124) with $0 \le \alpha^* < 1$, for sake of simplicity in the discussion.

We introduce the space $X_0 = (H^1(\Omega) \times L^2(\Omega))^2$. If γ and δ are two positive constants, we consider the following system defined on Ω,

$$\begin{aligned} u_{tt} + \gamma u_t - \Delta u + \delta u &= g_1(u,v) \\ v_{tt} + \gamma v_t - \Delta v + \delta v &= g_2(u,v) \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \partial\Omega \\ (u, u_t, v, v_t)(0,x) &= (u_0, u_1, v_0, v_1) \in X_0, \end{aligned} \tag{133}$$

which generates a local dynamical system $T_0(t) : X_0 \to X_0$ and is written as a first order system

$$U_t = B_0 U(t) + f(U)\,,\; U(0) = U_0 \in X_0,$$

where $U = (u, u_t, v, v_t)$, $f : U \in X_0 \mapsto f(U) = (0, g_1(u,v), 0, g_2(u,v)) \in (H^{s+1}(\Omega) \times H^s(\Omega))^2$, for any $0 \le s < 1$ and $B_0 U = (u_t, \Delta u - \delta u - \gamma u_t, v_t, \Delta v - \delta v - \gamma v_t)$.

Let A_0 be the operator $-\Delta + \delta I$ with domain $D(A_0) = \{u \in H^2(\Omega)\,|\,\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$ and set $E_0^s = D(A_0^{s/2})$ and $X_0^s = E_0^{s+1} \times E_0^s$, $0 \le s \le 1$.

Next, we introduce the thin domain Q_ε

$$Q_\varepsilon = \{(x,y) \in \mathbb{R}^3\,|\,x \in \Omega\,,\, 0 < y < \varepsilon h(x)\},$$

where $h : x \in \overline{\Omega} \mapsto h(x) \in \mathbb{R}^+$ is a C^2-function and $\varepsilon > 0$ is a small parameter. The corresponding system on Q_ε writes,

$$\begin{aligned} u^\varepsilon_{tt} + \gamma u^\varepsilon_t - \Delta u^\varepsilon + \delta u^\varepsilon &= g_1(u^\varepsilon, v^\varepsilon) \\ v^\varepsilon_{tt} + \gamma v^\varepsilon_t - \Delta v^\varepsilon + \delta v^\varepsilon &= g_2(u^\varepsilon, v^\varepsilon) \\ \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = \frac{\partial v^\varepsilon}{\partial \nu^\varepsilon} &= 0 \;\text{ on } \partial Q_\varepsilon \\ (u^\varepsilon, u^\varepsilon_t, v^\varepsilon, v^\varepsilon_t)(0,x) &= (u^\varepsilon_0, u^\varepsilon_1, v^\varepsilon_0, v^\varepsilon_1) \in X_\varepsilon, \end{aligned} \tag{134}$$

where $X_\varepsilon = (H^1(Q_\varepsilon) \times L^2(Q_\varepsilon))^2$. We define the operator $A_\varepsilon = -\Delta + \delta I$ with domain $D(A_\varepsilon) = \{u \in H^2(Q_\varepsilon)\,|\,\frac{\partial u}{\partial \nu_\varepsilon} = 0 \text{ on } \partial Q_\varepsilon\}$. The space $E_\varepsilon^s \equiv D(A_\varepsilon^{s/2})$ coincides with $H^s(Q_\varepsilon)$, for $0 \le s < 3/2$. Thus, for $0 < s < 1/2$, we may embed X_0^s into $X_\varepsilon^s \equiv (E_\varepsilon^{s+1} \times E_\varepsilon^s)^2$; this is no longer possible for $s \ge 1/2$, since boundary terms appear in the definition of these spaces.

The system (134) generates a local dynamical system $T_\varepsilon(t) : X_\varepsilon \to X_\varepsilon$, which writes as the first order system

$$U^\varepsilon_t = B_\varepsilon U^\varepsilon(t) + f(U^\varepsilon)\,,\; U^\varepsilon(0) = U^\varepsilon_0 \in X_\varepsilon,$$

where $U^\varepsilon = (u^\varepsilon, u^\varepsilon_t, v^\varepsilon, v^\varepsilon_t)$, and $B_\varepsilon U^\varepsilon = (u^\varepsilon_t, -A_\varepsilon u^\varepsilon - \gamma u^\varepsilon_t, v^\varepsilon_t, -A_\varepsilon v^\varepsilon - \gamma v^\varepsilon_t)$.

We notice that B_ε, $\varepsilon \ge 0$, is the generator of a linear C^0-group $e^{B_\varepsilon t}$ on X_ε^s, $0 \le s \le 1$. According to [10], there exist positive constants C_0, α and ε_0, such that, for $0 \le \varepsilon \le \varepsilon_0$, the inequality (127) holds.

Assume next that $T_0(t)$ admits a non-degenerate periodic solution $\mathcal{P}_0(t)$, of minimal period $\omega_0 > 0$. By applying a (slightly) generalized version of Theorem 2.9 (with obvious modifications due to the fact that we are working with different spaces X_0 and X_ε), we want to show that $T_\varepsilon(t)$ also admits a periodic solution $\mathcal{P}_\varepsilon(t)$ close to $\mathcal{P}_0(t)$ of minimal period ω_ε close to ω_0.

We first need to introduce an appropriate space Z_ε, which will also depend on ε. Because of the boundary conditions, which are not the same on the thin domain Q_ε and on Ω, we are limited in the choice of the space Z_ε. According to the discussion made in Sect. 2.3.1, we set $Z_\varepsilon = X_\varepsilon^{s_1}$, $Z \equiv Z_0 = X_0^{s_1}$, where $0 < s_1 < 1/2$ in the case $n = 1$ and where $s_1 < \inf(1/2, \frac{1-\alpha^*}{2})$ in the case $n = 2$.

If one wants to generalize Theorem 2.9 to the case considered here, one sees that the main changes in the proof occur in the key Lemma 2.5. There, we gave an estimate of $\|J_\varepsilon(\omega)w - J_0 w\|_{C_{\omega_0}(X)}$ when w belongs to $C_{\omega_0}(Z)$. Here, since J_ε and J_0 act on different spaces, we will need to replace this estimate by an estimate of $\|J_\varepsilon(\omega)W - J_0 MW\|_{C_{\omega_0}(X_\varepsilon)}$ for W in $C_{\omega_0}(Z_\varepsilon)$, where M is an appropriate continuous linear operator from X_ε into X_ε. The best choice for M is the *vertical mean value operator*. As in [11], we consider the vertical mean value operator $M \in L(L^2(Q_\varepsilon), L^2(\Omega))$, given by,

$$Mu = \frac{1}{\varepsilon h}\int_{Q_\varepsilon} u(x,y)dy, \quad \forall u \in L^2(Q_\varepsilon). \tag{135}$$

We still denote M the corresponding operator from $(L^2(Q_\varepsilon))^4$ into $(L^2(\Omega))^4$ given by $MU = \frac{1}{\varepsilon h}\int_{Q_\varepsilon} U(x,y)dy$. We briefly recall the needed comparison results (proved in [10, 11]). An elementary computation shows [11] that,

$$\begin{aligned} \|w - Mw\|_{L^2(Q_\varepsilon)} &\leq C\varepsilon \|w\|_{H^1(Q_\varepsilon)}, \quad \forall w \in H^1(Q_\varepsilon), \\ \|w - Mw\|_{H^1(Q_\varepsilon)} &\leq C\varepsilon \|w\|_{D(A_\varepsilon)}, \quad \forall w \in D(A_\varepsilon). \end{aligned} \tag{136}$$

From the inequalities (136), we deduce that, for any $W \in D(B_\varepsilon)$,

$$\|W - MW\|_{X_\varepsilon} \leq C\varepsilon \|W\|_{D(B_\varepsilon)}, \tag{137}$$

and that, for any $f^* \in L^2(Q_\varepsilon)$,

$$\|A_\varepsilon^{-1} f^* - A_\varepsilon^{-1} M f^*\|_{H^1(Q_\varepsilon)} \leq C\varepsilon \|f^*\|_{L^2(Q_\varepsilon)}. \tag{138}$$

In [11], we have proved that there exists a positive constant C such that, for any $Mf^* \in L^2(Q_\varepsilon)$,

$$\|A_\varepsilon^{-1} M f^* - A_0^{-1} M f^*\|_{H^1(Q_\varepsilon)} \leq C\varepsilon \|Mf^*\|_{L^2(Q_\varepsilon)}. \tag{139}$$

The properties (138) and (139) imply that,

$$\begin{aligned} \|B_\varepsilon^{-1} MV - B_0^{-1} MV\|_{X_\varepsilon} &\leq C\varepsilon \|MV\|_{X_\varepsilon}, \quad \forall MV \in X_\varepsilon, \\ \|B_\varepsilon^{-1} W - B_\varepsilon^{-1} MW\|_{X_\varepsilon} &\leq C\varepsilon \|W\|_{X_\varepsilon}, \quad \forall W \in X_\varepsilon. \end{aligned} \tag{140}$$

Let us now check that the Hypotheses (H1)–(H7) (or their generalizations) of Theorem 2.9 hold. Hypotheses (H1) and (H2) concerning the limiting system are satisfied (see Sect. 2.3.1). The Hypotheses (H3) and (H5) are obviously true. The hypothesis on the regularity of the periodic orbits of $T_\varepsilon(t)$ is proved like the corresponding property for $T_0(t)$ in Sect. 2.3.1. The second part of Hypothesis (H7) is replaced by the conditions that $B_\varepsilon^{-1}M - B_0^{-1}M$ and $B_\varepsilon^{-1} - B_\varepsilon^{-1}M$ tend to 0 in $L(X_\varepsilon, X_\varepsilon)$, when ε tends to 0. By (140), both properties are satisfied.

In the proof of the generalized version of Theorem 2.9, the "good" estimate of $\|e^{B_0 t}w - e^{B_\varepsilon t}w\|_X$ needs to be replaced by a "good estimate" of $\|e^{B_\varepsilon t}U - e^{B_0 t}MU\|_{X_\varepsilon}$, for any $U \in X_\varepsilon^s$. But this "good estimate" is a direct consequence of the following result of [10].

Proposition 2.12. *There exists a positive constant* C_1 *such that, for any* s, $0 < s \leq 1$, *for any* $U \in X_\varepsilon^s$, *we have*

$$\|e^{B_\varepsilon t}U - e^{B_0 t}MU\|_{X_\varepsilon} \leq C_1 \varepsilon^{s/2} e^{C_1 t} \|U\|_{X_\varepsilon^s}, \tag{141}$$

and also, for $0 \leq \sigma < 1/2$ *and* $0 \leq \sigma \leq s \leq 1$,

$$\|e^{B_\varepsilon t}U - e^{B_0 t}MU\|_{X_\varepsilon^\sigma} \leq C_1 \varepsilon^{(s-\sigma)/2} e^{C_1 t} \|U\|_{X_\varepsilon^s}. \tag{142}$$

We may now apply a generalized version of Theorem 2.9 and thereby obtain the existence and uniqueness of a periodic orbit $\Gamma_\varepsilon = \{\mathcal{P}_\varepsilon(t) \,|\, t \in [0, \omega_\varepsilon)\}$ of $T_\varepsilon(t)$, close to the image of $\mathcal{P}_0(t)$, of minimal period ω_ε close to ω_0.

3 A Simple Method Combining the Fredholm Alternative with a Lyapunov Schmidt Procedure

In the case of ODE's, one of the simplest methods for proving Theorem 1.1 consists in arguments combining the Fredholm alternative with a Lyapunov-Schmidt procedure. Actually, this method also uses the variation of constants (or Duhamel) formula and has some similarities with the integral method of Sect. 2. But the linearized equation (and the corresponding linear time-dependent operator) along the periodic solution of the unperturbed problem plays a bigger role (see the definition of the operator $\mathcal{K}_0$ below). In Sect. 3.1, we will describe this method in the case of ODE's and see that it is rather simple. In Sect. 3.2, we shall study the infinite-dimensional case. The extension of this method to the infinite dimensional case will turn out to be more involved (and less promising) than expected.

3.1 The Case of Ordinary Differential Equations

In order to have the same notations in the ODE and in the semilinear PDE cases, we may assume, without loss of generality, that in Eq. (1) the function g_0 is written as

$$g_0(x) = B_0 x + f_0(x), \tag{143}$$

where B_0 is a (continuous) linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$ and $f_0 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ is the non-linear part. We assume that $p_0(t)$ is a non-degenerate periodic solution of (minimal) period ω_0 of (1).

Like in the introduction, one is interested in obtaining a solution of (3), which is periodic of some period ω, close to ω_0 for ε small. When perturbations $h(\varepsilon,\cdot)$ of g_0 are introduced, we have already seen that the Eq. (3) will have a periodic solution p_ε of period ω, which will depend on ε in general. As in the previous section, ω will be considered as a parameter which must be determined along with a function which is periodic of period ω and satisfies the complete Eq. (3). To work in a fixed class of periodic functions for which the period is independent of ε, as in the previous section, we rescale time $t \mapsto (\omega/\omega_0)t$ so that the period always is ω_0 and that the period of the wanted solution becomes a parameter in the equation. With these remarks, one then considers the equation as a mapping on the space C_{ω_0} of continuous periodic functions of period ω_0 and one tries to find a zero of this mapping. Since p_{0t} is a solution of (3), we can use the Fredholm alternative for linear periodic systems and the method of Lyapunov-Schmidt to reduce the discussion of the existence of periodic solutions $p_\varepsilon = p_0 + w_\varepsilon$ of period ω_ε (for w_ε and ε small enough) to finding the zeros of a function which is obtained from a projection onto the one-dimensional subspace of C_{ω_0} spanned by p_{0t}. The period ω_ε is then determined by making this function zero and thereby we obtain a solution of the complete equation.

We now go into more details. We recall that

$$C_{\omega_0}(X) = \{w \in C^0(\mathbb{R}, X) \,|\, w \text{ is } \omega_0\text{-periodic}\},$$

and we equip this space with the norm $\|w\|_{C_{\omega_0}(X)} = \sup_{t\in[0,\omega_0]} \|w(t)\|_X$. As in Sect. 2, we rescale the time variable. Let $x(t)$ be a periodic solution of (3) of period ω. For any $\omega \in \mathbb{R}$, $\omega > 0$, the function $x^*(t) = x(\frac{\omega}{\omega_0}t)$ is periodic of period ω_0 if and only if $x(t)$ is periodic of period ω. As a consequence, if we use this rescaling of time, then $x^*(t)$ satisfies the equation

$$x_t^*(t) = \frac{\omega}{\omega_0} B_0 x^*(t) + \frac{\omega}{\omega_0}(f_0(x^*(t)) + h(\varepsilon, x^*(t))). \tag{144}$$

The objective is to find an ω_0-periodic solution of (144) for $\frac{\omega}{\omega_0} - 1$ and ε small. If $x^* = p_0 + z$, then z satisfies the equation

$$z_t(t) = A_0(t)z(t) + G(\varepsilon, \omega, t, z(t)), \tag{145}$$

where

$$A_0(t) = B_0 + Df_0(p_0(t)), \tag{146}$$

and

$$G(\varepsilon,\omega,t,z) = \frac{\omega}{\omega_0}[f_0(p_0+z) - f_0(p_0) - Df_0(p_0)z + h(\varepsilon,p_0+z)] \\ + (\frac{\omega}{\omega_0} - 1)\big(A_0 z + B_0 p_0 + f_0(p_0)\big). \tag{147}$$

Remark 3.1. We remark that $G(0,\omega_0,t,0) = 0$ and that $G(0,\omega_0,t,z)$ has the property that it is $o(|z|)$ as $z \to 0$. Therefore, the system (145) can be considered as a perturbation of (2). In most of the infinite-dimensional problems considered in this paper, $G(0,\omega_0,t,z)$ has no longer this property. And unfortunately, simple modifications of the linear term will not be sufficient.

Since we are assuming that $p_0(t)$ is nondegenerate, 1 is a simple eigenvalue of the period map $\Pi_0(\omega_0,0)$ and also of the adjoint map $(\Pi_0(\omega_0,0))^*$, that is, there exists a unique element $q_0(\omega_0) \in \mathbb{R}^n$ such that,

$$(\Pi_0(\omega_0,0))^* q_0(\omega_0) = q_0(\omega_0) = q_0(0), \tag{148}$$

and

$$(\Pi_0(\omega_0,0))^* q_0(\omega_0) \cdot p_{0t}(0) = q_0(\omega_0) \cdot p_{0t}(\omega_0) = 1. \tag{149}$$

Since

$$\frac{1}{\omega_0}\int_0^{\omega_0} \Pi_0(\omega_0,t) p_{0t}(t)dt = p_{0t}(\omega_0),$$

Property (149) implies that, if $q_0(t) = (\Pi_0(\omega_0,t))^* q_0(\omega_0)$,

$$\langle p_{0t}, q_0 \rangle \equiv \frac{1}{\omega_0}\int_0^{\omega_0} q_0(t) \cdot p_{0t}(t)\, dt = 1. \tag{150}$$

This means that there exists a unique ω_0-periodic solution $q_0(t)$ of the adjoint equation

$$w_t = -A_0(t)^* w, \tag{151}$$

such that (150) is satisfied. We next introduce the space

$$Z_{\omega_0} = \{\varphi \in C_{\omega_0}(X) \,|\, \langle \varphi, q_0 \rangle = 0\},$$

and decompose $C_{\omega_0}(X)$ into the direct sum

$$C_{\omega_0}(X) = \text{Vect}\,(p_{0t}) \oplus Z_{\omega_0}(X). \tag{152}$$

The following lemma is straightforward.

Lemma 3.2 (Fredholm alternative). *The equation*

$$w_t = A_0(t)w + L(t), \quad L \in C_{\omega_0}(X), \tag{153}$$

has a solution in $C_{\omega_0}(X)$ *if and only if* L *belongs to* Z_{ω_0}*; that is,* $\langle L, q_0\rangle = 0$.

Furthermore, if $L \in Z_{\omega_0}$*, then there is a unique solution* $\mathcal{K}_0 L \in Z_{\omega_0}$ *and the mapping* $\mathcal{K}_0$ *is a continuous linear operator from* Z_{ω_0} *to* Z_{ω_0}*, given by* (155) *below.*

Proof. We remark that $w(t)$ is a ω_0-periodic solution of (153) if and only if $w(\omega_0) - w(0) = 0$, that is, if and only if,

$$\tilde{L} \equiv \int_0^{\omega_0} \Pi_0(\omega_0, s) L(s)\, ds = (I - \Pi_0(\omega_0, 0)) w(0), \tag{154}$$

which means that $\tilde{L} \in \mathrm{R}(I - \Pi_0(\omega_0,0))$. But, $\mathrm{R}(I - \Pi_0(\omega_0,0)) = (\mathrm{Ker}(I - \Pi_0(\omega_0,0))^*))^{\perp}$ and thus $\tilde{L}$ belongs to $\mathrm{R}(I - \Pi_0(\omega_0,0))$ if and only if the scalar product in $\mathbb{R}^n$, $\tilde{L}\cdot y$ vanishes, for any y satisfying $(I - \Pi_0(\omega_0,0))^* y = 0$. Since the kernel of $(I - \Pi_0(\omega_0,0))^*$ is one-dimensional and generated by $q_0(\omega_0)$, $\tilde{L}$ belongs to $\mathrm{R}(I - \Pi_0(\omega_0,0))$ if and only if $\tilde{L}\cdot q_0(\omega_0) = 0$, that is, if and only if

$$\begin{aligned}\int_0^{\omega_0} \Pi_0(\omega_0, s) L(s)\cdot q_0(\omega_0)\, ds &= \int_0^{\omega_0} L(s)\cdot (\Pi_0(\omega_0, s))^* q_0(\omega_0)\, ds \\ &= \int_0^{\omega_0} L(s)\cdot q_0(s)\, ds = 0.\end{aligned}$$

Using the equality (154) together with the variation of constants formula, we obtain that

$$\mathcal{K}_0 L(t) = \tilde{\mathcal{K}}_0 L(t) - \langle \tilde{\mathcal{K}}_0 L, q_0\rangle p_{0t}(t), \tag{155}$$

where

$$\begin{aligned}\tilde{\mathcal{K}}_0 L(t) &= \Pi_0(t,0)((I - \Pi_0(\omega_0,0))_{|R(I-\Pi_0(\omega_0,0))})^{-1} \int_0^{\omega_0} \Pi_0(\omega_0, s) L(s) ds \\ &\quad + \int_0^t \Pi_0(t,s) L(s) ds\end{aligned}$$

One at once sees that $\mathcal{K}_0 L$ is the unique solution of (153) in Z_{ω_0}. Thus, Lemma 3.2 is proved. □

Let us now apply the classical method of Lyapunov-Schmidt to (145) by using Lemma 3.2. We thus see that we must determine $\omega > 0$, $z \in Z_{\omega_0}$ as solutions of the two equations:

$$\begin{aligned} &z_t = A_0(t) z + G(\varepsilon, \omega, t, z) - \langle G(\varepsilon, \omega, \cdot, z), q_0\rangle p_{0t} \\ &\langle G(\varepsilon, \omega, \cdot, z), q_0\rangle = 0.\end{aligned} \tag{156}$$

This implies that

$$z = \mathcal{K}_0\big(G(\varepsilon, \omega, \cdot, z) - \langle G(\varepsilon, \omega, \cdot, z), q_0\rangle p_{0t}\big). \tag{157}$$

For $\frac{\omega}{\omega_0} - 1, \varepsilon$ and z small enough, one can apply the implicit function theorem to obtain a unique function $\hat{z}(\omega,\varepsilon) \in Z_{\omega_0}$ satisfying (157) and $\hat{z}(\omega_0, 0) = 0$.

The function $\hat{z}(\omega,\varepsilon)$ will be a solution of (145) in Z_{ω_0} if and only if

$$\mathcal{G}(\varepsilon,\omega) \equiv \langle G(\varepsilon,\omega,\cdot,\hat{z}(\omega,\varepsilon)), q_0\rangle = 0. \tag{158}$$

It is clear that (158) is satisfied for $\omega = \omega_0$, $\varepsilon = 0$. An elementary computation shows that $D_\omega \mathcal{G}(0,\omega_0) = \omega_0^{-1}$ and thus is non-singular. The implicit function theorem implies that, for (ω,ε) close to $(\omega_0, 0)$, there is a unique solution $\omega(\varepsilon)$, $\omega(0) = \omega_0$, such that (158) is satisfied.

Tracing back through our transformations yields the proof of Theorem 1.1.

3.2 The Infinite-Dimensional Case

We first notice that, in the proof of Sect. 3.1, we have never used the fact that X is finite-dimensional. So the above proof is valid in the general case, where X is any Banach space, B_0 is a linear continuous mapping from X into X and $f_0(x) + h(\varepsilon, x)$ is a C^1-perturbation in (ε, x).

We notice that, in [15, Chapter 10, Theorem 4.1], Hale and Verduyn Lunel have used the method of Sect. 3.1 to prove that regular perturbations of RFDE's have still a periodic solution p_ε of period ω_ε, close to ω_0, if the unperturbed equation has a non-degenerate periodic solution $p_0(t)$ of period ω_0. In the introduction, we had noticed that, if the period ω_0 is strictly larger than the delay r, one could apply the Poincaré method. The method of Sect. 3.1 works even if the period ω_0 is less or equal to the delay r.

We now go back to the Eqs. (4) and (5) given in the introduction. We recall that B_0 and B_ε are generators of C^0- semigroups on X, and that f_ε, $\varepsilon \geq 0$, is a family of nonlinearities in $C^1(X,X)$, such that f_ε converges to f_0 in $C^1(O,X)$ for any bounded open subset O of X. Under these hypotheses, the Eqs. (4) and (5) define local nonlinear semigroups $T_0(t)$ and $T_\varepsilon(t)$ on X. We assume here that the (dense) domains $D(B_0)$ and $D(B_\varepsilon)$ are strict subspaces of X. Thus, the nonlinear term $G(\varepsilon,\omega,t,z)$ given in (147) does no longer satisfy the properties described in Remark 3.1, since the term $(\frac{\omega}{\omega_0} - 1)A_0 z$ does not belong to X for a general element $z \in X$. This means that the decomposition made in (145) into the sum of the linear term $A_0(t)$ and the nonlinear term $G(\varepsilon,\omega,t,z)$ is not longer appropriate.

Next we will quickly describe how one can extend the method of Sect. 3.1 to the Eqs. (4) and (5). Let $p_0(t) \in C^1(\mathbb{R},X)$ be a non-degenerate periodic solution of (4) of (minimal) period $\omega_0 > 0$. We recall that $p_0(t) \in C^1(\mathbb{R},X)$ if and only if $p_0(t) \in C^0(\mathbb{R},D(B_0))$. We recall that 1 is a non-degenerate (or simple) periodic solution of period ω_0 if 1 is a (algebraically) simple eigenvalue of the period map $\Pi_0(\omega_0,0)$, where $\Pi_0(t,0)$ is the linear evolution operator defined by the linearized equation

$$w_t(t) = (B_0 + Df_0(p_0(t)))w, \quad w(0) = w_0, \tag{159}$$

that is, $\Pi_0(t,0)w_0 = D_u(T_0(t)p_0(t))w_0 \equiv w(t) \in C^0((0,+\infty),X)$ is the (unique) solution of (159).

We suppose that there exist positive constants ε_0, α and C_0 such that, for $0 \leq \varepsilon \leq \varepsilon_0$ and for any $t \geq 0$,

$$\|e^{B_\varepsilon t}\|_{L(X,X)} \leq C_0 e^{-\alpha t}. \tag{160}$$

We also assume that the semigroups $e^{B_\varepsilon t}$ converge to $e^{B_0 t}$ in the sense that there exist a positive constant β and a subspace Z of X, continuously embedded into X such that, for any $0 \leq t \leq 2\omega_0$,

$$\|e^{B_\varepsilon t}v - e^{B_0 t}v\|_X \leq C_0 \varepsilon^\beta \|v\|_Z,\ \forall v \in Z. \tag{161}$$

The convergence of $e^{B_\varepsilon t}$ towards $e^{B_0 t}$ is called *non-regular* if Z is a proper subspace of X, in which case we assume that the injection of Z into X is compact.

In order to seek a periodic solution $p_\varepsilon(t) \in C^0(\mathbb{R},X)$, close to $p_0(t)$, with (minimal) period ω_ε near ω_0, we perform the same change of time variable $\frac{\omega}{\omega_0}t$ as in Sects 2.1 and 3.1. We are led thus to look for a ω_0-periodic solution $u^*(t) = u(\frac{\omega}{\omega_0}t)$ of the equation

$$u_t^*(t) = \frac{\omega}{\omega_0} B_\varepsilon u^*(t) + \frac{\omega}{\omega_0} f_\varepsilon(u^*(t)). \tag{162}$$

If $u^*(t) = p_0(t) + z(t)$, then $z(t)$ is an ω_0-periodic solution of the equation

$$z_t(t) = \frac{\omega}{\omega_0}\big(B_\varepsilon + Df_0(p_0(t))\big)z + G(\varepsilon,\omega,t,z), \tag{163}$$

where

$$G(\varepsilon,\omega,t,z) = \frac{\omega}{\omega_0}[f_\varepsilon(p_0+z) - f_0(p_0) - Df_0(p_0)z] + \left(\frac{\omega}{\omega_0} - 1\right) f_0(p_0) - B_0 p_0 + \frac{\omega}{\omega_0} B_\varepsilon p_0, \tag{164}$$

(we emphasize that we cannot choose the same non-linearity as in (147), since the term $(\frac{\omega}{\omega_0} - 1)A_0 z$ does not belong to X for a general element $z \in X$). The above equation makes sense if we assume, for example, that

- (A1) The periodic solution $p_0(t)$ belongs to the "more regular spaces" $D(B_\varepsilon)$, for $\varepsilon \geq 0$. In addition, $\sup_{0\leq\varepsilon\leq\varepsilon_0} \sup_{t\in[0,\omega_0)} \|p_0(t)\|_{D(B_\varepsilon)}$ is bounded by a positive constant C_0. Moreover, $B_\varepsilon p_0(t)$ converges to $B_0 p_0(t)$ (uniformly with respect to t) in the space X, as ε goes to 0.

The hypothesis that $p_0(t)$ belongs to the "more regular spaces" $D(B_\varepsilon)$, for $\varepsilon \geq 0$ is a strong assumption, since in some problems $D(B_\varepsilon) \neq D(B_0)$, for $\varepsilon > 0$. This is the case for problems in thin domains (see Sect. 2.3.3) or for the example with varying damping given in Sect. 2.3.2.

Here, contrary to the ODE's case, where we considered the linearized equation along $p_0(t)$ for $\varepsilon = 0$ (see (159)), we introduce the linear equation

$$w_t(t) = \frac{\omega}{\omega_0}\big(B_\varepsilon + Df_0(p_0(t))\big)w, \quad w(0) = w_0, \tag{165}$$

and consider the time-t map $\Pi(\varepsilon,\omega;t,0) : w_0 \in X \mapsto w(t) \equiv \Pi(\varepsilon,\omega;t,0)w_0 \in X$, which is the solution at time t of (165). We notice that $\Pi(0,\omega_0;t,0)$ is nothing else as the map $\Pi_0(t,0)$. If we want to follow the lines of proofs of Sect. 3.1, we need to know that $\Pi(\varepsilon,\omega;\omega_0,0)$ admits an eigenvalue $\lambda(\varepsilon,\omega)$ close to 1. More precisely, we assume that

- (A2) For $0 \le \varepsilon \le \varepsilon_0$ and for ω close to ω_0, the map $\Pi(\varepsilon,\omega;\omega_0,0)$ admits a unique eigenvalue $\lambda(\varepsilon,\omega)$ close to 1. This eigenvalue is isolated and (algebraically) simple, and $\lambda(\varepsilon,\omega)$ converges to 1, when ε and $\omega-\omega_0$ converge to zero. Moreover, there exists a family of associated eigenfunctions $\varphi(\varepsilon,\omega)(0)$ such that $\varphi(\varepsilon,\omega)(0)$ converges to $p_{0t}(0)$ as ε and $\omega-\omega_0$ tend to zero.

In the case where the operator B_ε is a non-regular perturbation of the operator B_0, the above hypothesis (A2) is strong and is maybe not satisfied.

We set $\varphi(\varepsilon,\omega)(t) = \Pi(\varepsilon,\omega;t,0)(\varphi(\varepsilon,\omega)(0))$ and

$$\mu(\varepsilon,\omega) = \frac{-\log\lambda(\varepsilon,\omega)}{\omega_0}. \tag{166}$$

Then, the function $\psi(\varepsilon,\omega)(t) \equiv e^{\mu(\varepsilon,\omega)t}\varphi(\varepsilon,\omega)(t)$ satisfies

$$\psi(\varepsilon,\omega)(\omega_0) = e^{\mu(\varepsilon,\omega)\omega_0}\lambda(\varepsilon,\omega)\varphi(\varepsilon,\omega)(0) = \psi(\varepsilon,\omega)(0), \tag{167}$$

and

$$\frac{d}{dt}\psi(\varepsilon,\omega)(t) = \frac{\omega}{\omega_0}\big(B_\varepsilon + Df_0(p_0(t))\big)\psi(\varepsilon,\omega)(t) + \mu(\varepsilon,\omega)\psi(\varepsilon,\omega)(t). \tag{168}$$

In other terms, 1 is an eigenvalue of the time ω_0-map $\Pi_\mu(\varepsilon,\omega;\omega_0,0)$ with associated eigenfunction $\psi(\varepsilon,\omega)(0)$, where $\Pi_\mu(\varepsilon,\omega;t,0)v_0 = v(t)$ is the unique solution of the linear equation

$$v_t(t) = \frac{\omega}{\omega_0}\big(B_\varepsilon + Df_0(p_0(t))\big)v(t) + \mu(\varepsilon,\omega)v(t) \equiv A_\mu(\varepsilon,\omega;t)v(t), \quad v(0) = v_0. \tag{169}$$

We notice that 1 is a (isolated) simple eigenvalue of the linear map $\Pi_\mu(\varepsilon,\omega;\omega_0,0)$.

The above considerations lead us to look for an ω_0- periodic solution $u^*(t) = p_0(t) + z(t)$, where $z(t)$ is an ω_0-periodic solution of the equation

$$z_t(t) = A_\mu(\varepsilon,\omega;t)z + \big(G(\varepsilon,\omega,t,z) - \mu(\varepsilon,\omega)z\big). \tag{170}$$

With the above changes, we can now mimic the arguments of the end of Sect. 3.1. Since 1 is a simple eigenvalue of the period map $\Pi_\mu(\varepsilon,\omega;\omega_0,0)$, and hence of the adjoint map $(\Pi_\mu(\varepsilon,\omega;\omega_0,0))^*$, there exists a unique element $q(\varepsilon,\omega;\omega_0) \in X^*$ such that,

$$(\Pi_\mu(\varepsilon,\omega;\omega_0,0))^* q(\varepsilon,\omega;\omega_0) = q(\varepsilon,\omega;\omega_0), \tag{171}$$

and

$$q(\varepsilon,\omega;\omega_0)\cdot\psi(\varepsilon,\omega)(0) = 1. \tag{172}$$

Properties (171) and (172) imply that $q(\varepsilon,\omega;t) \equiv (\Pi_\mu(\varepsilon,\omega;\omega_0,t))^* q(\varepsilon,\omega;\omega_0)$ is the unique ω_0-periodic solution of

$$w_t = -A_\mu(\varepsilon,\omega;t)^* w, \tag{173}$$

satisfying the condition

$$\langle \psi(\varepsilon,\omega)(\cdot), q(\varepsilon,\omega;\cdot)\rangle \equiv \frac{1}{\omega_0}\int_0^{\omega_0} \psi(\varepsilon,\omega)(t)\cdot q(\varepsilon,\omega;t)\,dt = 1. \tag{174}$$

We next introduce the space

$$Z_{\omega_0}(\varepsilon,\omega) = \{\varphi \in C_{\omega_0}(X) \,|\, \langle \varphi, q(\varepsilon,\omega;\cdot)\rangle = 0\},$$

and decompose $C_{\omega_0}(X)$ into the direct sum

$$C_{\omega_0}(X) = \text{Vect}\,(\psi(\varepsilon,\omega)(\cdot)) \oplus Z_{\omega_0}(\varepsilon,\omega).$$

As in Lemma 3.2, one shows that, the equation

$$w_t = A_\mu(\varepsilon,\omega;t)w + L(t), \quad L \in C_{\omega_0}(X), \tag{175}$$

has a solution in $C_{\omega_0}(X)$ if and only if L belongs to $Z_{\omega_0}(\varepsilon,\omega)$. Furthermore, if $L \in Z_{\omega_0}(\varepsilon,\omega)$, then there is a unique solution $\mathcal{K}(\varepsilon,\omega)L \in Z_{\omega_0}(\varepsilon,\omega)$ and the mapping $\mathcal{K}(\varepsilon,\omega)$ is a continuous linear operator from $Z_{\omega_0}(\varepsilon,\omega)$ into itself. Moreover, using the variation of constants formula, one proves that, if L belongs to $Z_{\omega_0}(\varepsilon,\omega)$, then,

$$(\mathcal{K}(\varepsilon,\omega)L)(t) = (\tilde{\mathcal{K}}(\varepsilon,\omega)L)(t) - \langle \tilde{\mathcal{K}}(\varepsilon,\omega)L, q(\varepsilon,\omega,\cdot)\rangle \psi(\varepsilon,\omega)(t), \tag{176}$$

where

$$\begin{aligned}
&(\tilde{\mathcal{K}}(\varepsilon,\omega)L)(t) \\
&\quad = \Pi_\mu(\varepsilon,\omega;t,0)((I-\Pi_\mu(\varepsilon,\omega;\omega_0,0))_{|R(I-\Pi_\mu(\varepsilon,\omega;\omega_0,0))})^{-1}\int_0^{\omega_0} \Pi_\mu(\varepsilon,\omega;\omega_0,s)L(s)ds \\
&\qquad + \int_0^t \Pi_\mu(\varepsilon,\omega;\omega_0,s)L(s)ds
\end{aligned}$$

As in Sect. 3.1, we now apply the method of Lyapunov-Schmidt to (170), that is, we determine ω close to ω_0, $z \in Z_{\omega_0}(\varepsilon,\omega)$ as solutions of the two equations

$$z_t = A_\mu(\varepsilon,\omega;t)z + (G(\varepsilon,\omega;t,z) - \mu(\varepsilon,\omega)z) - \langle G(\varepsilon,\omega;\cdot,z), q(\varepsilon,\omega;\cdot)\rangle \psi(\varepsilon,\omega)$$
$$\langle G(\varepsilon,\omega,\cdot,z), q(\varepsilon,\omega;\cdot)\rangle = 0. \tag{177}$$

This implies that

$$z = \mathcal{K}(\varepsilon,\omega)\big(G(\varepsilon,\omega;t,z) - \mu(\varepsilon,\omega)z) - \langle G(\varepsilon,\omega;\cdot,z), q(\varepsilon,\omega;\cdot)\rangle \psi(\varepsilon,\omega)\big). \tag{178}$$

In the ODE's case, a straightforward application of the implicit function theorem implied that, for $\frac{\omega}{\omega_0} - 1$ and ε small enough, there exists a unique C^1-function $\hat{z}(\omega,\varepsilon) \in Z_{\omega_0}$ satisfying (157) and $\hat{z}(\omega_0,0) = 0$. Here the complication comes from the fact that the operator $\mathcal{K}(\varepsilon,\omega)$ depends on ε and ω. In view of the definition (176) of $\mathcal{K}(\varepsilon,\omega)$, one sees that solving the equality (178) is not simpler than solving the first equality in (54) or finding a fixed point of the mapping $\mathcal{L}_\varepsilon(\omega,\cdot)$ given in (55) of Sect. 2. Since the expressions of $\tilde{\mathcal{K}}(\varepsilon,\omega)$ and $J_\varepsilon(\omega)$ are both consequences of the variation of constants formula, it is not surprising that $\tilde{\mathcal{K}}(\varepsilon,\omega)$ is obtained from $J_\varepsilon(\omega)$ by replacing $S_{\varepsilon,\omega}(t)$ by $\Pi_\mu(\varepsilon,\omega;t,0)$.

The function $\hat{z}(\omega,\varepsilon)$ will be a solution of (170) in $Z_{\omega_0}(\varepsilon,\omega)$ if and only if

$$\mathcal{G}(\varepsilon,\omega) \equiv \langle G(\varepsilon,\omega,\cdot,\hat{z}(\omega,\varepsilon)), q(\varepsilon,\omega;\cdot)\rangle = 0. \tag{179}$$

Again, the Eq. (179) is more delicate than the corresponding equality (158) of the ODE's case. There, solving (158) was a simple consequence of the implicit theorem. We emphasize that solving the Eq. (179) is not simpler than solving the Eq. (78) in Sect. 2.

A more careful analysis shows that we can solve the Eqs. (178) and (179) by requiring that the above hypotheses (A1), (A2) together with the Hypotheses (H1)–(H7) of Sect. 2 hold, and by arguing as in Sect. 2. We do not give any detail of the proof since, in general, this method is more complicated than the integral method and requires the more restrictive hypothesis (A1). To conclude, in general the integral method is superior to this method.

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Spectral Theory for Forward Nonautonomous Parabolic Equations and Applications

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Dedicated to Professor George Sell on the occasion of his 70th birthday

Abstract We introduce the concept of the principal spectrum for linear forward nonautonomous parabolic partial differential equations. The principal spectrum is a nonempty compact interval. Fundamental properties of the principal spectrum for forward nonautonomous equations are investigated. The paper concludes with applications of the principal spectrum theory to the problem of uniform persistence in some population growth models.

Mathematics Subject Classification (2010): Primary 35K15, 35P05; Secondary 35K55, 35P15, 37B55, 92D25

1 Introduction

The current paper is devoted to the study of principal spectrum of the following linear nonautonomous parabolic equation:

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$$
\begin{aligned}
u_t = & \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a_{ij}(t,x)\frac{\partial u}{\partial x_j} + a_i(t,x)u\right) \\
& + \sum_{i=1}^{N} b_i(t,x)\frac{\partial u}{\partial x_i} + c_0(t,x)u, \quad t > s, \quad x \in D,
\end{aligned} \tag{1}
$$

endowed with the boundary condition

$$
\mathcal{B}(t)u = 0, \quad t > s, \quad x \in \partial D, \tag{2}
$$

where $D \subset \mathbb{R}^N$, $s \geq 0$, a_{ij}, a_i, b_i, and c_0 are appropriate functions on $[0,\infty) \times D$, and $\mathcal{B}$ is a boundary operator of either the Dirichlet or Neumann or Robin type, that is,

$$
\mathcal{B}(t)u = \begin{cases} u & \text{(Dirichlet)} \\ \sum_{i=1}^{N}\Big(\sum_{j=1}^{N} a_{ij}(t,x)\partial_j u + a_i(t,x)u\Big)\nu_i & \text{(Neumann)} \\ \sum_{i=1}^{N}\Big(\sum_{j=1}^{N} a_{ij}(t,x)\partial_j u + a_i(t,x)u\Big)\nu_i + d_0(t,x)u, & \text{(Robin)} \end{cases} \tag{3}
$$

where d_0 is an appropriate function on $[0,\infty) \times \partial D$. Let $a = ((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, d_0)$ with $d_0 \equiv 0$ in the Dirichlet or Neumann boundary condition case. To indicate the dependence of (1)+(2) on a, we may write (1)+(2) as $(1)_a$+$(2)_a$.

Among others, (1)+(2) arise from linearization of nonautonomous nonlinear parabolic equations at a global solution (i.e., a solution which exists for all $t \geq 0$) as well as from linearization of autonomous nonlinear parabolic equations at a global time dependent solution.

Concerning the linearization of a nonlinear problem at a global solution, it is of great importance to study the dynamical behavior of solutions of (1) + (2) as $s \to \infty$ and $t - s \to \infty$, where s represents the initial time. This paper is focused on the study of the least upper bound of exponential growth rates of solutions of (1) + (2) as $s \to \infty$ and $t - s \to \infty$, which is equivalent to the study of so called principal spectrum of (1) + (2) introduced in this paper.

Observe that (1)+(2) is called forward nonautonomous because, first, we are mainly interested in the properties of solutions as $s \to \infty$, $t - s \to \infty$, and a_{ij}, a_i, b_i, c_0, and d_0 are not necessarily defined for $t < 0$, and, second, the set of forward limiting equations can contain elements depending on time.

Principal spectrum for nonautonomous parabolic equations defined for all $t \in \mathbb{R}$ is well studied in several works (see [9–12, 14–18, 21], and references therein) and has also found great applications (see [8, 13, 19, 22, 27], etc.). Principal spectrum for such nonautonomous parabolic equations reflects the growth rates of solutions as $t - s \to \infty$, where s represents the initial time.

As the focus for forward nonautonomous parabolic equations is on the study of the behavior of solutions as $s \to \infty$ and $t-s \to \infty$, the principal spectral theory developed for nonautonomous parabolic equations defined for all $t \in \mathbb{R}$ cannot be applied to forward nonautonomous ones directly. The objective of this paper is to establish some principal spectral theory for forward nonautonomous parabolic equations, and discuss its applications to nonlinear parabolic equations of Kolmogorov type.

In order to do so, we first in Sect. 2 introduce the assumptions and the notion of weak solutions of (1)+(2) and present some basic properties of weak solutions.

In Sect. 3, we give the definition of principal spectrum of (1)+(2) and establish some fundamental properties. Let $U_a(t,s)u_0$ denote the weak solution of (1)+(2) with initial condition $u(s) = U_a(s,s)u_0 = u_0 \in L_2(D)$ $(s \geq 0)$. Roughly speaking, the *principal spectrum* of (1)+(2) is the complement in $\mathbb{R}$ of all the $\lambda \in \mathbb{R}$ satisfying either of the following conditions:

- There are $\eta > 0$, $M \geq 1$, and $T > 0$ such that

$$\|U_a(t,s)\| \leq Me^{(\lambda-\eta)(t-s)} \quad \text{for } t > s \geq T;$$

- There are $\eta > 0$, $M \in (0,1]$, and $T > 0$ such that

$$\|U_a(t,s)\| \geq Me^{(\lambda+\eta)(t-s)} \quad \text{for } t > s \geq T$$

(see Definition 3.2). Among others, it is proved in Sect. 3 that

- The principal spectrum of (1)+(2) is a compact interval $[\lambda_{\min}(a), \lambda_{\max}(a)]$ (see Theorem 3.1).
- $\lambda_{\min}(a) = \liminf\limits_{\substack{s\to\infty \\ t-s\to\infty}} \dfrac{\ln\|U_a(t,s)\|}{t-s}$ and $\lambda_{\max}(a) = \limsup\limits_{\substack{s\to\infty \\ t-s\to\infty}} \dfrac{\ln\|U_a(t,s)\|}{t-s}$ (see Theorem 3.2).

In Sect. 4, we investigate the relation of the principal spectrum of (1)+(2) and the principal spectrum, denoted by $[\tilde{\lambda}_{\min}(a), \tilde{\lambda}_{\max}(a)]$, of its forward limiting equations, and show that if some extension of (1)+(2) together with its limiting equations admits a so-called exponential separation, then

- $[\lambda_{\min}(a), \lambda_{\max}(a)] = [\tilde{\lambda}_{\min}(a), \tilde{\lambda}_{\max}(a)]$ (see Theorem 4.3);
- $\lambda_{\min}(a) = \liminf\limits_{\substack{s\to\infty \\ t-s\to\infty}} \dfrac{\ln\|U_a(t,s)u_0\|}{t-s}$ and $\lambda_{\max}(a) = \limsup\limits_{\substack{s\to\infty \\ t-s\to\infty}} \dfrac{\ln\|U_a(t,s)u_0\|}{t-s}$ for any nonzero nonnegative $u_0 \in L_2(D)$ (see Theorem 4.3);
- If, moreover, (1)+(2) is asymptotically uniquely ergodic, which includes the asymptotically periodic as a special case, then $[\lambda_{\min}(a), \lambda_{\max}(a)]$ is a singleton, i.e., $\lambda_{\min}(a) = \lambda_{\max}(a)$, and in the asymptotically periodic case, $\lambda_{\max}(a)$ equals the principal eigenvalue of the forward limiting periodic parabolic equation (see Corollary 4.5).

In Sect. 5, we establish more properties of the principal spectrum $[\lambda_{\min}(a), \lambda_{\max}(a)]$ of (1)+(2), including

- $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$ continuously depend on a in the norm topology (see Theorem 5.1);
- When a_{ij}, a_i and b_i depend only on x, the principal spectrum of (1)+(2) is greater than or equal to that of its time-averaged equations (see Theorem 5.3).

The properties mentioned above provide some important tools for the principal spectrum analysis as well as its computation.

We remark that the theories and techniques developed in this paper would have applications to the study of long time behavior in various forward nonautonomous nonlinear equations arising from biology and chemistry. In particular, they would have applications to the extensions of the existing dynamical theories for asymptotically periodic systems (see [28–31], etc.) to asymptotically uniquely ergodic ones, which include asymptotically periodic and almost periodic systems as special cases. In the last section (i.e., Sect. 6), we discuss applications of the principal spectrum theory for forward nonautonomous parabolic equations to the asymptotic dynamics of nonlinear parabolic equations of Kolmogorov type. In particular, we provide sufficient conditions for the uniform persistence (see Theorem 6.1).

Throughout the paper $D \subset \mathbb{R}^N$ is a bounded domain (an open and connected subset).

The norm in $L_2(D)$ is denoted by $\|\cdot\|$. Also, the norm in the Banach space $\mathcal{L}(L_2(D), L_2(D))$ of bounded linear operators from $L_2(D)$ into $L_2(D)$ is denoted by $\|\cdot\|$.

For the meaning of some symbols, like $C^{k+\alpha,l+\beta}(E_1 \times E_2)$, or $\mathcal{D}(E)$, etc., the reader is referred to the authors' monograph [18].

2 Assumptions and Weak Solutions

In this section, we state the assumptions, introduce the definition of weak solutions, and present some basic properties of weak solutions.

2.1 Assumptions

Consider (1)+(2). Our first assumption is on the regularity of the domain D.

(A1) (Boundary regularity) *For Dirichlet boundary conditions, D is a bounded domain. For Neumann or Robin boundary conditions, D is a bounded domain with Lipschitz boundary.*

If (A1) holds, D is always considered with the N-dimensional Lebesgue measure, whereas, in the case of Robin boundary conditions, ∂D is considered with the $(N-1)$-dimensional Hausdorff measure, which is equivalent to the surface measure.

The second assumption regards boundedness of the coefficients of the equations (and of the boundary conditions):

(A2) (Boundedness) $a = ((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, d_0)$ *belongs to* $L_\infty([0,\infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([0,\infty) \times \partial D, \mathbb{R})$ (*in the Dirichlet or Neumann case* d_0 *is set to be constantly equal to zero*).

We may write $a = (a_{ij}, a_i, b_i, c_0, d_0)$ for $a = ((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, d_0)$ if no confusion occurs.

The next assumption is about the uniform ellipticity.

(A3) (Uniform ellipticity) *There exists* $\alpha_0 > 0$ *such that there holds*

$$\sum_{i,j=1}^N a_{ij}(t,x)\,\xi_i\,\xi_j \geq \alpha_0 \sum_{i=1}^N \xi_i^2 \quad \text{for a.e. } (t,x) \in [0,\infty) \times D \text{ and all } \xi \in \mathbb{R}^N,$$
$$a_{ij}(t,x) = a_{ji}(t,x) \quad \text{for a.e. } (t,x) \in [0,\infty) \times D, \quad i,j = 1,2,\dots,N. \tag{4}$$

Sometimes we will use the forward limit equations to study the principal spectrum of (1)+(2). For any $t \geq 0$ we define the *time-translate* $a \cdot t$ of a by

$$a \cdot t := ((a_{ij} \cdot t)_{i,j=1}^N, (a_i \cdot t)_{i=1}^N, (b_i \cdot t)_{i=1}^N, c_0 \cdot t, d_0 \cdot t),$$

where $a_{ij} \cdot t(\tau, x) := a_{ij}(\tau + t, x)$ for $\tau \in [-t, \infty)$, $x \in D$, etc.

For a given sequence $(t_n) \subset [0,\infty)$ with $t_n \to T^*$ ($T^* \leq \infty$) and $\tilde{a} = ((\tilde{a}_{ij})_{i,j=1}^N, (\tilde{a}_i)_{i=1}^N, (\tilde{b}_i)_{i=1}^N, \tilde{c}_0, \tilde{d}_0) \in L_\infty((-T^*,\infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty((-T^*,\infty) \times \partial D, \mathbb{R})$, we say that $a \cdot t_n$ *converges to* $\tilde{a}$ *in the weak-* topology* if for any $T > -T^*$, $a \cdot t_n \to \tilde{a}$ in the weak-* topology of $L_\infty([T,\infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([T,\infty) \times \partial D, \mathbb{R})$.

Recall that the Banach space $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ is the dual of $L_1(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_1(\mathbb{R} \times \partial D, \mathbb{R})$. We denote the duality pairing by $\langle \cdot, \cdot \rangle_{L_1, L_\infty}$.

We fix a countable dense subset $\{g_1, g_2, \dots\}$ of the unit ball in $L_1(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_1(\mathbb{R} \times \partial D, \mathbb{R})$ such that for each $k \in \mathbb{N}$ there exists $K = K(k) > 0$ with the property that $g_k(t, \cdot) = 0$ for a.e. $t \in \mathbb{R} \setminus [-K, K]$.

For any $\tilde{a}^{(1)}, \tilde{a}^{(2)} \in L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ put

$$d(\tilde{a}^{(1)}, \tilde{a}^{(2)}) := \sum_{k=1}^\infty \frac{1}{2^k} |\langle g_k, (\tilde{a}^{(1)} - \tilde{a}^{(2)}) \rangle_{L_1, L_\infty}|. \tag{5}$$

For any $\tilde{a} \in L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$, $\tilde{a} = (\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0, \tilde{d}_0)$, and any $t \in \mathbb{R}$ we define the *time-translate* $\tilde{a} \cdot t$ of $\tilde{a}$ by

$$\tilde{a}\cdot t := ((\tilde{a}_{ij}\cdot t)_{i,j=1}^N, (\tilde{a}_i\cdot t)_{i=1}^N, (\tilde{b}_i\cdot t)_{i=1}^N, \tilde{c}_0\cdot t, \tilde{d}_0\cdot t),$$

where $\tilde{a}_{ij}\cdot t(\tau,x) := \tilde{a}_{ij}(\tau+t,x)$ for $\tau\in\mathbb{R}$, $x\in D$, etc.

We may extend a to functions belonging to $L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R})$ to study the forward limits of a. A function $\bar{a}\in L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R})$, $\bar{a} = ((\bar{a}_{ij})_{i,j=1}^N, (\bar{a}_i)_{i=1}^N, (\bar{b}_i)_{i=1}^N, \bar{c}_0, \bar{d}_0)$, is called an *extension of a* if $\bar{a}_{ij}(t,x) = a_{ij}(t,x)$, $\bar{a}_i(t,x) = a_i(t,x)$, $\bar{b}_i(t,x) = b_i(t,x)$, and $\bar{c}_0(t,x) = c_0(t,x)$ for a.e. $(t,x)\in[0,\infty)\times D$, and $\bar{d}_0(t,x) = d_0(t,x)$ for a.e. $(t,x)\in[0,\infty)\times\partial D$.

The lemma below will be instrumental in showing that the forward limits of a do not depend on the extension of a to a function in $L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R})$.

Lemma 2.1. *Let $\bar{a}^{(1)} = (\bar{a}_{ij}^{(1)}, \bar{a}_i^{(1)}, \bar{b}_i^{(1)}, \bar{c}_0^{(1)}, \bar{d}_0^{(1)})$ and $\bar{a}^{(2)} = (\bar{a}_{ij}^{(2)}, \bar{a}_i^{(2)}, \bar{b}_i^{(2)}, \bar{c}_0^{(2)}, \bar{d}_0^{(2)})$ be extensions of $a\in L_\infty([0,\infty)\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty([0,\infty)\times\partial D,\mathbb{R})$. Then, for any $t_n\to\infty$, one has $d(\bar{a}^{(1)}\cdot t_n, \bar{a}^{(2)}\cdot t_n)\to 0$. In particular, $\bar{a}^{(1)}\cdot t_n$ converges in the weak-* topology to $\tilde{a}$ ($\in L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R})$) if and only if $\bar{a}^{(2)}\cdot t_n$ converges in the weak-* topology to $\tilde{a}$.*

Proof. For $\varepsilon>0$, take $k_0\in\mathbb{N}$ such that

$$\sum_{k=k_0}^\infty \frac{1}{2^k} < \frac{\varepsilon}{2M},$$

where M denotes the maximum of the $(L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R}))$-norms of $\bar{a}^{(1)}$ and $\bar{a}^{(2)}$. Then we have

$$\sum_{k=k_0}^\infty \frac{1}{2^k}|\langle g_k, (\bar{a}^{(1)}\cdot\tau - \bar{a}^{(2)}\cdot\tau)\rangle_{L_1,L_\infty}| < \varepsilon$$

for each $\tau\in\mathbb{R}$. Let $K>0$ be such that $g_k(t,x)=0$ for a.e. t outside $[-K,K]$, for all $k=1,2,\dots,k_0-1$. We have

$$\sum_{k=1}^{k_0-1} \frac{1}{2^k}|\langle g_k, (\bar{a}^{(1)}\cdot t_n - \bar{a}^{(2)}\cdot t_n)\rangle_{L_1,L_\infty}| = 0$$

for $n\in\mathbb{N}$ so large that $t_n>K$. As a result, $d(\bar{a}^{(1)}\cdot t_n, \bar{a}^{(2)}\cdot t_n)<\varepsilon$ for such n. Therefore

$$d(\bar{a}^{(1)}\cdot t_n, \bar{a}^{(2)}\cdot t_n)\to 0 \quad\text{as}\quad n\to\infty,$$

and then $\bar{a}^{(1)}\cdot t_n$ converges in the weak-* topology to $\tilde{a}$ ($\in L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R})$) if and only if $\bar{a}^{(2)}\cdot t_n$ converges in the weak-* topology to $\tilde{a}$. □

For an extension $\bar{a}$ of a, the set $\{\bar{a}\cdot t : t\in\mathbb{R}\}$ is (norm-)bounded, hence has compact closure in the weak-* topology. We define

$$Y(\bar{a}) := \mathrm{cl}\{\bar{a}\cdot t : t\in\mathbb{R}\}, \tag{6}$$

where the closure is taken in the weak-* topology. When not remarked to the contrary, $Y(\bar{a})$ is considered with the weak-* topology. $Y(\bar{a})$ is a compact metrizable space, with a metric given by $d(\cdot,\cdot)$.

For $\tilde{a}\in Y(\bar{a})$ and $t\in\mathbb{R}$ we write $\sigma_t\tilde{a} := \tilde{a}\cdot t$. $(Y(\bar{a}),\{\sigma_t\}_{t\in\mathbb{R}})$ is a compact flow (i.e., $\sigma_t\tilde{a}$ is continuous in $t\in\mathbb{R}$ and $\tilde{a}\in Y(\bar{a})$, and $\sigma_0=\mathrm{Id}$, $\sigma_{t+s}=\sigma_t\circ\sigma_s$ for any $t,s\in\mathbb{R}$).

Let $\bar{a}$ be an extension of a. Put

$$Y_0(\bar{a}) := \bigcap_{s\geq 0}\mathrm{cl}\{\bar{a}\cdot t : t\in[s,\infty)\}. \tag{7}$$

In other words, $Y_0(\bar{a})$ equals the ω-limit set of $\bar{a}$ for the compact flow $(Y(\bar{a}),\{\sigma_t\}_{t\in\mathbb{R}})$. By standard results in the theory of topological dynamical systems, $Y_0(\bar{a})$ is invariant, nonempty, compact and connected. Also, $\tilde{a}\in Y_0(\bar{a})$ if and only if there is a sequence $t_n\to\infty$ such that $\bar{a}\cdot t_n\to\tilde{a}$ as $n\to\infty$.

In view of Lemma 2.1, $Y_0(\bar{a})$ does not depend on the choice of extension $\bar{a}$ of a. We can (and will) thus write $Y_0(a)$. Further, $\tilde{a}\in Y_0(a)$ if and only if there is a sequence $t_n\to\infty$ such that $a\cdot t_n\to\tilde{a}$ as $n\to\infty$.

The next assumption will be instrumental in proving the continuous dependence of solutions on parameters.

(A4) (Convergence almost everywhere)
In the Dirichlet or Neumann case:

(A4a) *For any sequence* $(t_n)\subset[0,\infty)$ *with* $t_n\to T^*$ $(T^*\leq\infty)$ *such that* $a\cdot t_n$ *converges to* $\tilde{a}$ *in the weak-* topology we have that* $a_{ij}\cdot t_n\to\tilde{a}_{ij}$, $a_i\cdot t_n\to\tilde{a}_i$, $b_i\cdot t_n\to\tilde{b}_i$ *pointwise a.e. on* $[T,\infty)\times D$, *for any* $T>-T^*$,

and

(A4b) *for any sequence* $(\tilde{a}^{(n)})\subset Y_0(a)$ *converging to* $\tilde{a}$ *in the weak-* topology we have that* $\tilde{a}^{(n)}_{ij}\to\tilde{a}_{ij}$, $\tilde{a}^{(n)}_i\to\tilde{a}_i$, $\tilde{b}^{(n)}_i\to\tilde{b}_i$ *pointwise a.e. on* $\mathbb{R}\times D$.

In the Robin case:

(A4a) *For any sequence* $(t_n)\subset[0,\infty)$ *with* $t_n\to T^*$ $(T^*\leq\infty)$ *such that* $a\cdot t_n$ *converges to* $\tilde{a}$ *in the weak-* topology we have that* $a_{ij}\cdot t_n\to\tilde{a}_{ij}$, $a_i\cdot t_n\to\tilde{a}_i$, $b_i\cdot t_n\to\tilde{b}_i$ *pointwise a.e. on* $[T,\infty)\times D$, *and* $d_0\cdot t_n\to\tilde{d}_0$ *pointwise a.e. on* $[T,\infty)\times\partial D$, *for any* $T>-T^*$,

and

(A4b) *for any sequence* $(\tilde{a}^{(n)})\subset Y_0(a)$ *converging to* $\tilde{a}$ *in the weak-* topology we have that* $\tilde{a}^{(n)}_{ij}\to\tilde{a}_{ij}$, $\tilde{a}^{(n)}_i\to\tilde{a}_i$, $\tilde{b}^{(n)}_i\to\tilde{b}_i$ *pointwise a.e. on* $\mathbb{R}\times D$, *and* $\tilde{d}^{(n)}_0\to\tilde{d}_0$ *pointwise a.e. on* $\mathbb{R}\times\partial D$.

To study the continuous dependence of the weak solutions and principal spectrum of (1)+(2) with respect to its coefficients, we may imbed the extensions of a into a subset Y of $L_\infty(\mathbb{R}\times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R}\times\partial D, \mathbb{R})$ satisfying

(A2)′ (Boundedness and invariance) *Y is a bounded subset of $L_\infty(\mathbb{R}\times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R}\times\partial D, \mathbb{R})$ and is closed (hence, compact) in the weak-* topology of $L_\infty(\mathbb{R}\times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R}\times\partial D, \mathbb{R})$. Moreover, Y is invariant: For any $\tilde{a}\in Y$ and any $t\in\mathbb{R}$ there holds $\tilde{a}\cdot t\in Y$.*

(It should be remarked here that, under Assumption (A2)′, $(Y, \{\sigma_t\}_{t\in\mathbb{R}})$, where $\sigma_t\tilde{a} := \tilde{a}\cdot t$, is a compact flow.)

(A3)′ (Uniform ellipticity) *There exists $\alpha_0 > 0$ such that for any $\tilde{a}\in Y$ there holds*

$$\sum_{i,j=1}^{N} \tilde{a}_{ij}(t,x)\,\xi_i\,\xi_j \geq \alpha_0 \sum_{i=1}^{N} \xi_i^2 \quad \text{for a.e. } (t,x)\in\mathbb{R}\times D \text{ and all } \xi\in\mathbb{R}^N,$$

$$\tilde{a}_{ij}(t,x) = \tilde{a}_{ji}(t,x) \quad \text{for a.e. } (t,x)\in\mathbb{R}\times D, \quad i,j=1,2,\ldots,N. \tag{8}$$

At some places, we may assume

(A4)′ (Convergence almost everywhere)

In the Dirichlet or Neumann case:

For any sequence $(\tilde{a}^{(n)})\subset Y$ converging to $\tilde{a}$ in the weak- topology we have that $\tilde{a}_{ij}^{(n)} \to \tilde{a}_{ij}$, $\tilde{a}_i^{(n)} \to \tilde{a}_i$, $\tilde{b}_i^{(n)} \to \tilde{b}_i$ pointwise a.e. on $\mathbb{R}\times D$.*

In the Robin case:

For any sequence $(\tilde{a}^{(n)})\subset Y$ converging to $\tilde{a}$ in the weak- topology we have that $\tilde{a}_{ij}^{(n)} \to \tilde{a}_{ij}$, $\tilde{a}_i^{(n)} \to \tilde{a}_i$, $\tilde{b}_i^{(n)} \to \tilde{b}_i$ pointwise a.e. on $\mathbb{R}\times D$, and $\tilde{d}_0^{(n)} \to \tilde{d}_0$ pointwise a.e. on $\mathbb{R}\times\partial D$.*

Observe that for a given a satisfying (A2) and (A3), $Y = Y_0(a)$ satisfies (A2)′ and (A3)′.

For $a\in L_\infty([0,\infty)\times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([0,\infty)\times\partial D, \mathbb{R})$ satisfying (A2) and (A3) we denote by $\bar{\bar{a}} = (\bar{\bar{a}}_{ij}, \bar{\bar{a}}_i, \bar{\bar{b}}_i, \bar{\bar{c}}_0, \bar{\bar{d}}_0)$ the extension of a given by

$$\begin{aligned}
\bar{\bar{a}}_{ij}(t,x) &:= \alpha_0\delta_{ij} && \text{for } t<0,\ x\in D,\\
\bar{\bar{a}}_i(t,x) &:= 0 && \text{for } t<0,\ x\in D,\\
\bar{\bar{b}}_i(t,x) &:= 0 && \text{for } t<0,\ x\in D,\\
\bar{\bar{c}}_0(t,x) &:= 0 && \text{for } t<0,\ x\in D,\\
\bar{\bar{d}}_0(t,x) &:= 0 && \text{for } t<0,\ x\in \partial D
\end{aligned} \tag{9}$$

(δ_{ij} denotes the Kronecker delta).

Sometimes, for a fulfilling (A2) and (A3), we pick up some extension $\bar{a}$ of a so that $Y = Y(\bar{a})$ satisfies (A2)′ and (A3)′. We may say that such $\bar{a}$ *satisfies* (A2)′ *and* (A3)′. If $Y = Y(\bar{a})$ satisfies (A4)′, we say $\bar{a}$ *satisfies* (A4)′.

Clearly $\bar{\bar{a}}$ defined by (9) satisfies (A2)$'$ and (A3)$'$.

Lemma 2.2. *For a satisfying* (A2)–(A4) *the extension* $\bar{\bar{a}}$ *given by* (9) *satisfies* (A4)$'$.

Proof. In the following, the expression "$\tilde{a}^{(n)}$ converges pointwise a.e. to $\tilde{a}$" means that $\tilde{a}_{ij}^{(n)} \to \tilde{a}_{ij}$, $\tilde{a}_i^{(n)} \to \tilde{a}_i$, $\tilde{b}_i^{(n)} \to \tilde{b}_i$ pointwise a.e. on $\mathbb{R} \times D$, and $\tilde{d}_0^{(n)} \to \tilde{d}_0$ pointwise a.e. on $\mathbb{R} \times \partial D$.

Note that the proof reduces to proving the following subcases:

(i) *For any real sequence* (t_n) *with* $\lim_{n\to\infty} t_n = -\infty$ *we have that* $\bar{\bar{a}} \cdot t_n$ *converges pointwise a.e. to* $(\alpha_0 \delta_{ij}, 0, 0, 0, 0)$.

This is straightforward.

(ii) *For any real sequence* (t_n) *with* $\lim_{n\to\infty} t_n = T \in \mathbb{R}$ *we have that* $\bar{\bar{a}} \cdot t_n$ *converges pointwise a.e. to* $\bar{\bar{a}} \cdot T$.

The fact that the corresponding coefficients converge pointwise a.e. on $[T, \infty) \times D$ (resp. pointwise a.e. on $[T, \infty) \times \partial D$) is a consequence of (A4a). The pointwise convergence a.e. on $(-\infty, T) \times D$ (resp. on $(-\infty, T) \times \partial D$) follows by the construction of $\bar{\bar{a}}$.

(iii) *For any real sequence* (t_n) *with* $\lim_{n\to\infty} t_n = \infty$ *such that* $\bar{\bar{a}} \cdot t_n$ *converges to* $\tilde{a} \in Y_0(a)$ *in the weak-* topology we have that* $\bar{\bar{a}} \cdot t_n$ *converges pointwise a.e. to* $\tilde{a}$.

This is a consequence of (A4a).

(iv) *For any sequence* $(a^{(n)}) \subset Y_0(a)$ *convergent to* $\tilde{a} \in Y_0(a)$ *in the weak-* topology we have that* $a^{(n)}$ *converges pointwise a.e. to* $\tilde{a}$.

This is just (A4b). □

The next result is a consequence of the Ascoli–Arzelà theorem.

Lemma 2.3. *Assume that the boundary* ∂D *of* D *is of class* C^{β}, *for some* $\beta > 0$.

(1) If a_{ij}, a_i, b_i, $c_0 \in C^{\beta_1,\beta_2}([0,\infty) \times D)$, *and* $d_0 \in C^{\beta_1,\beta_2}([0,\infty) \times \partial D)$, *where* $0 < \beta_2 \le \beta$, *then* $a = (a_{ij}, a_i, b_i, c_0, d_0)$ *satisfies* (A4).

(2) Assume that Y *satisfies* (A2)$'$. *If for each* $\tilde{a} = (\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0, \tilde{d}_0) \in Y$ *there holds* $\tilde{a}_{ij}$, $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_0 \in C^{\beta_1,\beta_2}(\mathbb{R} \times D)$, *and* $\tilde{d}_0 \in C^{\beta_1,\beta_2}(\mathbb{R} \times \partial D)$, *where* $0 < \beta_2 \le \beta$, *and the* $C^{\beta_1,\beta_2}(\mathbb{R} \times D)$*-norms of* $\tilde{a}_{ij}$, $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_0$ *are bounded uniformly in* $\tilde{a} \in Y$ *and the* $C^{\beta_1,\beta_2}(\mathbb{R} \times \partial D)$*-norms of* $\tilde{d}_0$ *are bounded uniformly in* $\tilde{a} \in Y$, *then* Y *satisfies* (A4)$'$.

2.2 Weak Solutions: Definition

Throughout this subsection, D satisfies (A1) and Y is a subset of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ satisfying (A2)$'$–(A3)$'$.

Let a satisfy (A2), (A3), and let $\bar{a}$ be an extension of a such that $Y(\bar{a}) \subset Y$. In particular, $\bar{a}$ satisfies (A2)$'$ and (A3)$'$.

We define V as follows:

$$V := \begin{cases} \mathring{W}_2^1(D) & \text{(Dirichlet)} \\ W_2^1(D) & \text{(Neumann)} \\ W_{2,2}^1(D, \partial D), & \text{(Robin)} \end{cases} \tag{10}$$

where $\mathring{W}_2^1(D)$ is the closure of $\mathcal{D}(D)$ in $W_2^1(D)$ and $W_{2,2}^1(D,\partial D)$ is the completion of

$$V_0 := \{\, v \in W_2^1(D) \cap C(\bar{D}) : v \text{ is } C^\infty \text{ on } D \text{ and } \|v\|_V < \infty \,\}$$

with respect to the norm $\|v\|_V := (\|\nabla v\|_2^2 + \|v\|_{2,\partial D}^2)^{1/2}$, where $\mathcal{D}(D)$ is the space of smooth real functions having compact support in D.

If no confusion occurs, we will write $\langle u, u^* \rangle$ for the duality between V and V^*, where $u \in V$ and $u^* \in V^*$.

For $s \le t$, let

$$W = W(s,t;V,V^*) := \{\, v \in L_2((s,t),V) : \dot{v} \in L_2((s,t),V^*) \,\} \tag{11}$$

equipped with the norm

$$\|v\|_W := \Big(\int_s^t \|v(t)\|_V^2 \, dt + \int_s^t \|\dot{v}(t)\|_{V^*}^2 \, dt \Big)^{\frac{1}{2}},$$

where $\dot{v} := dv/dt$ is the time derivative in the sense of distributions taking values in V^* (see [5, Chap. XVIII] for definitions).

For a given $\tilde{a} = (\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0, \tilde{d}_0) \in Y$, consider

$$u_t = \sum_{i=1}^N \frac{\partial}{\partial x_i} \Big(\sum_{j=1}^N \tilde{a}_{ij}(t,x) \frac{\partial u}{\partial x_j} + \tilde{a}_i(t,x) u \Big) + \sum_{i=1}^N \tilde{b}_i(t,x) \frac{\partial u}{\partial x_i} + \tilde{c}_0(t,x) u, \quad x \in D, \tag{12}$$

endowed with the boundary condition

$$\tilde{\mathcal{B}}(t) u = 0, \quad x \in \partial D, \tag{13}$$

where $\tilde{\mathcal{B}}$ is a boundary operator of either the Dirichlet or Neumann or Robin type, that is, $\tilde{\mathcal{B}}(t)u = \mathcal{B}(t)u$, where $\mathcal{B}(t)u$ is as in (3) with a being replaced by $\tilde{a}$. Sometimes we write the nonautonomous problem (12)+(13) as $(12)_{\tilde{a}}$+$(13)_{\tilde{a}}$.

Denote by $B_{\tilde{a}} = B_{\tilde{a}}(t,\cdot,\cdot)$ the bilinear form on V associated with $\tilde{a} \in Y$,

$$B_{\tilde{a}}(t,u,v) := \int_D \Big((\tilde{a}_{ij}(t,x) \partial_j u + \tilde{a}_i(t,x) u) \partial_i v - (\tilde{b}_i(t,x) \partial_i u + \tilde{c}_0(t,x) u) v \Big) \, dx, \tag{14}$$

$(u,v \in V)$ in the Dirichlet and Neumann boundary condition cases, and

$$B_{\tilde{a}}(t,u,v) := \int_D \big((\tilde{a}_{ij}(t,x)\partial_j u + \tilde{a}_i(t,x)u)\partial_i v - (\tilde{b}_i(t,x)\partial_i u + \tilde{c}_0(t,x)u)v\big)\,dx + \int_{\partial D} \tilde{d}_0(t,x)uv\,dH_{N-1}, \tag{15}$$

$(u,v \in V)$ in the Robin boundary condition case, where H_{N-1} stands for the $(N-1)$-dimensional Hausdorff measure, which is, by (A1), equivalent to the surface measure (we used the summation convention in the above).

Definition 2.1 (Weak solution). (1) Let $\tilde{a} \in Y$. A function $u \in L_2((s,t),V)$ is a *weak solution of* $(12)_{\tilde{a}}$+$(13)_{\tilde{a}}$ *on* $[s,t] \times D$*,* $s < t$*, with initial condition* $u(s) = u_0$ if

$$-\int_s^t \langle u(\tau),v\rangle\,\dot{\phi}(\tau)\,d\tau + \int_s^t B_{\tilde{a}}(\tau,u(\tau),v)\phi(\tau)\,d\tau - \langle u_0,v\rangle\,\phi(s) = 0 \tag{16}$$

for all $v \in V$ and $\phi \in \mathcal{D}([s,t))$, where $\mathcal{D}([s,t))$ is the space of all smooth real functions having compact support in $[s,t)$.

(2) If $\bar{a}$ is an extension of a and $s \geq 0$, a weak solution $u \in L_2((s,t),V)$ of $(12)_{\bar{a}}$+$(13)_{\bar{a}}$ on $[s,t] \times D$ with initial condition $u(s) = u_0$ is called a *weak solution of* (1)+(2) *on* $[s,t] \times D$ *with initial condition* $u(s) = u_0$.

Definition 2.2 (Global weak solution). (1) Let $\tilde{a} \in Y$. A function $u \in L_{2,\mathrm{loc}}((s,\infty),V)$ is a *global weak solution of* $(12)_{\tilde{a}}$+$(13)_{\tilde{a}}$ *with initial condition* $u(s) = u_0$, $s \in \mathbb{R}$, if for each $t > s$ its restriction $u|_{[s,t]}$ is a weak solution of (12)+(13) on $[s,t] \times D$ with initial condition $u(s) = u_0$.

(2) If $\bar{a}$ is an extension of a and $s \geq 0$, a global solution of $(12)_{\bar{a}}$+$(13)_{\bar{a}}$ on $[s,\infty)$ is called *a global solution of* (1)+(2) *on* $[s,\infty)$.

We remark that the (global) weak solutions of (1)+(2) are independent of the choices of the extensions of a. Sometimes we will write of (global) weak solutions of $(1)_a$+$(2)_a$.

2.3 Weak Solutions: Basic Properties

Throughout this subsection, D satisfies (A1) and Y is a subset of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ satisfying (A2)′–(A4)′.

Let a satisfy (A2)–(A4), and let $\bar{a}$ be an extension of a such that $Y(\bar{a}) \subset Y$.

We recall some basic properties of weak solutions of (12)+(13) ((1)+(2)) from [18] to be used in later sections. The reader is referred to [18] for various other important properties.

Proposition 2.4 (Existence of global solution). *For any $\tilde{a} \in Y$, $s \in \mathbb{R}$, and any $u_0 \in L_2(D)$ there exists a unique global weak solution $u(t;s,\tilde{a},u_0)$ of $(12)_{\tilde{a}}$+$(13)_{\tilde{a}}$ with initial condition $u(s;s,\tilde{a},u_0) = u_0$.*

Proof. See [3, Theorem 2.4]. □

As, for $s \leq t$ and $\tilde{a} \in Y$ fixed, the assignment $[L_2(D) \ni u_0 \mapsto u(t;s,\tilde{a},u_0) \in L_2(D)]$ is linear, we write $U_{\tilde{a}}(t,s)u_0$ for $u(t;s,\tilde{a},u_0)$.

Proposition 2.5. *(i) For any $s \leq t$ and any $\tilde{a} \in Y$ there holds*

$$U_{\tilde{a}}(t,s) = U_{\tilde{a}\cdot s}(t-s,0).$$

(ii) For any $s \leq t_1 \leq t_2$ and any $\tilde{a} \in Y$ there holds

$$U_{\tilde{a}}(t_2,s) = U_{\tilde{a}}(t_2,t_1) \circ U_{\tilde{a}}(t_1,s).$$

As a consequence, for any $s \leq t$ and any $\tilde{a} \in Y$ there holds

$$U_{\tilde{a}}(s+t,0) = U_{\tilde{a}\cdot s}(t,0) \circ U_{\tilde{a}}(s,0).$$

Proof. See [18, Propositions 2.1.6, 2.1.7 and 2.1.8]. □

We may write $U_{\tilde{a}}(t,s)$ as $U_a(t,s) = U_{a\cdot s}(t-s,0)$ if $t \geq s \geq 0$.

Proposition 2.6 (L_2–L_2 estimates). *There are constants $M > 0$ and $\gamma > 0$ such that*

$$\|U_{\tilde{a}}(t,0)\| \leq Me^{\gamma t} \tag{17}$$

for $\tilde{a} \in Y$ and $t > 0$.

Proof. See [18, Proposition 2.2.2]. □

Proposition 2.7 (Compactness). *For any given $0 < t_1 \leq t_2$, if E is a bounded subset of $L_2(D)$ then $\{U_{\tilde{a}}(t,0)u_0 : \tilde{a} \in Y,\ t \in [t_1,t_2],\ u_0 \in E\}$ is relatively compact in $L_2(D)$.*

Proof. See [18, Proposition 2.2.5]. □

For $u, v \in L_2(D)$ we write $u \leq v$ (or $v \geq u$) if $u(x) \leq v(x)$ for a.e. $x \in D$. We denote $L_2(D)^+ := \{u \in L_2(D) : u \geq 0\}$.

Proposition 2.8 (Monotonicity on initial data). *Let $\tilde{a} \in Y$, $t > 0$ and $u_1, u_2 \in L_2(D)$.*

(1) If $u_1 \leq u_2$ then $U_{\tilde{a}}(t,0)u_1 \leq U_{\tilde{a}}(t,0)u_2$.
(2) If $u_1 \leq u_2$, $u_1 \neq u_2$, then $(U_{\tilde{a}}(t,0)u_1)(x) < (U_{\tilde{a}}(t,0)u_2)(x)$ for $x \in D$.

Proof. See [18, Proposition 2.2.9]. □

Lemma 2.9. *Let $\tilde{a} \in Y$ and $t > 0$. Then $\|U_{\tilde{a}}(t,0)\| = \sup\{U_{\tilde{a}}(t,0)u_0 : u_0 \in L_2(D)^+, \|u_0\| = 1\}$.*

Proof. See [18, Lemma 3.1.1]. □

Proposition 2.10 (Monotonicity on coefficients). *Assume that $a^{(1)}$ and $a^{(2)}$ satisfy* (A2)–(A4).

(1) *Assume the Dirichlet boundary condition. Let, for some $T \geq 0$, $a_{ij}^{(1)} = a_{ij}^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, but $c_0^{(1)} \leq c_0^{(2)}$, where equalities and inequalities are to be understood a.e. on $[T,\infty) \times D$. Then*

$$U_{a^{(1)}}(t,s)u_0 \leq U_{a^{(2)}}(t,s)u_0$$

for any $t > s \geq T$ and any $u_0 \in L_2(D)^+$.

(2) *Assume the Neumann or Robin boundary condition. Let, for some $T \geq 0$, $a_{ij}^{(1)} = a_{ij}^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, but $c_0^{(1)} \leq c_0^{(2)}$, $d_0^{(1)} \geq d_0^{(2)}$, where equalities and inequalities are to be understood a.e. on $[T,\infty) \times D$ or a.e. on $[T,\infty) \times \partial D$. Then*

$$U_{a^{(1)}}(t,s)u_0 \leq U_{a^{(2)}}(t,s)u_0$$

for any $t > s \geq T$ and any $u_0 \in L_2(D)^+$.

(3) *Let, for some $T \geq 0$, $a_{ij}^{(1)} = a_{ij}^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, $c_0^{(1)} = c_0^{(2)}$, but $d_0^{(1)} \geq 0$, $d_0^{(2)} = 0$, where equalities and inequalities are to be understood a.e. on $[T,\infty) \times D$ or a.e. on $[T,\infty) \times \partial D$. Then*

$$U^{\mathrm{R}}_{a^{(1)}}(t,s)u_0 \leq U^{\mathrm{N}}_{a^{(2)}}(t,s)u_0$$

for any $t > s \geq T$ and any $u_0 \in L_2(D)^+$, where $U^{\mathrm{R}}_a(t,s)u_0$ and $U^{\mathrm{N}}_a(t,s)u_0$ denote the solutions of $(1)_a$+$(2)_a$ *with Robin and Neumann boundary conditions, respectively.*

(4) *Let, for some $T \geq 0$, $a_{ij}^{(1)} = a_{ij}^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, $c_0^{(1)} = c_0^{(2)}$, but $d_0^{(2)} \geq 0$, where equalities and inequalities are to be understood a.e. on $[T,\infty) \times D$ or a.e. on $[T,\infty) \times \partial D$. Then*

$$U^{\mathrm{D}}_{a^{(1)}}(t,s)u_0 \leq U^{\mathrm{R}}_{a^{(2)}}(t,s)u_0$$

for any $t > s \geq T$ and any $u_0 \in L_2(D)^+$, where $U^{\mathrm{D}}_a(t,s)u_0$ and $U^{\mathrm{R}}_a(t,s)u_0$ denote the solutions of $(1)_a$+$(2)_a$ *with Dirichlet and Robin boundary conditions, respectively.*

Proof. Compare [18, Proposition 2.2.10]. □

Proposition 2.11 (Continuous dependence). *For any real sequences $(s_n)_{n=1}^{\infty}$ with $s_n \to \infty$ and $(t_n)_{n=1}^{\infty}$ with $t_n \to t \in (0,\infty)$, if*

$$\lim_{n\to\infty} a \cdot s_n = \tilde{a},$$

then for any $u_0 \in L_2(D)$, $U_a(s_n+t_n,s_n)u_0 = U_{a\cdot s_n}(t_n,0)u_0$ *converges to* $U_{\tilde{a}}(t,0)u_0$ *in* $L_2(D)$.

Proof. It follows from the arguments of [18, Theorem 2.4.1]. □

Proposition 2.12 (Continuous dependence). *For any sequence* $(\tilde{a}^{(n)})_{n=1}^{\infty} \subset Y$, *any real sequence* $(t_n)_{n=1}^{\infty}$ *and any sequence* $(u_n)_{n=1}^{\infty} \subset L_2(D)$, *if* $\lim_{n\to\infty}\tilde{a}^{(n)} = \tilde{a}$, $\lim_{n\to\infty} t_n = t$, *where* $t \in (0,\infty)$, *and* $\lim_{n\to\infty} u_n = u_0$ *in* $L_2(D)$, *then* $U_{\tilde{a}^{(n)}}(t_n,0)u_n$ *converges in* $L_2(D)$ *to* $U_{\tilde{a}}(t,0)u_0$.

Proof. It follows from [18, Theorem 2.4.1]. □

We denote by $\Pi(Y) = \{\Pi(Y)_t\}_{t\geq 0}$ the *topological linear skew-product semiflow generated by* the family $(12)_{\tilde{a}}$+$(13)_{\tilde{a}}$, $\tilde{a} \in Y$, on the product bundle $L_2(D)\times Y$:

$$\Pi(Y)(t;u_0,\tilde{a}) = \Pi(Y)_t(u_0,\tilde{a}) := (U_{\tilde{a}}(t,0)u_0, \sigma_t\tilde{a}) \quad (t\geq 0,\ \tilde{a}\in Y,\ u_0\in L_2(D)).$$

For $Y = Y(\bar{a})$, we will denote $\Pi(Y)$ by $\Pi(\bar{a})$.

3 Principal Spectrum

In this section, we introduce the definition of the principal spectrum of (1)+(2) and establish some fundamental properties of it. Throughout the present section, we assume that D and a satisfy (A1)–(A4). Let $\bar{a}$ be an extension of a such that it satisfies (A2)′–(A4)′.

3.1 Definition

Definition 3.1 (Principal resolvent). A real number λ belongs to the *principal resolvent* of $(1)_a$+$(2)_a$ or $\{U_a(t,s)\}_{t\geq s\geq 0}$, denoted by $\rho(a)$, if either of the following conditions holds:

- There are $\eta > 0$, $M \geq 1$, and $T > 0$ such that

$$\|U_a(t,s)\| \leq Me^{(\lambda-\eta)(t-s)} \quad \text{for } t > s \geq T$$

  (such λ are said to belong to the *upper principal resolvent*, denoted by $\rho_+(a)$),
- There are $\eta > 0$, $M \in (0,1]$, and $T > 0$ such that

$$\|U_a(t,s)\| \geq Me^{(\lambda+\eta)(t-s)} \quad \text{for } t > s \geq T$$

  (such λ are said to belong to the *lower principal resolvent*, denoted by $\rho_-(a)$).

Definition 3.2 (Principal spectrum). The *principal spectrum* of $(1)_a$+$(2)_a$ or $\{U_a(t,s)\}_{t\geq s\geq 0}$, denoted by $\Sigma(a)$, equals the complement in $\mathbb{R}$ of the principal resolvent $\rho(a)$.

3.2 Fundamental Properties

Theorem 3.1. *The principal spectrum $\Sigma(a)$ of $(1)_a$+$(2)_a$ is a compact nonempty interval $[\lambda_{\min}(a), \lambda_{\max}(a)]$.*

In the following, $[\lambda_{\min}(a), \lambda_{\max}(a)]$ denotes the principal spectrum $\Sigma(a)$ of $(1)_a$+$(2)_a$ unless otherwise specified.

Theorem 3.2.

$$\lambda_{\min}(a) = \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)\|}{t-s} \leq \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)\|}{t-s} = \lambda_{\max}(a).$$

Theorem 3.3. *Assume that there is $T \geq 0$ such that there holds: $a_i(t,x) = b_i(t,x) = 0$ for a.e. $(t,x) \in [T,\infty) \times D$, and $c_0(t,x) \leq 0$ for a.e. $(t,x) \in [T,\infty) \times D$. Then $[\lambda_{\min}(a), \lambda_{\max}(a)] \subset (-\infty, 0]$.*

Theorem 3.4. *In the case of the Dirichlet boundary condition, assume that there is $T \geq 0$ such that there holds: $a_i(t,x) = b_i(t,x) = 0$ for a.e. $(t,x) \in [T,\infty) \times D$, and $c_0(t,x) \leq 0$ for a.e. $(t,x) \in [T,\infty) \times D$. Then $\lambda_{\max}(a) < 0$.*

To prove the above theorems, we first prove some lemmas.

Lemma 3.5. *(1) For any $t_0 > 0$ there is $K_1 = K_1(t_0) \geq 1$ such that $\|U_a(s+t,s)\| \leq K_1$ for all $s \geq 0$ and all $t \in [0,t_0]$.*
(2) For any $t_0 > 0$ there is $K_2 = K_2(t_0) > 0$ such that $\|U_a(s+t,s)\| \geq K_2$ for all $s \geq 0$ and all $t \in [0,t_0]$.

Proof. See [18, Lemma 3.1.2]. □

Lemma 3.6. *(1) A real number λ belongs to the upper principal resolvent if and only if there are $\delta_0 > 0$, $T > 0$, $\eta > 0$ and $\tilde{M} > 0$ such that*

$$\|U_a(t,s)\| \leq \tilde{M} e^{(\lambda-\eta)(t-s)} \quad \text{for } t-s \geq \delta_0 \text{ and } s \geq T.$$

(2) A real number λ belongs to the lower principal resolvent if and only if there are $\delta_0 > 0$, $T > 0$, $\eta > 0$ and $\tilde{M} > 0$ such that

$$\|U_a(t,s)\| \geq \tilde{M} e^{(\lambda+\eta)(t-s)} \quad \text{for } t-s \geq \delta_0 \text{ and } s \geq T.$$

Proof. The "only if" parts follow from Definition 3.1 in a straightforward way.

To prove the “if” part in (1), it suffices to notice that, by Lemma 3.5(1), there is $K_1 = K_1(\delta_0) > 0$ such that $\|U_a(t,s)\| \le K_1 \le (K_1 \max\{1, e^{-\delta_0(\lambda-\eta)}\})e^{(\lambda-\eta)(t-s)}$ for all $t > s \ge T$ with $t - s \le \delta_0$.

To prove the “if” part in (2), it suffices to notice that, by Lemma 3.5(2), there is $K_2 = K_2(\delta_0) > 0$ such that $\|U_a(t,s)\| \ge K_2 \ge (K_2 \min\{1, e^{-\delta_0(\lambda+\eta)}\})e^{(\lambda+\eta)(t-s)}$ for all $t > s \ge T$ with $t - s \le \delta_0$. □

Lemma 3.7. *There exist $\delta_1 > 0$, $M_1 > 0$ and a real $\underline{\lambda}$ such that $\|U_a(t,s)\| \ge M_1 e^{\underline{\lambda}(t-s)}$ for all $s \ge 0$ and all $t - s \ge \delta_1$.*

Proof. See [18, Lemma 3.1.4]. □

Proof of Theorem 3.1. We prove first that the upper principal resolvent $\rho_+(a)$ is nonempty. Indeed, by the L_2–L_2 estimates (Proposition 2.6), there are $M > 0$ and $\gamma > 0$ such that $\|U_a(t,s)\| \le Me^{\gamma(t-s)}$ for all $t \ge s \ge 0$. Consequently, $\gamma + 1 \in \rho_+(a)$. Further, it follows from the definition that $\rho_+(a)$ is a right-unbounded open interval $(\lambda_{\max}(a), \infty)$.

The lower principal resolvent $\rho_-(a)$ is nonempty, too, since it contains, by Lemma 3.7 combined with Lemma 3.6(2), the real number $\underline{\lambda} - 1$. Further, it follows from the definition that $\rho_-(a)$ is a left-unbounded open interval $(-\infty, \lambda_{\min}(a))$.

As $\rho_-(a) \cap \rho_+(a) = \emptyset$, one has $\Sigma(a) = \mathbb{R} \setminus \rho(a) = [\lambda_{\min}(a), \lambda_{\max}(a)]$. □

Proof of Theorem 3.2. First, by Definition 3.2, for any sequences $(t_n)_{n=1}^{\infty} \subset (0,\infty)$, $(s_n)_{n=1}^{\infty} \subset (0,\infty)$, such that $s_n \to \infty$ and $t_n - s_n \to \infty$ as $n \to \infty$ there holds

$$\lambda_{\min}(a) \le \liminf_{n\to\infty} \frac{\ln\|U_a(t_n,s_n)\|}{t_n - s_n} \le \limsup_{n\to\infty} \frac{\ln\|U_a(t_n,s_n)\|}{t_n - s_n} \le \lambda_{\max}(a). \tag{18}$$

Notice that, since $\lambda_{\min}(a) \notin \rho_-(a)$, it follows from Definition 3.1 that for each $n \in \mathbb{N}$ there are $n \le s_{n,1} < t_{n,1}$ with the property that

$$\|U_a(t_{n,1}, s_{n,1})\| < \tfrac{1}{n} \exp\left((\lambda_{\min}(a) + \tfrac{1}{n})(t_{n,1} - s_{n,1})\right).$$

We claim that $\lim_{n\to\infty}(t_{n,1} - s_{n,1}) = \infty$ as $n \to \infty$. Indeed, if not then there is a bounded subsequence $(t_{n_k,1} - s_{n_k,1})_{k=1}^{\infty}$, $n_k \to \infty$ as $k \to \infty$. It follows that $\|U_a(t_{n_k,1}, s_{n_k,1})\| \to 0$ as $k \to \infty$, which contradicts Lemma 3.5(2). Thus we have

$$\limsup_{n\to\infty} \frac{\ln\|U_a(t_{n,1},s_{n,1})\|}{t_{n,1} - s_{n,1}} \le \lambda_{\min}(a). \tag{19}$$

Notice also that, since $\lambda_{\max}(a) \notin \rho_+(a)$, it follows from Definition 3.1 that for each $n \in \mathbb{N}$ there are $n \le s_{n,2} < t_{n,2}$ with the property that

$$\|U_a(t_{n,2}, s_{n,2})\| > n \exp\left((\lambda_{\max}(a) - \tfrac{1}{n})(t_{n,2} - s_{n,2})\right).$$

We claim that $\lim_{n\to\infty}(t_{n,2} - s_{n,2}) = \infty$ as $n \to \infty$. Indeed, if not then there is a bounded subsequence $(t_{n_k,2} - s_{n_k,2})_{k=1}^{\infty}$, $n_k \to \infty$ as $k \to \infty$. It follows that $\|U_a(t_{n_k,2}, s_{n_k,2})\| \to \infty$ as $k \to \infty$, which contradicts Lemma 3.5(1). Thus we have

$$\liminf_{n\to\infty} \frac{\ln \|U_a(t_{n,2}, s_{n_k,2})\|}{t_{n,2} - t_{n_k,2}} \geq \lambda_{\max}(a). \tag{20}$$

The theorem then follows from (18)–(20). □

Proof of Theorem 3.3. Fix $u_0 \in L_2(D)^+$ with $\|u_0\| = 1$, and put $u(t,x) := (U_a(t,T)u_0)(x)$, $t \geq T$, $x \in D$. It follows from [18, Proposition 2.1.4] that

$$\begin{aligned}\|u(t,\cdot)\|^2 - \|u(s,\cdot)\|^2 &= -2\int_s^t B_a(\tau, u(\tau,\cdot), u(\tau,\cdot))\,d\tau \\ &\leq -2\int_s^t \int_D \Big(\sum_{i,j=1}^N a_{ij}(\tau,x)\,\partial_{x_i} u(\tau,x)\,\partial_{x_j} u(\tau,x)\Big)\,dx\,d\tau \leq 0\end{aligned}$$

for any $T \leq s < t$. Consequently, with the help of Lemma 2.9 we have $\|U_a(t,s)\| \leq 1$ for any $T \leq s < t$. Therefore $(0,\infty) \subset \rho_+(Y_0)$. □

Proof of Theorem 3.4. It follows by the Poincaré inequality (see [6, Theorem 3 in Sect. 5.6]) that there is $\alpha_1 > 0$ such that $\|u\| \leq \alpha_1 \|\nabla u\|$ for any $u \in \mathring{W}_2^1(D)$.

Starting as in the proof of Theorem 3.3 we estimate

$$\begin{aligned}\|u(t,\cdot)\|^2 - \|u(s,\cdot)\|^2 &= -2\int_s^t B_a(\tau, u(\tau,\cdot), u(\tau,\cdot))\,d\tau \\ &\leq -2\int_s^t \int_D \Big(\sum_{i,j=1}^N a_{ij}(\tau,x)\,\partial_{x_i} u(\tau,x)\,\partial_{x_j} u(\tau,x)\Big)\,dx\,d\tau \\ &\overset{\text{by (A2)}}{\leq} -2\alpha_0 \int_s^t \|\nabla u(\tau,\cdot)\|^2\,d\tau \leq \frac{-2\alpha_0}{(\alpha_1)^2}\int_s^t \|u(\tau,\cdot)\|^2\,d\tau\end{aligned}$$

for $T \leq s < t$. An application of the regular Gronwall inequality and Lemma 2.9 gives that

$$\|U_a(t,s)\| \leq e^{-\lambda_0(t-s)}$$

for any $T \leq s < t$, where $\lambda_0 := \alpha_0/\alpha_1^2 > 0$. Consequently, $[-\lambda_0,\infty) \subset \rho_+(a)$ and $\lambda_{\max}(a) < -\lambda_0$. □

3.3 Monotonicity and Continuity with Respect to Zero Order Terms

In this subsection, we explore the monotonicity and continuity of the principal spectrum of (1)+(2) with respect to the zero order terms.

Let $a^{(1)}$, $a^{(2)}$ be such that they satisfy properties (A1)–(A4).

We assume that there is $T \geq 0$ such that the following assumptions are satisfied:

(MC1) $a_{ij}^{(1)}(\cdot,\cdot) = a_{ij}^{(2)}(\cdot,\cdot)$, $a_i^{(1)}(\cdot,\cdot) = a_i^{(2)}(\cdot,\cdot)$, $b_i^{(1)}(\cdot,\cdot) = b_i^{(2)}(\cdot,\cdot)$, for a.e. $(t,x) \in [T,\infty) \times D$.
(MC2) $d_0^{(2)}(\cdot,\cdot) = d_0^{(1)}(\cdot,\cdot)$ for a.e. $(t,x) \in [T,\infty) \times \partial D$.
(MC3) One of the following conditions, (a), (b), (c), (d), or (e) holds:

(a) Both $(1)_{a^{(1)}}$ and $(1)_{a^{(2)}}$ are endowed with the Dirichlet boundary conditions, and

- $c_0^{(1)}(\cdot,\cdot) \leq c_0^{(2)}(\cdot,\cdot)$ for a.e. $(t,x) \in [T,\infty) \times D$,

(b) Both $(1)_{a^{(1)}}$ and $(1)_{a^{(2)}}$ are endowed with the Robin boundary conditions, and

- $c_0^{(1)}(\cdot,\cdot) \leq c_0^{(2)}(\cdot,\cdot)$ for a.e. $(t,x) \in [T,\infty) \times D$,
- $d_0^{(1)}(\cdot,\cdot) \geq d_0^{(2)}(\cdot,\cdot)$ for a.e. $(t,x) \in [T,\infty) \times \partial D$.

(c) Both $(1)_{a^{(1)}}$ and $(1)_{a^{(2)}}$ are endowed with the Neumann boundary conditions, and

- $c_0^{(1)}(\cdot,\cdot) \leq c_0^{(2)}(\cdot,\cdot)$ for a.e. $(t,x) \in [T,\infty) \times D$,

(d) $(1)_{a^{(1)}}$ is endowed with the Dirichlet boundary condition and $(1)_{a^{(2)}}$ is endowed with the Robin boundary condition, and $d_0^{(2)}(\cdot,\cdot) \geq 0$

- $c_0^{(1)}(\cdot,\cdot) = c_0^{(2)}(\cdot,\cdot)$ for a.e. $(t,x) \in [T,\infty) \times D$.

(e) $(1)_{a^{(1)}}$ is endowed with the Robin boundary condition and $(1)_{a^{(2)}}$ is endowed with the Neumann boundary condition, and $d_0^{(2)}(\cdot,\cdot) \geq 0$

- $c_0^{(1)}(\cdot,\cdot) = c_0^{(2)}(\cdot,\cdot)$ for a.e. $(t,x) \in [T,\infty) \times D$.

Theorem 3.8. *Assume that* (MC1) *and* (MC3) *hold. Then* $\lambda_{\min}(a^{(1)}) \leq \lambda_{\min}(a^{(2)})$ *and* $\lambda_{\max}(a^{(1)}) \leq \lambda_{\max}(a^{(2)})$.

Proof. We prove only the first inequality, the proof of the other being similar.

By Theorem 3.2, there are sequences $(s_n)_{n=1}^{\infty}$, $(t_n)_{n=1}^{\infty}$, with $0 < s_n < t_n$, $s_n \to \infty$ and $t_n - s_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{\ln \|U_{a^{(2)}}(t_n,s_n)\|}{t_n - s_n} = \lambda_{\min}(a^{(2)}).$$

Proposition 2.10 implies that for each $u_0 \in L_2(D)^+$ there holds

$$\|U_{a^{(1)}}(t_n,s_n)u_0\| \leq \|U_{a^{(2)}}(t_n,s_n)u_0\|$$

for $T \leq s_n < t_n$, which implies, via Lemma 2.9, that $\|U_{a^{(1)}}(t_n,s_n)\| \leq \|U_{a^{(2)}}(t_n,s_n)\|$ for sufficiently large n. By Theorem 3.2,

$$\lambda_{\min}(a^{(1)}) \le \liminf_{n\to\infty} \frac{\ln \|U_{a^{(1)}}(t_n,s_n)\|}{t_n - s_n} \le \lim_{n\to\infty} \frac{\ln \|U_{a^{(2)}}(t_n,s_n)\|}{t_n - s_n} = \lambda_{\min}(a^{(2)}). \qquad \square$$

Theorem 3.9. *Assume that* (MC1) *and* (MC2) *hold. Then* $|\lambda_{\min}(a^{(1)}) - \lambda_{\min}(a^{(2)})| \le r$ *and* $|\lambda_{\max}(a^{(1)}) - \lambda_{\max}(a^{(2)})| \le r$*, where* $r := \lim_{\tau\to\infty} \operatorname{ess\,sup}\{\,|c_0^{(1)}(t,x) - c_0^{(2)}(t,x)| : t \in (\tau,\infty),\ x \in D\}$.

Proof. For $m \in \mathbb{N}$, put $a^{(1)} \pm (r + \frac{1}{m})$ to be $a^{(1)}$ with $c_0^{(1)}$ replaced by $c_0^{(1)} \pm (r + \frac{1}{m})$.

By using arguments as in the proof of [18, Lemma 4.3.1] we see that

$$U_{a^{(k)} \pm (r+\frac{1}{m})}(t,s) = e^{\pm(r+\frac{1}{m})(t-s)} U_{a^{(k)}}(t,s) \qquad (0 \le s < t)$$

for $k = 1,2$. Consequently, by Theorem 3.2,

$$\lambda_{\text{ext}}(a^{(1)} \pm (r + \tfrac{1}{m})) = \lambda_{\text{ext}}(a^{(1)}) \pm (r + \tfrac{1}{m}),$$

where λ_{ext} stands for $\lambda_{\min}$ or $\lambda_{\max}$.

Observe that for any $m \in \mathbb{N}$ there is $T_m > 0$ such that

$$c_0^{(1)}(t,x) - (r + \tfrac{1}{m}) \le c_0^{(2)}(t,x) \le c_0^{(1)}(t,x) + (r + \tfrac{1}{m})$$

for a.e. $(t,x) \in (T_m,\infty) \times D$.

It then follows from Theorem 3.8 that

$$\lambda_{\text{ext}}(a^{(1)} - (r + \tfrac{1}{m})) \le \lambda_{\text{ext}}(a^{(2)}) \le \lambda_{\text{ext}}(a^{(1)} + (r + \tfrac{1}{m})),$$

hence

$$\lambda_{\text{ext}}(a^{(1)}) - (r + \tfrac{1}{m}) \le \lambda_{\text{ext}}(a^{(2)}) \le \lambda_{\text{ext}}(a^{(1)}) + (r + \tfrac{1}{m}).$$

As $m \in \mathbb{N}$ is arbitrary, this gives the desired result. $\square$

4 Exponential Separation and Equivalent Definition

In this section, we investigate the relation between the principal spectrum of (1)+(2) and that of the forward limit equations of (1)+(2). To do so, we employ the so-called exponential separation theory for general time dependent linear parabolic equations, which together with principal spectrum theory extends principal eigenvalue and principal eigenfunction theory for time periodic parabolic equations.

4.1 Definitions and Characterizations

We first introduce the principal spectrum of (12)$_a$+(13)$_a$ over $Y_0(a)$ and the exponential separation of $\Pi(Y)$ over Y. We then show that the principal spectrum of (1)$_a$+(2)$_a$ equals that of (12)$_a$+(13)$_a$ over $Y_0(a)$ provided that $\Pi(\bar{a})$ admits an exponential separation on $Y(\bar{a})$.

Throughout the present subsection, Y is a subset of $L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R})$ satisfying (A2)$'$–(A4)$'$.

Let D and a satisfy (A1)–(A4), and let $\bar{a}$ be an extension of a such that $Y(\bar{a})\subset Y$. In particular, $\bar{a}$ satisfies (A2)$'$–(A4)$'$.

Definition 4.1. $\lambda\in\mathbb{R}$ belongs to the *principal resolvent of* $Y_0(a)$ or the *principal resolvent of* (12)$_a$+(13)$_a$ *over* $Y_0(a)$, denoted by $\tilde{\rho}(a)$, if either of the following conditions is satisfied:

- There are $\eta>0$ and $M\geq 1$ such that
$$\|U_{\tilde{a}}(t,0)\|\leq Me^{(\lambda-\eta)t}\quad\text{for}\quad t>0,\quad \tilde{a}\in Y_0(a)$$
(such λ are said to belong to the *upper principal resolvent of* $Y_0(a)$, denoted by $\tilde{\rho}_+(a)$),
- There are $\eta>0$ and $M\in(0,1]$ such that
$$\|U_{\tilde{a}}(t,0)\|\geq Me^{(\lambda+\eta)t}\quad\text{for}\quad t>0,\quad \tilde{a}\in Y_0(a)$$
(such λ are said to belong to the *lower principal resolvent of* $Y_0(a)$, denoted by $\tilde{\rho}_-(a)$).

Definition 4.2. The *principal spectrum of* (12)$_a$+(13)$_a$ *over* $Y_0(a)$, denoted by $\tilde{\Sigma}(a)$, equals the complement in $\mathbb{R}$ of the principal resolvent of (12)$_a$+(13)$_a$ over $Y_0(a)$.

Remark 4.1. In the terminology of the monograph [18], the principal resolvent of $Y_0(a)$ (resp. the principal spectrum of $Y_0(a)$) is called the *principal resolvent of* $\Pi(\bar{a})$ *over* $Y_0(a)$ (resp. the *principal spectrum of* $\Pi(\bar{a})$ *over* $Y_0(a)$).

Theorem 4.2. $\tilde{\Sigma}(a)$ *is a nonempty interval* $[\tilde{\lambda}_{\min}(a),\tilde{\lambda}_{\max}(a)]$.

Proof. See [18, Theorem 3.1.1]. □

Definition 4.3. Let Y' be a closed invariant subset of Y. We say that $\Pi(Y)$ admits an *exponential separation with separating exponent* $\gamma_0>0$ *over* Y' if there are an invariant one-dimensional subbundle X_1 of $L_2(D)\times Y'$ with fibers $X_1(\tilde{a})=\operatorname{span}\{w(\tilde{a})\}$, $\|w(\tilde{a})\|=1$, and an invariant complementary one-codimensional subbundle X_2 of $L_2(D)\times Y'$ with fibers $X_2(\tilde{a})=\{v\in L_2(D):\langle v,w^*(\tilde{a})\rangle=0\}$ having the following properties:

(i) $w(\tilde{a})\in L_2(D)^+$ for all $\tilde{a}\in Y'$,
(ii) $X_2(\tilde{a})\cap L_2(D)^+=\{0\}$ for all $\tilde{a}\in Y'$,

(iii) There is $M \geq 1$ such that for any $\tilde{a} \in Y'$ and any $v \in X_2(\tilde{a})$ with $\|v\| = 1$,

$$\|U_{\tilde{a}}(t,0)v\| \leq Me^{-\gamma_0 t}\|U_{\tilde{a}}(t,0)w(\tilde{a})\| \quad (t > 0).$$

For more on bundles, etc., see [18, Sect. 3.2].

Let (A5) stand for the following assumption.

(A5) $\Pi(\bar{a})$ *admits an exponential separation over* $Y(\bar{a})$*, for some extension* $\bar{a}$ *of* a.

In the next subsection, we will show that if both D and a are sufficiently smooth, (A5) is satisfied.

Theorem 4.3. *Assume* (A5). *Then*

(i)

$$\lambda_{\min}(a) = \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} = \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)u_0\|}{t-s}$$
$$= \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)\|}{t-s}$$

for each nonzero $u_0 \in L_2(D)^+$,

(ii)

$$\lambda_{\max}(a) = \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} = \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)u_0\|}{t-s}$$
$$= \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)\|}{t-s}$$

for each nonzero $u_0 \in L_2(D)^+$.

(iii) $\Sigma(a) = \tilde{\Sigma}(a)$*, i.e.,* $\lambda_{\min}(a) = \tilde{\lambda}_{\min}(a)$ *and* $\lambda_{\max}(a) = \tilde{\lambda}_{\max}(a)$.

Before proving the above theorem, we first recall a lemma from [18].

Lemma 4.4. *Assume* (A5). *Then*

(1) $\lambda \in \mathbb{R}$ *belongs to* $\tilde{\rho}_+(a)$ *if and only if there are* $\eta > 0$ *and* $M \geq 1$ *such that*

$$\|U_{\tilde{a}}(t,0)w(\tilde{a})\| \leq Me^{(\lambda-\eta)t} \quad \textit{for } t > 0 \textit{ and } \tilde{a} \in Y_0(a),$$

(2) $\lambda \in \mathbb{R}$ *belongs to* $\tilde{\rho}_-(a)$ *if and only if there are* $\eta > 0$ *and* $M \in (0,1)$ *such that*

$$\|U_{\tilde{a}}(t,0)w(\tilde{a})\| \geq Me^{(\lambda+\eta)t} \quad \textit{for } t > 0 \textit{ and } \tilde{a} \in Y_0(a).$$

Proof. See [18, Lemma 3.2.6]. □

We remark that the complement of the set of those $\lambda \in \mathbb{R}$ for which either of the conditions in Lemma 4.4 holds is called the *dynamical spectrum* or the *Sacker–Sell spectrum* of $\Pi|_{X_1 \cap (L_2(D) \times Y_0(a))}$. The reader is referred to [23–26] for the fundamental spectral theory for nonautonomous linear evolution equations.

Proof of Theorem 4.3 First of all, by [18, Lemma 3.2.5], we have

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\ln \|U_a(t_n,s_n)u_0\|}{t_n - s_n} \\
&\quad = \liminf_{n\to\infty} \frac{\ln \|U_a(t_n,s_n)\|}{t_n - s_n} \\
&\quad = \liminf_{n\to\infty} \frac{\ln \|U_a(t_n,s_n)w(\bar{a}\cdot s_n)\|}{t_n - s_n} \\
&\quad \le \limsup_{n\to\infty} \frac{\ln \|U_a(t_n,s_n)w(\bar{a}\cdot s_n)\|}{t_n - s_n} \\
&\quad = \limsup_{n\to\infty} \frac{\ln \|U_a(t_n,s_n)\|}{t_n - s_n} = \limsup_{n\to\infty} \frac{\ln \|U_a(t_n,s_n)u_0\|}{t_n - s_n} \qquad (21)
\end{aligned}$$

for any $(s_n)_{n=1}^{\infty}$, $(t_n)_{n=1}^{\infty}$ such that $s_n \to \infty$ and $t_n - s_n \to \infty$, and any nonzero $u_0 \in L_2(D)^+$. By Theorem 3.2, there holds

$$\lambda_{\min}(a) = \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)\|}{t-s} \le \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)\|}{t-s} = \lambda_{\max}(a). \qquad (22)$$

(i) and (ii) then follow from (21) and (22).

Next, we prove (iii). We first prove

$$\tilde{\lambda}_{\min}(a) \le \lambda_{\min}(a) \le \lambda_{\max}(a) \le \tilde{\lambda}_{\max}(a). \qquad (23)$$

Fix, for the moment, $\varepsilon > 0$. As $\tilde{\lambda}_{\min}(a) - \varepsilon \in \tilde{\rho}_-(a)$, it follows from Lemma 4.4(2) that there is $T > 0$ such that for any $t \ge T$ and $\tilde{a} \in Y_0(a)$ there holds

$$\ln \|U_{\tilde{a}}(t,0)w(\tilde{a})\| > (\tilde{\lambda}_{\min}(a) - \varepsilon)t. \qquad (24)$$

By Proposition 2.12, there is $\delta > 0$ such that for any $\tilde{a}^{(1)}, \tilde{a}^{(2)} \in Y(\bar{a})$ with $d(\tilde{a}^{(1)}, \tilde{a}^{(2)}) < \delta$ there holds

$$-\varepsilon T \le \ln \|U_{\tilde{a}^{(1)}}(T,0)w(\tilde{a}^{(1)})\| - \ln \|U_{\tilde{a}^{(2)}}(T,0)w(\tilde{a}^{(2)})\| \le \varepsilon T. \qquad (25)$$

For the above $\delta > 0$ there is $T_1 > 0$ such that for any $s \ge T_1$ there is $\tilde{a} \in Y_0(a)$ such that

$$d(\bar{a}\cdot s, \tilde{a}) < \delta.$$

It then follows from (24) and (25) that

$$\ln \|U_a(T+s,s)w(\bar{a}\cdot s)\| \geq (\tilde{\lambda}_{\min}(a)-2\varepsilon)T,$$

and hence

$$\|U_a(T+s,s)w(\bar{a}\cdot s)\| \geq e^{(\tilde{\lambda}_{\min}(a)-2\varepsilon)T}, \tag{26}$$

for any $s \geq T_1$.

We then have, applying Proposition 2.5,

$$\begin{aligned}
&\|U_a(nT+s,s)w(\bar{a}\cdot s)\| \\
&\quad = \|U_a((n-1)T+s,T+s)U_a(T+s,s)w(\bar{a}\cdot s)\| \\
&\quad \geq \|U_a((n-1)T+s,T+s)w(\bar{a}\cdot(T+s))\| \cdot e^{(\tilde{\lambda}_{\min}(a)-2\varepsilon)T} \\
&\quad \geq \|U_a((n-2)T+s,2T+s)w(\bar{a}\cdot(2T+s))\| \cdot e^{2(\tilde{\lambda}_{\min}(a)-2\varepsilon)T} \\
&\quad \geq \dots \\
&\quad \geq e^{n(\tilde{\lambda}_{\min}(a)-2\varepsilon)T}
\end{aligned} \tag{27}$$

for any $s \geq T_1$ and $n \in \mathbb{N}$.

Therefore for any $s \geq T_1$ and $t > 0$ with $t-s = nT+\tau$ for some $n \in \{0,1,2,\dots\}$ and $0 \leq \tau < T$ there holds

$$\begin{aligned}
\|U_a(t,s)w(\bar{a}\cdot s)\| &= \|U_a(t,nT+s)U_a(nT+s,s)w(\bar{a}\cdot s)\| \\
&\geq \|U_a(t,nT+s)w(\bar{a}\cdot(nT+s))\| \cdot e^{n(\tilde{\lambda}_{\min}(a)-2\varepsilon)T} \\
&\geq Me^{n(\tilde{\lambda}_{\min}(a)-2\varepsilon)T},
\end{aligned} \tag{28}$$

where $M := \inf\{\, \|U_{\tilde{a}}(\tau,0)w(\tilde{a})\| : 0 \leq \tau \leq T,\ \tilde{a} \in Y(\bar{a})\,\} > 0$. This implies that

$$\liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} \geq \tilde{\lambda}_{\min}(a) - 2\varepsilon.$$

Letting $\varepsilon \to 0$ we get

$$\liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} \geq \tilde{\lambda}_{\min}(a). \tag{29}$$

Similarly we prove that

$$\limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} \leq \tilde{\lambda}_{\max}(a). \tag{30}$$

(23) then follows from (22), (29), and (30).

Next, we prove

$$\lambda_{\min}(a) \le \tilde{\lambda}_{\min}(a) \le \tilde{\lambda}_{\max}(a) \le \lambda_{\max}(a). \tag{31}$$

Let $\lambda \in \rho_+(a)$. By Definition 3.1, there are $\eta > 0$, $M \ge 1$ and $T > 0$ such that

$$\|U_a(t,s)\| \le Me^{(\lambda-\eta)(t-s)} \quad \text{for } t > s \ge T.$$

In particular, $\|U_a(t,s)w(\bar{a}\cdot s)\| \le Me^{(\lambda-\eta)(t-s)}$ for any $t > s \ge T$.

For each $\tilde{a} \in Y_0(a)$ there is $(s_n)_{n=1}^{\infty} \subset (0,\infty)$ with $s_n \to \infty$ such that $\bar{a}\cdot s_n \to \tilde{a}$. Then by Proposition 2.12, for any $t > 0$

$$U_a(t+s_n,s_n)w(\bar{a}\cdot s_n) \to U_{\tilde{a}}(t,0)w(\tilde{a})$$

as $n \to \infty$. Hence

$$\|U_{\tilde{a}}(t,0)w(\tilde{a})\| \le Me^{(\lambda-\eta)t}$$

for any $t > 0$. It then follows via Lemma 4.4(1) that $\lambda \in \tilde{\rho}_+(a)$. Consequently,

$$\tilde{\lambda}_{\max}(a) \le \lambda_{\max}(a). \tag{32}$$

Let $\lambda \in \rho_-(a)$. By Definition 3.1, there are $\eta > 0$, $M \in (0,1)$ and $T > 0$ such that

$$\|U_a(t,s)\| \ge Me^{(\lambda+\eta)(t-s)} \quad \text{for } t > s \ge T.$$

By [18, Lemma 3.2.3], there is $M_2 \ge 1$ such that $\|U_a(t,s)\| \le M_2\|U_a(t,s)w(\bar{a}\cdot s)\|$ for all $t > s$. Therefore, $\|U_a(t,s)w(\bar{a}\cdot s)\| \ge \tilde{M}e^{(\lambda+\eta)(t-s)}$ for any $t > s \ge T$, where $\tilde{M} := M/M_2 \in (0,1)$.

For each $\tilde{a} \in Y_0(a)$ there is $(s_n)_{n=1}^{\infty} \subset (0,\infty)$ with $s_n \to \infty$ such that $\bar{a}\cdot s_n \to \tilde{a}$. Then by Proposition 2.12, for any $t > 0$

$$U_a(t+s_n,s_n)w(\bar{a}\cdot s_n) \to U_{\tilde{a}}(t,0)w(\tilde{a})$$

as $n \to \infty$. Hence

$$\|U_{\tilde{a}}(t,0)w(\tilde{a})\| \ge \tilde{M}e^{(\lambda+\eta)t}$$

for any $t > 0$. It then follows via Lemma 4.4(2) that $\lambda \in \tilde{\rho}_-(a)$. Consequently,

$$\tilde{\lambda}_{\min}(a) \ge \lambda_{\min}(a). \tag{33}$$

(31) follows from (32) and (33).

By (23) and (31), $\Sigma(a) = \tilde{\Sigma}(a)$, i.e., (iii) holds. □

Corollary 4.5. *Assume* (A5). *If a is asymptotically uniquely ergodic* (*i.e., $Y_0(a)$ is uniquely ergodic*)*, then $\lambda_{\min}(a) = \lambda_{\max}(a)$. If, furthermore, a is asymptotically* *periodic with period T (i.e., $Y_0(a) = \{\tilde{a}\cdot t : t \in [0,T]\}$ for some $\tilde{a}$), then $\lambda := \lambda_{\min}(a)(= \lambda_{\max}(a))$ is the principal eigenvalue of the following periodic eigenvalue problem,*

$$\begin{cases} -u_t + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} \tilde{a}_{ij}(t,x)\frac{\partial u}{\partial x_j} + \tilde{a}_i(t,x)u\right) & \\ \qquad + \sum_{i=1}^{N} \tilde{b}_i(t,x)\frac{\partial u}{\partial x_i} + \tilde{c}_0(t,x)u = \lambda u, & x \in D, \\ \tilde{\mathcal{B}}(t)u = 0, & x \in \partial D, \\ u(t+T,x) = u(t,x). & \end{cases} \tag{34}$$

Proof. By [18, Corollary 3.2.2], we have $\tilde{\lambda}_{\max}(a) = \tilde{\lambda}_{\min}(a)$. It then follows from Theorem 4.3 that $\lambda_{\max}(a) = \lambda_{\min}(a)$. □

4.2 The Classical Case: An Example

In this subsection, we consider the so-called classical case, i.e., both D and the coefficients of (1)+(2) are sufficiently smooth (see (SM1) and (SM2) in the following) and show that for such a case (A5) is satisfied.

(SM1) (Boundary regularity) *$D \subset \mathbb{R}^N$ is a bounded domain, where its boundary ∂D is an $(N-1)$-dimensional manifold of class $C^{3+\alpha}$ for some $\alpha > 0$.*

(SM2) (Smoothness) *There is $\alpha > 0$ such that the functions a_{ij} $(= a_{ji})$ and a_i belong to $C^{2+\alpha,3+\alpha}([0,\infty)\times\bar{D})$, the functions b_i and c_0 belong to $C^{2+\alpha,1+\alpha}([0,\infty)\times\bar{D})$, and the function d_0 belongs to $C^{2+\alpha,3+\alpha}([0,\infty)\times\partial D)$.*

(SM3) (Ellipticity) *There exists $\alpha_0 > 0$ such that*

$$\sum_{i,j=1}^{N} a_{ij}(t,x)\,\xi_i\,\xi_j \geq \alpha_0 \sum_{i=1}^{N} \xi_i^2 \text{ for all } x \in \bar{D}, \xi \in \mathbb{R}^N \text{ and } t \geq 0.$$

We extend a to $\bar{a}$ by putting $\bar{a}_{ij}(t,x) := a_{ij}(0,x)$ $(i,j = 1,2,\ldots,N)$, $\bar{a}_i(t,x) := a_i(0,x)$ $(i = 1,2,\ldots,N)$, $\bar{b}_i(t,x) := b_i(0,x)$ $(i = 1,2,\ldots,N)$, $\bar{c}_0(t,x) := c_0(0,x)$, for all $t < 0$ and $x \in \bar{D}$, and $\bar{d}_0(t,x) := d_0(0,x)$ for all $t < 0$ and $x \in \partial D$.

(SM1) implies the fulfillment of (A1), (SM2) implies the fulfillment of (A2), and (SM3) is just (A3). By Lemma 2.3(1), (SM1) and (SM2) imply (A4). (SM2) and (SM3) together with the construction of $\bar{a}$ give (A2)′–(A3)′ with $Y = Y(\bar{a})$. The satisfaction of (A4)′ with $Y = Y(\bar{a})$ follows now from (SM2) via Lemma 2.3(2).

We claim that the problem (1)+(2) satisfies (A5). We have

$$Y(\bar{a}) = \alpha(\bar{a}) \cup \{\bar{a}\cdot t : t \in \mathbb{R}\} \cup Y_0(a),$$

where $\alpha(\bar{a}) = \{(a_{ij}(0,\cdot), a_i(0,\cdot), b_i(0,\cdot), c_0(0,\cdot), d_0(0,\cdot))\}$.

Let $\tilde{a} = (\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0, \tilde{d}_0) \in Y(\bar{a})$.

- Assume $\tilde{a} \in Y_0(a)$, or $\tilde{a} \in \alpha(\bar{a})$. It follows by the Ascoli–Arzelà theorem that the functions $\tilde{a}_{ij}$ and $\tilde{a}_i$ belong to $C^{2+\alpha,3+\alpha}(\mathbb{R}\times\bar{D})$, the functions $\tilde{b}_i$ and $\tilde{c}_0$ belong to $C^{2+\alpha,1+\alpha}(\mathbb{R}\times\bar{D})$ and the function $\tilde{d}_0$ belongs to $C^{2+\alpha,3+\alpha}(\mathbb{R}\times\partial D)$. Applying the theory in [1] (see [1, Corollary 15.3]), we have that $U_{\tilde{a}}(\cdot,0)u_0$ is a classical solution on $[t_0,\infty)$, for any $t_0 > 0$ and $u_0 \in L_2(D)$.
- Assume $\tilde{a} = \bar{a}\cdot\tau$ for some $\tau \geq 0$. Then the functions $\tilde{a}_{ij}$ and $\tilde{a}_i$ belong to $C^{2+\alpha,3+\alpha}([0,\infty)\times\bar{D})$, the functions $\tilde{b}_i$ and $\tilde{c}_0$ belong to $C^{2+\alpha,1+\alpha}([0,\infty)\times\bar{D})$ and the function $\tilde{d}_0$ belongs to $C^{2+\alpha,3+\alpha}([0,\infty)\times\partial D)$. Again applying the theory in [1], we have that $U_{\tilde{a}}(\cdot,0)u_0$ is a classical solution on $[t_0,\infty)$, for any $t_0 > 0$ and $u_0 \in L_2(D)$.
- Assume $\tilde{a} = \bar{a}\cdot\tau$ for some $\tau < 0$. Applying the theory in [1] and the theory in [18], we have that $[(0,T)\times D \ni (t,x) \mapsto (U_{\tilde{a}}(t,0)u_0)(x)] \in W_p^{1,2}((0,T)\times D)$ for any $T > 0$ and $p > 1$, and $U_{\tilde{a}}(t,0)u_0$ is a strong solution on (t_0,T), for any $0 < t_0 < T$ and $u_0 \in L_2(D)$.

Then in the Dirichlet case, by [11, Theorem 2.1 and Lemma 3.9], there hold

(HI1) (Harnack type inequality for quotients) *For each* $\delta_1 > 0$ *there is* $C_1 = C_1(\delta_1) > 1$ *with the property that*

$$\sup_{x\in D}\frac{(U_{\tilde{a}}(t,0)u_0^{(1)})(x)}{(U_{\tilde{a}}(t,0)u_0^{(2)})(x)} \leq C_1 \inf_{x\in D}\frac{(U_{\tilde{a}}(t,0)u_0^{(1)})(x)}{(U_{\tilde{a}}(t,0)u_0^{(2)})(x)}$$

for any $\tilde{a} \in Y(\bar{a})$, $t \geq \delta_1$ *and any* $u_0^{(1)}, u_0^{(2)} \in L_2(D)^+$ *with* $u_0^{(2)} \neq 0$.

(HI2) (Pointwise Harnack inequality) *There is* $\varsigma \geq 0$ *such that for each* $\delta_2 > 0$ *there is* $C_2 = C_2(\delta_2) > 0$ *with the property that*

$$(U_{\tilde{a}}(t,0)u_0)(x) \geq C_2(d(x))^{\varsigma}\|U_{\tilde{a}}(t,0)u_0\|_\infty \tag{35}$$

for any $\tilde{a} \in Y(\bar{a})$, $t \geq \delta_2$, $u_0 \in L_2(D)^+$ *and* $x \in D$, *where* $d(x)$ *denotes the distance of* $x \in D$ *from the boundary* ∂D *of* D.

In the Neumann or Robin cases, [9, Theorem 2.5] states that (HI2) is satisfied with $\varsigma = 0$, which implies, via [18, Lemma 3.3.1], the fulfillment of (HI1). The above reasoning can be repeated for the adjoint equation, hence, by [18, Theorem 3.3.3], the topological linear skew-product semiflow $\Pi(\bar{a})$ admits an exponential separation over $Y(\bar{a})$.

For $t \geq 0$ we define

$$\kappa(t) := -B_{\bar{a}\cdot t}(0, w(\bar{a}\cdot t), w(\bar{a}\cdot t)),$$

that is,

$$\kappa(t) = -\sum_{i=1}^{N} \int_D \Big(\sum_{j=1}^{N} a_{ij}(t,x) \partial_j w(\bar{a}\cdot t) + a_i(t,x) w(\bar{a}\cdot t) \Big) \partial_i w(\bar{a}\cdot t)\, dx$$
$$+ \int_D \Big(\sum_{i=1}^{N} b_i(t,x) \partial_i w(\bar{a}\cdot t) + c_0(t,x) w(\bar{a}\cdot t) \Big) w(\bar{a}\cdot t)\, dx$$

in the Dirichlet and Neumann boundary condition cases, and

$$\kappa(t) = -\sum_{i=1}^{N} \int_D \Big(\sum_{j=1}^{N} a_{ij}(t,x) \partial_j w(\bar{a}\cdot t) + a_i(t,x) w(\bar{a}\cdot t) \Big) \partial_i w(\bar{a}\cdot t)\, dx$$
$$+ \int_D \Big(\sum_{i=1}^{N} b_i(t,x) \partial_i w(\bar{a}\cdot t) + c_0(t,x) w(\bar{a}\cdot t) \Big) w(\bar{a}\cdot t)\, dx$$
$$- \int_{\partial D} d_0(t,x) (w(\bar{a}\cdot t))^2\, dH_{N-1}$$

in the Robin boundary condition case, where H_{N-1} stands for the $(N-1)$-dimensional Hausdorff measure (which is, under our assumption (SM1), equivalent to the $(N-1)$-dimensional Lebesgue measure).

Observe that the function $\kappa\colon [0,\infty) \to \mathbb{R}$ is well defined and continuous (see [18] for detail).

Lemma 4.6. *Assume* (SM1)–(SM3). *For* $0 \le s < t$ *put* $\eta(t;s) := \|U_a(t,s) w(\bar{a}\cdot s)\|$. *Then*

$$\eta_t(t;s) = \kappa(t)\eta(t;s)$$

for any $0 \le s < t$.

Proof. See the proof of [18, Lemma 3.5.3]. □

In view of Lemma 4.6 we have the following extension of Theorem 4.3.

Theorem 4.7. *Assume* (SM1)–(SM3). *For any nonzero* $u_0 \in L_2(D)^+$ *there holds*

$$\lambda_{\min}(a) = \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)u_0\|}{t-s}$$
$$= \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s)\|}{t-s} = \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln \|U_a(t,s) w(\bar{a}\cdot s)\|}{t-s}$$
$$= \liminf_{\substack{s\to\infty \\ t-s\to\infty}} \frac{1}{t-s} \int_s^t \kappa(\tau)\, d\tau \le \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{1}{t-s} \int_s^t \kappa(\tau)\, d\tau$$

$$= \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} = \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)\|}{t-s}$$

$$= \limsup_{\substack{s\to\infty \\ t-s\to\infty}} \frac{\ln\|U_a(t,s)u_0\|}{t-s} = \lambda_{\max}(a).$$

5 More Properties of Principal Spectrum

5.1 Continuity with Respect to the Coefficients

In the present subsection we investigate continuous dependence of the principal spectrum on the whole of the coefficients.

Assume (A1). We let Y be a subset of $L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R})$ satisfying (A2)′–(A4)′.

Throughout the present subsection we make also the following assumption.

(A5)′ $\Pi(Y)$ *admits an exponential separation over* Y.

Let $d_{\mathrm{norm}}(\cdot,\cdot)$ denote the metric on $L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times\partial D,\mathbb{R})$ generated by the norm, and let $d(\cdot,\cdot)$ be given by (5).

For $a^{(1)},a^{(2)}\in L_\infty([0,\infty)\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty([0,\infty)\times\partial D,\mathbb{R})$ and $s\ge 0$, by $d^s_{\mathrm{norm}}(a^{(1)},a^{(2)})$ we denote the $L_\infty([s,\infty)\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty([s,\infty)\times\partial D,\mathbb{R})$-norm of the difference of the restrictions of $a^{(1)}$, $a^{(2)}$ to $[s,\infty)\times D$ ($[s,\infty)\times\partial D$).

Definition 5.1. We say that $a\in L_\infty([0,\infty)\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty([0,\infty)\times\partial D,\mathbb{R})$ is Y*-admissible* if a satisfies (A2)–(A4) and, moreover, $Y_0(a)\subset Y$.

We remark here that, for a Y-admissible a, it follows from [18, Theorem 3.2.3] (the uniqueness of exponential separation) that the restrictions to $Y_0(a)$ of the one-dimensional subbundles (resp. one-codimensional subbundles) appearing in the definition of an exponential separation over $Y(\bar{a})$ and over Y are the same.

For the rest of the subsection we fix a Y-admissible $a^{(0)}$.

Theorem 5.1. *For each* $\varepsilon>0$ *there is* $\eta>0$ *such that for any* Y*-admissible* a*, if* $\limsup_{s\to\infty} d^s_{\mathrm{norm}}(a,a^{(0)})<\eta$ *then*

$$|\lambda_{\min}(a)-\lambda_{\min}(a^{(0)})|\le\varepsilon \quad \textit{and} \quad |\lambda_{\max}(a)-\lambda_{\max}(a^{(0)})|\le\varepsilon.$$

Lemma 5.2. *For each* $\varepsilon>0$ *there is* $\eta>0$ *with the following property. Let* $\hat{a},\check{a}\in Y$ *be such that* $d(\hat{a}\cdot t,\check{a}\cdot t)<\eta$ *for all* $t\in\mathbb{R}$*. Then, for any integer sequences* $(k_n)_{n=1}^\infty$*,* $(l_n)_{n=1}^\infty$*, such that* $l_n-k_n\to\infty$ *as* $n\to\infty$ *and*

$$\lim_{n\to\infty}\frac{\ln\|U_{\hat{a}}(l_n,k_n)w(\hat{a}\cdot k_n)\|}{l_n-k_n}=\lambda,$$

one has

$$\lambda - \varepsilon \le \liminf_{n\to\infty} \frac{\ln \|U_{\check{a}}(l_n,k_n)w(\check{a}\cdot k_n)\|}{l_n-k_n} \le \limsup_{n\to\infty} \frac{\ln \|U_{\check{a}}(l_n,k_n)w(\check{a}\cdot k_n)\|}{l_n-k_n} \le \lambda+\varepsilon.$$

Proof. It follows from [18, Lemma 4.4.2]. □

Proof of Theorem 5.1. Fix $\varepsilon > 0$, and take a Y-admissible a such that $\limsup_{s\to\infty} d^s_{\mathrm{norm}}(a,a^{(0)}) < \eta$, where $\eta > 0$ is as in Lemma 5.2.

By Theorem 4.3 and [18, Theorems 3.2.5 and 3.2.6], there exist an ergodic invariant measure $\mu_{\min}$ for the compact flow $(Y_0(a^{(0)}),\{\sigma_t\})$ and a Borel set $Y_1 \subset Y_0(a^{(0)})$ with $\mu_{\min}(Y_1) = 1$ such that

$$\lim_{t\to\infty} \frac{\ln \|U_{\tilde{a}}(t,0)w(\tilde{a})\|}{t} = \lambda_{\min}(a^{(0)})$$

for any $\tilde{a} \in Y_1$. Fix some $\tilde{a} \in Y_1$. Let $(t_n)_{n=1}^{\infty}$ be a sequence with $\lim_{n\to\infty} t_n = \infty$ such that $\bar{a}^{(0)}\cdot t_n$ converges to $\tilde{a}$. We can extract a subsequence (t_{n_k}) such that $\bar{a}\cdot t_{n_k}$ converges, as $k\to\infty$, to some $\check{a}$.

We claim that $d(\tilde{a}\cdot t,\check{a}\cdot t) < \eta$ for all $t\in\mathbb{R}$. Denote $\eta_1 := \limsup_{s\to\infty} d^s_{\mathrm{norm}}(a,a^{(0)})$ $(<\eta)$, and let M_1 stand for the maximum of the $L_\infty(\mathbb{R}\times D,\mathbb{R}^{N^2+2N+1})\times L_\infty(\mathbb{R}\times \partial D,\mathbb{R})$-norms of $\bar{a}$ and $\bar{a}^{(0)}$. Fix $t\in\mathbb{R}$. Take $k_0\in\mathbb{N}$ so large that $1/2^{k_0-1} < M_1(\eta-\eta_1)/6$. Then we have

$$\sum_{k=k_0}^{\infty} \frac{1}{2^k} |\langle g_k, \bar{a}\cdot\tau - \bar{a}^{(0)}\cdot\tau\rangle_{L_1,L_\infty}| < \frac{\eta-\eta_1}{3} \tag{36}$$

for all $\tau\in\mathbb{R}$. Take $M>0$ such that $g_k(\tau,\cdot)=0$ for all $\tau\le -M$ and all $k=1,2,\dots,k_0-1$. Further, take $s_0>0$ such that $d^s_{\mathrm{norm}}(a,a^{(0)}) < (\eta+2\eta_1)/3$ for all $s>s_0-M$. Finally, let $l_0\in\mathbb{N}$ be such that $t+t_{n_l} > s_0-M$ for all $l>l_0$.

Then we have

$$|\langle g_k, \bar{a}\cdot(t+t_{n_l}) - \bar{a}^{(0)}\cdot(t+t_{n_l})\rangle_{L_1,L_\infty}| \le (\eta+2\eta_1)/3$$

for $k=1,2,\dots,k_0-1$ and all $l>l_0$, hence

$$\sum_{k=1}^{k_0-1} \frac{1}{2^k} |\langle g_k, \bar{a}\cdot(t+t_{n_l}) - \bar{a}^{(0)}\cdot(t+t_{n_l})\rangle_{L_1,L_\infty}| < \frac{\eta+2\eta_1}{3} \tag{37}$$

for all $l>l_0$.

Taking (36) and (37) into account we see that $d(\bar{a}\cdot(t+t_{n_l}),\bar{a}^{(0)}\cdot(t+t_{n_l})) < (2\eta+\eta_1)/3$ for sufficiently large l. By letting l go to infinity we have $d(\tilde{a}\cdot t,\check{a}\cdot t) \le (2\eta+\eta_1)/3 < \eta$.

By Lemma 5.2,

$$\lambda_{\min}(a^{(0)}) - \varepsilon \le \liminf_{n\to\infty} \frac{\ln \|U_{\check{a}}(n,0)w(\check{a}\cdot n)\|}{n} \le \limsup_{n\to\infty} \frac{\ln \|U_{\check{a}}(n,0)w(\check{a}\cdot n)\|}{n} \le \lambda_{\min}(a^{(0)}) + \varepsilon.$$

As a consequence of Theorem 4.3 and [18, Theorem 3.1.2 and Lemma 3.2.5], both $\liminf_{n\to\infty} \frac{1}{n} \ln \|U_{\check{a}}(n,0)w(\check{a}\cdot n)\|$ and $\limsup_{n\to\infty} \frac{1}{n} \ln \|U_{\check{a}}(n,0)w(\check{a}\cdot n)\|$ are in $\Sigma(a)$. Hence we have found $\lambda \in [\lambda_{\min}(a), \lambda_{\max}(a)]$ with $|\lambda - \lambda_{\min}(a^{(0)})| \le \varepsilon$. By interchanging the rôles of $a^{(0)}$ and a we obtain that there is $\lambda \in [\lambda_{\min}(a^{(0)}), \lambda_{\max}(a^{(0)})]$ with $|\lambda - \lambda_{\min}(a)| \le \varepsilon$.

We proceed in the same way with $\lambda_{\max}$, obtaining that the Hausdorff distance between $[\lambda_{\min}(a), \lambda_{\max}(a)]$ and $[\lambda_{\min}(a^{(0)}), \lambda_{\max}(a^{(0)})]$ is not bigger than ε, which is equivalent to the statement of Theorem 5.1. □

5.2 *Time Averaging*

In the present subsection we assume that $a_{ij}(t,x) \equiv a_{ij}(x)$, $a_i(t,x) \equiv a_i(x)$, $b_i(t,x) \equiv b_i(x)$, and D and a satisfy (SM1)–(SM3).

Let $\bar{a}$ be the extension of a as in Sect. 4.2. $\Pi(\bar{a})$ admits an exponential separation over $Y(\bar{a})$.

We call $\hat{a} = (a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0)$ a *time-averaged function* of a if

$$\hat{c}_0(x) = \lim_{n\to\infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} c_0(t,x)\,dt \qquad \text{for all } x \in \bar{D},$$

and

$$\hat{d}_0(x) = \lim_{n\to\infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} d_0(t,x)\,dt \qquad \text{for all } x \in \partial D,$$

for some real sequences $(s_n)_{n=1}^\infty$, $(t_n)_{n=1}^\infty$ with $s_n \to \infty$ and $t_n - s_n \to \infty$ as $n \to \infty$.

The time independent equation

$$\begin{cases} u_t = \displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^N a_{ij}(x)\frac{\partial u}{\partial x_j} + a_i(x)u\Big) + \sum_{i=1}^N b_i(x)\frac{\partial u}{\partial x_i} + \hat{c}_0(x)u, & x \in D, \\ \mathcal{B}_{\hat{a}} u = 0, & x \in \partial D, \end{cases} \tag{38}$$

where

$$\mathcal{B}_{\hat{a}}u = \begin{cases} u & \text{(Dirichlet)} \\ \sum_{i=1}^{N}\Big(\sum_{j=1}^{N} a_{ij}(x)\partial_j u + a_i(x)u\Big)\nu_i & \text{(Neumann)} \\ \sum_{i=1}^{N}\Big(\sum_{j=1}^{N} a_{ij}(x)\partial_j u + a_i(x)u\Big)\nu_i + \hat{d}_0(x)u, & \text{(Robin),} \end{cases}$$

is called a *time-averaged equation* of (1)+(2) if $\hat{a} = (a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0)$ is a time-averaged function of a.

The eigenvalue problem associated to (38) reads as

$$\begin{cases} \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j} + a_i(x)u\Big) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + \hat{c}_0(x)u = \lambda u, & x \in D, \\ \mathcal{B}_{\hat{a}}u = 0, & x \in \partial D. \end{cases} \tag{39}$$

It is well known that (39) has a unique eigenvalue, denoted by $\lambda_{\mathrm{princ}}(\hat{a})$, which is real, simple, has an eigenfunction $\varphi_{\mathrm{princ}}(\hat{a}) \in L_2(D)^+$ associated to it, and for any other eigenvalue λ of (39), $\mathrm{Re}\,\lambda < \lambda_{\mathrm{princ}}(\hat{a})$ (see [2, 4]). We call $\lambda_{\mathrm{princ}}(\hat{a})$ the *principal eigenvalue* of (38) and $\varphi_{\mathrm{princ}}(\hat{a})$ a *principal eigenfunction* (in the literature, sometimes, $-\lambda_{\mathrm{princ}}(\hat{a})$ is called the principal eigenvalue of (38)).

Let

$$\hat{Y}(a) := \Big\{ \hat{a} : \exists 0 \le s_n < t_n \quad \text{with} \quad s_n \to \infty \quad \text{and} \quad t_n - s_n \to \infty \quad \text{such that}$$
$$\hat{c}_0(x) = \lim_{n\to\infty} \frac{1}{t_n - s_n}\int_{s_n}^{t_n} c_0(t,x)\,dt \quad \text{for all} \quad x \in \bar{D},$$
$$\hat{d}_0(x) = \lim_{n\to\infty} \frac{1}{t_n - s_n}\int_{s_n}^{t_n} d_0(t,x)\,dt \quad \text{for all} \quad x \in \partial D \Big\}.$$

It follows from our assumptions, via the Ascoli–Arzelà theorem, that $\hat{Y}(a)$ is nonempty, and consists of functions belonging to $C^{3+\alpha}(\bar{D}, \mathbb{R}^{N^2+N}) \times C^{1+\alpha}(\bar{D}, \mathbb{R}^{N+1}) \times C^{3+\alpha}(\partial D)$, with their $C^{3+\alpha}(\bar{D}, \mathbb{R}^{N^2+N}) \times C^{1+\alpha}(\bar{D}, \mathbb{R}^{N+1}) \times C^{3+\alpha}(\partial D)$-norms uniformly bounded. Moreover, the convergence in the definition of $\hat{Y}(a)$ is uniform in $x \in \bar{D}$ (resp. uniform in $x \in \partial D$).

Theorem 5.3. *(1) There is $\hat{a} \in \hat{Y}(a)$ such that $\lambda_{\min}(a) \ge \lambda_{\mathrm{princ}}(\hat{a})$.*
(2) $\lambda_{\max}(a) \ge \lambda_{\mathrm{princ}}(\hat{a})$ for any $\hat{a} \in \hat{Y}(a)$.
(3) Assume moreover that a is asymptotically uniquely ergodic. Then $\hat{Y}(a)$ is a singleton $\{\hat{a}\}$, $\lambda_{\max}(a) = \lambda_{\min}(a) \ge \lambda_{\mathrm{princ}}(\hat{a})$, and $\lambda_{\min}(a) = \lambda_{\mathrm{princ}}(\hat{a})$ if and

only if there is a sequence $(s_n)_{n=1}^{\infty} \subset [0,\infty)$ *with* $\lim_{n\to\infty} s_n = \infty$ *with the property that the following two conditions are satisfied:*

- *There are a continuous function* $c_{01}\colon \bar{D} \to \mathbb{R}$ *and a bounded continuous function* $c_{02}\colon (-\infty,\infty) \to \mathbb{R}$ *such that* $c_0(t+s_n,x)$ *converges, as* $n\to\infty$*, to* $c_{01}(x)+c_{02}(t)$*, uniformly on compact subsets of* $\mathbb{R}\times\bar{D}$*,*
- *There is a continuous function* $d_{01}\colon \partial D \to \mathbb{R}$ *such that* $d_0(t+s_n,x)$ *converges, as* $n\to\infty$*, to* $d_{01}(x)$*, uniformly on compact subsets of* $\mathbb{R}\times\partial D$*.*

To prove the above theorem, we first recall a lemma from [18].

Lemma 5.4. *Let* $\tilde{v}(t,x) := w(\bar{a}\cdot t)(x)$ $(t \ge 0,\ x\in\bar{D})$ *and*

$$\hat{w}(x;s,t) := \exp\left(\frac{1}{t-s}\int_s^t \ln w(\bar{a}\cdot\tau)(x)\,d\tau\right)$$

$(0 \le s < t,\ x\in D)$*. Then* $\hat{w}(x;s,t)$ *satisfies*

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a_{ij}(x)\frac{\partial\hat{w}}{\partial x_j}+a_i(x)\hat{w}\right)+\sum_{i=1}^{N} b_i(x)\frac{\partial\hat{w}}{\partial x_i}$$
$$\le\left(\frac{1}{t-s}\int_s^t\frac{1}{\tilde{v}}\frac{\partial\tilde{v}}{\partial\tau}(\tau,x)\,d\tau\right)\hat{w}+\left(\frac{1}{t-s}\int_s^t\kappa(\tau)\,d\tau-\frac{1}{t-s}\int_s^t c_0(\tau,x)\,d\tau\right)\hat{w} \tag{40}$$

for $x\in D$ *and*

$$\hat{\mathcal{B}}(s,t)\hat{w}=0$$

for $x\in\partial D$*, where*

$$\hat{\mathcal{B}}(s,t)\hat{w} := \begin{cases} \hat{w} & (\textit{Dirichlet}) \\ \displaystyle\sum_{i=1}^{N}\Big(\sum_{j=1}^{N} a_{ij}(x)\partial_{x_j}\hat{w}+a_i(x)\hat{w}\Big)\nu_i & (\textit{Neumann}) \\ \displaystyle\sum_{i=1}^{N}\Big(\sum_{j=1}^{N} a_{ij}(x)\partial_{x_j}\hat{w}+a_i(x)\hat{w}\Big)\nu_i & \\ \qquad +\Big(\frac{1}{t-s}\int_s^t d_0(\tau,x)\,d\tau\Big)\hat{w} & (\textit{Robin}). \end{cases} \tag{41}$$

Proof. See the proof of [18, Lemma 5.2.1]. □

Proof of Theorem 5.3(1) and (2). (1) For given $0 \le s < t$ put

$$\eta(t;s) := \|U_a(t,s)w(\bar{a}\cdot s)\|$$

and

$$\hat{w}(x;s,t) := \exp\left(\frac{1}{t-s}\int_s^t \ln w(\bar{a}\cdot\tau)(x)\,d\tau\right), \qquad x \in \bar{D}.$$

By Theorem 4.7, there are sequences $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ with $s_n \to \infty$ and $t_n - s_n \to \infty$ such that

$$\frac{\ln \eta(t_n;s_n)}{t_n - s_n} = \frac{1}{t_n - s_n}\int_{s_n}^{t_n} \kappa(t)\,dt \to \lambda_{\min}(a).$$

It follows from (SM2) with the help of the Ascoli–Arzelà theorem that (after possibly taking a subsequence and relabeling) $\lim\limits_{n\to\infty} \frac{1}{t_n-s_n}\int_{s_n}^{t_n} c_0(t,x)\,dt$ and $\lim\limits_{n\to\infty} \frac{1}{t_n-s_n}\int_{s_n}^{t_n} d_0(t,x)\,dt$ exist, and the limits are uniform in $x \in \bar{D}$ and in $x \in \partial D$, respectively. Denote these limits by $\hat{c}_0(x)$ and $\hat{d}_0(x)$. Let $\hat{a} := (a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0)$. It then follows by arguments similar to those in [18, Theorem 5.2.2(1)] that $\lambda_{\min}(a) \geq \lambda_{\text{princ}}(\hat{a})$.

(2) For any $\hat{a} = (a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0) \in \hat{Y}(a)$ there are $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ with $s_n \to \infty$ and $t_n - s_n \to \infty$ such that

$$\frac{1}{t_n - s_n}\int_{s_n}^{t_n} c_0(t,x)\,dt \to \hat{c}_0(x) \quad \text{and} \quad \frac{1}{t_n - s_n}\int_{s_n}^{t_n} d_0(t,x)\,dt \to \hat{d}_0(x)$$

uniformly in $x \in \bar{D}$ and in $x \in \partial D$, respectively. By passing (if necessary) to subsequences and relabeling we can assume that there is λ_0 such that

$$\lim_{n\to\infty} \frac{1}{t_n - s_n}\int_{s_n}^{t_n} \kappa(t)\,dt = \lambda_0.$$

By arguments similar to those in the proof of (1), $\lambda_0 \geq \lambda_{\text{princ}}(\hat{a})$. It follows from Theorem 4.7 that $\lambda_{\max}(a) \geq \lambda_0$. Then we have $\lambda_{\max}(a) \geq \lambda_{\text{princ}}(\hat{a})$. □

Before proving Theorem 5.3(3) we formulate and prove the following auxiliary result.

Lemma 5.5. *Assume that a is asymptotically uniquely ergodic. Then*

(i) For each $x \in \bar{D}$ and each $\tilde{a} = (a_{ij}, a_i, b_i, \tilde{c}_0, \tilde{d}_0) \in Y_0(a)$ the limits

$$\lim_{\substack{s\to\infty \\ t-s\to\infty}} \frac{1}{t-s}\int_s^t c_0(\tau,x)\,d\tau \qquad \textit{and} \qquad \lim_{t-s\to\infty} \frac{1}{t-s}\int_s^t \tilde{c}_0(\tau,x)\,d\tau \tag{42}$$

exist and are equal, and

(ii) For each $x \in \bar{D}$ and each $\tilde{a} = (a_{ij}, a_i, b_i, \tilde{c}_0, \tilde{d}_0) \in Y_0(a)$ the limits

$$\lim_{\substack{s\to\infty\\ t-s\to\infty}} \frac{1}{t-s}\int_s^t d_0(\tau,x)\,d\tau \qquad \textit{and} \qquad \lim_{t-s\to\infty} \frac{1}{t-s}\int_s^t \tilde{d}_0(\tau,x)\,d\tau \tag{43}$$

exist and are equal.

In particular, it follows that $\hat{Y}(a) = \{(a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0)\}$*, where* $\hat{c}_0$ *is the common limit in* (42) *and* $\hat{d}_0$ *is the common limit in* (43)*.*

Proof. Let $\mathbb{P}$ be the unique ergodic invariant measure for the compact flow $(Y_0(a), \{\sigma_t\}_{t\in\mathbb{R}})$. $\mathbb{P}$ is also the unique ergodic invariant measure for the compact semiflow $(\tilde{Y}_0(\bar{a}), \{\sigma_t\}_{t\geq 0})$, where $\tilde{Y}_0(\bar{a}) := \{\bar{a}\cdot t : t \geq 0\} \cup Y_0(a) = \mathrm{cl}\{\bar{a}\cdot t : t \geq 0\}$.

For each $x \in \bar{D}$ (resp. for each $x \in \partial D$) we define a function $\tilde{c}_0[x]\colon \tilde{Y}_0(\bar{a}) \to \mathbb{R}$ (resp. a function $\tilde{d}_0[x]\colon \tilde{Y}_0(\bar{a}) \to \mathbb{R}$) as:

$$\tilde{c}_0[x](\tilde{a}) := \tilde{c}_0(0,x), \quad \tilde{a} \in \tilde{Y}_0(\bar{a}),$$
$$\tilde{d}_0[x](\tilde{a}) := \tilde{d}_0(0,x), \quad \tilde{a} \in \tilde{Y}_0(\bar{a}).$$

where $\tilde{a} = (a_{ij}, a_i, b_i, \tilde{c}_0, \tilde{d}_0)$. The functions $\tilde{c}_0[x]\colon \tilde{Y}_0(\bar{a}) \to \mathbb{R}$ (resp. $\tilde{d}_0[x]\colon \tilde{Y}_0(\bar{a}) \to \mathbb{R}$) are, for each $x \in \bar{D}$ (resp. for each $x \in \partial D$), continuous.

As $(\tilde{Y}_0(\bar{a}), \{\sigma_t\}_{t\geq 0})$ is uniquely ergodic, it follows from the results in [20] that for any continuous $g\colon \tilde{Y}_0(\bar{a}) \to \mathbb{R}$ and any $\varepsilon > 0$ there is $T_0 = T_0(g,\varepsilon) > 0$ such that

$$\left| \frac{1}{t}\int_0^t g(\tilde{a}\cdot\tau)\,d\tau - \int_{\tilde{Y}_0(\bar{a})} g(\cdot)\,d\mathbb{P}(\cdot) \right| < \varepsilon$$

for each $t > T_0$ and each $\tilde{a} \in \tilde{Y}_0(a)$. In particular, for any continuous $g\colon \tilde{Y}_0(\bar{a}) \to \mathbb{R}$ there holds

$$\lim_{\substack{s\to\infty\\ t-s\to\infty}} \frac{1}{t-s}\int_s^t g(\bar{a}\cdot\tau)\,d\tau = \lim_{t-s\to\infty} \frac{1}{t-s}\int_s^t g(\tilde{a}\cdot\tau)\,d\tau = \int_{\tilde{Y}_0(\bar{a})} g(\cdot)\,d\mathbb{P}(\cdot),$$

for each $\tilde{a} \in \tilde{Y}_0(a)$. By substituting in the above, for a fixed $x \in \bar{D}$, the function $\tilde{c}_0[x]$ for g we have

$$\lim_{\substack{s\to\infty\\ t-s\to\infty}} \frac{1}{t-s}\int_s^t c_0(\tau,x)\,d\tau = \lim_{t-s\to\infty} \frac{1}{t-s}\int_s^t \tilde{c}_0(\tau,x)\,d\tau = \int_{\tilde{Y}_0(\bar{a})} \tilde{c}_0[x](\cdot)\,d\mathbb{P}(\cdot),$$

for each $\tilde{a} \in \tilde{Y}_0(a)$. Similarly, by substituting, for a fixed $x \in \partial D$, the function $\tilde{d}_0[x]$ for g we have

$$\lim_{\substack{s\to\infty\\ t-s\to\infty}} \frac{1}{t-s}\int_s^t d_0(\tau,x)\,d\tau = \lim_{t-s\to\infty} \frac{1}{t-s}\int_s^t \tilde{c}_0(\tau,x)\,d\tau = \int_{\tilde{Y}_0(\bar{a})} \tilde{d}_0[x](\cdot)\,d\mathbb{P}(\cdot),$$

for each $\tilde{a} \in \tilde{Y}_0(a)$. □

Proof of Theorem 5.3(3). By Theorem [18, Theorem 5.2.2(3)] and Theorem 4.3, we have

$$\lambda_{\max}(a) = \tilde{\lambda}_{\max}(a) = \tilde{\lambda}_{\min}(a) = \lambda_{\min}(a).$$

Let $Y_1 \subset Y_0(a)$ be a minimal invariant set. By the unique ergodicity of $(Y_0(a), \{\sigma_t\})$, the compact flow $(Y_1, \{\sigma_t\})$ is both minimal and uniquely ergodic. Let $\tilde{a} \in Y_1$. In view of Lemma 5.5 we can apply [18, Theorem 5.2.2(3)] to have that $\tilde{\lambda}_{\max}(a) = \lambda(\hat{a})$ if and only if there are c_{01}, c_{02}, and d_{01} such that

$$\tilde{c}_0(t,x) = c_{01}(t) + c_{02}(x) \quad \text{and} \quad \tilde{d}_0(t,x) = d_{01}(x).$$

Note that there is $s_n \to \infty$ such that $a \cdot s_n \to \tilde{a}$. Therefore, $\tilde{\lambda}_{\max}(a) = \lambda(\hat{a})$ if and only if there is $s_n \to \infty$ such that

$$\lim_{n\to\infty} c_0(t + s_n, x) = c_{01}(t) + c_{02}(x) \quad \text{and} \quad \lim_{n\to\infty} d_0(t,x) = d_{01}(x),$$

where the convergence is uniform on compact subsets of $\mathbb{R} \times \bar{D}$ (resp. on compact subsets of $\mathbb{R} \times \partial D$). □

5.3 *Space-Averaging*

In the present subsection we assume that $a_{ij}(t,x) \equiv a_{ij}(t)$, $a_i(t,x) \equiv 0$, $b_i(t,x) \equiv 0$, and the boundary condition is Neumann. We also assume that D and a satisfy (SM1)–(SM3).

Let $\check{c}_0(t) := \frac{1}{|D|}\int_D c_0(t,x)\,dx$, $t \geq 0$. We call $\check{a} := (a_{ij}, 0, 0, \check{c}_0, 0)$ the *space-average* of a, and call the problem

$$\begin{cases} u_t = \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a_{ij}(t)\dfrac{\partial u}{\partial x_j}\right) + \check{c}_0(t)u, & t > s \geq 0,\ x \in D, \\ \sum_{i=1}^{N}\left(\sum_{j=1}^{N} a_{ij}(t)\partial_j u\right)\nu_i = 0, & t > s \geq 0,\ x \in \partial D \end{cases} \tag{44}$$

the *space-averaged equation* of (1)+(2).

The theory presented in Sect. 4.2 applies to (44).

Denote by $[\lambda_{\min}(\check{a}), \lambda_{\max}(\check{a})]$ the principal spectrum interval of (44).

Theorem 5.6. *(1)* $[\lambda_{\min}(\check{a}), \lambda_{\max}(\check{a})] = \{\,\lambda : \exists s_n < t_n$ *with* $s_n \to \infty$ *and* $t_n - s_n \to \infty$ *such that* $\lambda = \lim_{n\to\infty} \frac{1}{t_n - s_n}\int_{s_n}^{t_n} \check{c}_0(t)\,dt\,\}$.
(2) $\lambda_{\min}(a) \geq \lambda_{\min}(\check{a})$ *and* $\lambda_{\max}(a) \geq \lambda_{\max}(\check{a})$.

Proof. (1) Observe that the function $u\colon [0,\infty) \times \bar{D} \to \mathbb{R}$ defined as $u(t,x) := \exp\left(\int_0^t \check{c}_0(\tau)\,d\tau\right)$, $t \geq 0$, $x \in \bar{D}$, is a solution of (44) satisfying $u(0,\cdot) \in L_2(D)^+ \setminus \{0\}$, and apply Theorem 4.7 to obtain (1).

To prove (2), we use the following inequality, which was proved as a part of the proof of [18, Theorem 5.3.1(2)]:

$$\frac{1}{t-s}\int_s^t \check{c}_0(\tau)\,d\tau \le \frac{\ln\|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} + \frac{1}{|D|}\frac{1}{t-s}\int_D \ln\frac{w(\bar{a}\cdot t)(x)}{w(\bar{a}\cdot s)(x)}\,dx,\ 0\le s<t. \tag{45}$$

It follows from [18, Lemma 5.2.3(2)] that the set $\{w(\tilde{a}^{(1)})(x)/w(\tilde{a}^{(2)})(x) : \tilde{a}^{(1)},\tilde{a}^{(2)} \in Y(\bar{a}),\ x\in D\}$ is bounded and bounded away from zero. Therefore the limit, as $s\to\infty$ and $t-s\to\infty$, of the second term on the right-hand side of (45) equals zero. Consequently,

$$\lambda_{\min}(\check{a}) = \liminf_{\substack{s\to\infty\\ t-s\to\infty}} \frac{1}{t-s}\int_s^t \check{c}_0(\tau)\,d\tau \le \liminf_{\substack{s\to\infty\\ t-s\to\infty}} \frac{\ln\|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} = \lambda_{\min}(a)$$

and

$$\lambda_{\max}(\check{a}) = \limsup_{\substack{s\to\infty\\ t-s\to\infty}} \frac{1}{t-s}\int_s^t \check{c}_0(\tau)\,d\tau \le \limsup_{\substack{s\to\infty\\ t-s\to\infty}} \frac{\ln\|U_a(t,s)w(\bar{a}\cdot s)\|}{t-s} = \lambda_{\max}(a).$$

This concludes the proof of (2). □

6 Applications to Nonlinear Equations of Kolmogorov Type

In this section we study the asymptotic dynamics of nonlinear parabolic equations of Kolmogorov type. In particular, we provide conditions for (forward) uniform persistence of the nonlinear Kolmogorov equations by utilizing the principal spectrum associated to proper forward nonautonomous linear parabolic equations.

Throughout the present section we make the following assumption.

(NA1) *$D\subset\mathbb{R}^N$ is a bounded domain, where its boundary ∂D is an $(N-1)$-dimensional manifold of class $C^{3+\alpha}$, for some $\alpha>0$.*

Further, $\mathcal{B}$ will stand for the boundary operator either of the Dirichlet type

$$\mathcal{B}u = u \quad \text{on } \partial D,$$

or of the Neumann type

$$\mathcal{B}u = \frac{\partial u}{\partial \boldsymbol{\nu}} \quad \text{on } \partial D,$$

where $\boldsymbol{\nu}$ denotes the unit normal vector pointing out of D.

Let φ_{princ} be the unique (nonnegative) principal eigenfunction of the elliptic boundary value problem

$$\begin{cases} \Delta u = \lambda u & \text{on } D, \\ \mathcal{B}u = 0 & \text{on } \partial D, \end{cases} \tag{46}$$

normalized so that $\sup\{\varphi_{\mathrm{princ}}(x) : x \in \bar{D}\} = 1$. By the elliptic strong maximum principle and the Hopf boundary point principle, in the Dirichlet case $\varphi_{\mathrm{princ}}(x) > 0$ for each $x \in D$ and $(\partial \varphi_{\mathrm{princ}}/\partial \boldsymbol{\nu})(x) < 0$ for each $x \in \partial D$. In the Neumann case $\varphi_{\mathrm{princ}} \equiv 1$.

Let X be a fractional power space of the Laplacian operator Δ in $L_p(D)$ with the boundary condition $\mathcal{B}u = 0$ such that X is compactly imbedded into $C^1(\bar{D})$. We denote the norm in X by $\|\cdot\|_X$.

Denote $X^+ := \{u \in X : u(x) \geq 0 \text{ for all } x \in \bar{D}\}$. The interior X^{++} of X^+ is nonempty, and is characterized in the following way: In the case of Dirichlet boundary conditions, $X^{++} = \{u \in X^+ : u(x) > 0 \text{ for all } x \in D \text{ and } (\partial u/\partial \boldsymbol{\nu})(x) < 0 \text{ for all } x \in \partial D\}$, and in the case of Neumann boundary conditions, $X^{++} = \{u \in X^+ : u(x) > 0 \text{ for all } x \in \bar{D}\}$ (see [18, Lemma 7.1.8]). In particular, observe that $\varphi_{\mathrm{princ}} \in X^{++}$.

For $u_1, u_2 \in X$ we write $u_1 \ll u_2$ (or $u_2 \gg u_1$) if $u_2 - u_1 \in X^{++}$.

Consider the following nonautonomous partial differential equation of Kolmogorov type:

$$u_t = \Delta u + f(t,x,u)u, \qquad x \in D, \tag{47}$$

with $f\colon [0,\infty) \times \bar{D} \times [0,\infty) \to \mathbb{R}$, endowed with the boundary conditions

$$\mathcal{B}u = 0, \qquad x \in \partial D. \tag{48}$$

We assume the following.

(NA2) *For any $M > 0$ the restrictions to $[0,\infty) \times \bar{D} \times [0,M]$ of the function f and its derivatives up to order two belong to $C^{1-,1-,1-}([0,\infty) \times \bar{D} \times [0,M])$.*

(NA3) *There are $P > 0$ and a continuous function $m\colon [P,\infty) \to (0,\infty)$ such that $f(t,x,u) \leq -m(u)$ for any $t \geq 0$, any $x \in \bar{D}$ and any $u \geq P$.*

By the theory in [7], for each $t_0 \geq 0$ and each $u_0 \in X^+$ there is a (classical) solution $u(\cdot;t_0,u_0)$ of (47)+(48), defined on $[t_0,\infty)$, with initial condition $u(t_0;t_0,u_0)(x) = u_0(x)$, such that $u(t;t_0,u_0) \in X$ for all $t \geq t_0$. By the comparison principle, there holds $u(t;t_0,u_0) \in X^+$ for all $t \geq t_0$.

Definition 6.1. Equation (47)+(48) is said to be *forward uniformly persistent* if there is $\eta > 0$ such that for any $u_0 \in X^+ \setminus \{0\}$ there is $\tau(u_0) \geq 0$ with the property that

$$u(t;t_0,u_0) \geq \eta \varphi_{\mathrm{princ}}$$

for all $t_0 \geq 0$ and all $t \geq \tau(u_0) + t_0$.

Note that $u \equiv 0$ is the solution of (47)+(48).

Consider the linearization of (47)+(48) along 0,

$$\begin{cases} v_t = \Delta v + f_0(t,x)v, & x \in D, \\ \mathcal{B}v = 0, & x \in \partial D, \end{cases} \tag{49}$$

where $f_0(t,x) = f(t,x,0)$. We also have that for each $t_0 \geq 0$ and each $v_0 \in X$ there is a (classical) solution $v(\cdot;t_0,v_0)$ of (49), defined on $[t_0,\infty)$, with initial condition $v(t_0;t_0,v_0)(x) = v_0(x)$, such that $v(t;t_0,v_0) \in X$ for all $t \geq t_0$.

It follows from (NA1) and (NA2) that the assumptions (SM1) through (SM3) are satisfied for (49), with $a = (\delta_{ij},0,0,f_0,0)$. Consequently, the theory presented in Sect. 4.2 applies.

Let $[\lambda_{\min},\lambda_{\max}]$ stand for the principal spectrum interval of (49). We then have

Theorem 6.1. *If $\lambda_{\min} > 0$ then (47)+(48) is forward uniformly persistent.*

For any function $g\colon \mathbb{R}\times\bar{D}\times[0,\infty)\to\mathbb{R}$ and any $t\in\mathbb{R}$ we write $g\cdot t(\tau,x,u) := g(\tau+t,x,u)$, $\tau\in\mathbb{R}$, $x\in\bar{D}$, $u\geq 0$.

We extend the function f to a function $\bar{f}\colon \mathbb{R}\times\bar{D}\times[0,\infty)\to\mathbb{R}$ by putting $\bar{f}(t,x,u) := f(0,x,u)$ for $t<0$, $x\in\bar{D}$ and $u\geq 0$.

Put

$$Z := \mathrm{cl}\{\,\bar{f}\cdot t : t\in\mathbb{R}\,\} \tag{50}$$

with the open-compact topology, where the closure is taken in the open-compact topology. By the Ascoli–Arzelà theorem, the set Z is a compact metrizable space. Further, if $g\in Z$ and $t\in\mathbb{R}$ then $g\cdot t =: \zeta_t g\in Z$. Hence $(Z,\{\zeta_t\}_{t\in\mathbb{R}})$ is a compact flow.

Put

$$Z_0 := \bigcap_{s\geq 0}\mathrm{cl}\{\,\bar{f}\cdot t : t\in[s,\infty)\,\}. \tag{51}$$

Z_0, as the ω-limit set of a forward orbit in the compact flow $(Z,\{\zeta_t\}_{t\in\mathbb{R}})$, is nonempty, compact, connected and invariant.

Put

$$\tilde{Z}_0 := \{\,\bar{f}\cdot t : t\geq 0\,\}\cup Z_0 = \mathrm{cl}\{\,\bar{f}\cdot t : t\geq 0\,\}. \tag{52}$$

The set $\tilde{Z}_0$ is a closed, hence compact, subset of Z. Further, it is *forward invariant*: for any $g\in\tilde{Z}_0$ and any $t\geq 0$ there holds $g\cdot t\in\tilde{Z}_0$.

For any $g\in\tilde{Z}_0$, consider the following semilinear second order parabolic equation of Kolmogorov type,

$$\begin{cases} u_t = \Delta u + g(t,x,u)u, & t>0,\ x\in D, \\ \mathcal{B}u = 0, & t>0,\ x\in\partial D. \end{cases} \tag{53}$$

By the theory in [7], the following holds.

Proposition 6.2. *For each $u_0 \in X^+$ and each $g \in \tilde{Z}_0$ there exists a unique solution $u(\cdot;u_0,g)$ of (53), defined on $[0,\infty)$, satisfying the initial condition $u(0;u_0,g) = u_0$, such that $u(t;u_0,g) \in X^+$ for all $t \geq 0$. That solution is classical. Further, the mapping*

$$[\,[0,\infty) \times X^+ \times \tilde{Z}_0 \ni (t,u_0,g) \mapsto u(t;u_0,g) \in X\,]$$

is continuous.

Observe that $u(\cdot+t_0;t_0,u_0) = u(\cdot;u_0,\bar{f}_0 \cdot t_0)$ for $t_0 \geq 0$.

Let Y_0 and $\tilde{Y}_0$ be defined as follows,

$$Y_0 := \{\, g_0 : \exists g \in Z_0 \text{ such that } g_0(t,x) = g(t,x,0),\ t \in \mathbb{R},\ x \in \bar{D}\,\},$$

and

$$\tilde{Y}_0 := \{\, g_0 : \exists g \in \tilde{Z}_0 \text{ such that } g_0(t,x) = g(t,x,0),\ t \in \mathbb{R},\ x \in \bar{D}\,\}.$$

The sets Y_0 and $\tilde{Y}_0$ are considered endowed with the open-compact topology. As the images of the compact sets Z_0 and $\tilde{Z}_0$, respectively, under restriction, they are compact.

For $t_0 \in \mathbb{R}$ and $g_0 \in \tilde{Y}_0$ consider

$$\begin{cases} v_t = \Delta v + g_0(t,x)v, & t > t_0,\ x \in D, \\ \mathcal{B}v = 0, & t > t_0,\ x \in \partial D. \end{cases} \tag{54}$$

By the theory in [7], for any $v_0 \in X$, $t_0 \in \mathbb{R}$ and $g_0 \in \tilde{Y}_0$, (54) has a unique (classical) solution $v(t;t_0,v_0,g_0)$, defined on $[t_0,\infty)$, with $v(t_0;t_0,v_0,g_0) = v_0$, such that $v(t;t_0,v_0,g_0) \in X$ for all $t \geq t_0$.

Observe that for any $g \in \tilde{Z}_0$, $u \equiv 0$ is the solution of (53) and (54) with $g_0(t,x) = g(t,x,0)$ is the linearization of (53) along $u \equiv 0$. Put $U_{g_0}(t,t_0)v_0 := v(t;t_0,v_0,g_0)$. If $g_0 = \bar{f}_0 \cdot t_0$ and $t_0 \geq 0$, we write $U_{g_0}(t,t_0)$ as $U(t,t_0)$.

Lemma 6.3. *For each $t > 0$ there holds*

$$\frac{\|u(t;\rho u_0,g) - \rho U_{g_0}(t,0)u_0\|_X}{\rho} \to 0 \quad \text{as } \rho \to 0^+$$

uniformly in $g \in \tilde{Z}_0$ and $u_0 \in X^+$ with $\|u_0\|_X = 1$, where $g_0(t,x) = g(t,x,0)$.

Proof. It follows from [18, Theorem 7.1.5]. □

Lemma 6.4. *Assume that $\lambda_{\min} > 0$. Then there is $T > 0$ such that*

$$U_{g_0}(T,0)\varphi_{\text{princ}} \gg 2\varphi_{\text{princ}} \quad \text{for all} \quad g_0 \in Y_0.$$

Proof. Let $[\tilde{\lambda}_{\min},\tilde{\lambda}_{\max}]$ be the principal spectrum of (54) over Y_0. By Theorem 4.3, $\lambda_{\min} = \tilde{\lambda}_{\min}$ and hence $\tilde{\lambda}_{\min} > 0$. The lemma then follows from [18, Lemma 7.1.16]. □

Proof of Theorem 6.1. Let $T > 0$ be as in Lemma 6.4. As the mapping $[Y_0 \ni g_0 \mapsto U_{g_0}(T,0)\varphi_{\rm princ} \in X]$ is continuous and Y_0 is compact, the set $\{\, U_{g_0}(T,0)\varphi_{\rm princ} - 2\varphi_{\rm princ} : g_0 \in Y_0 \,\}$ is compact, too. Further, this set is, by Lemma 6.4, contained in the open set X^{++}. Therefore $\varepsilon_0 := \inf\{\, \|(U_{g_0}(T,0)\varphi_{\rm princ} - 2\varphi_{\rm princ}) - v\|_X : g_0 \in Y_0,\ v \in \partial X^+ \,\}$ is positive. By linearity,

$$\inf\{\, \|(rU_{g_0}(T,0)\varphi_{\rm princ} - 2r\varphi_{\rm princ}) - v\|_X : g_0 \in Y_0,\ v \in \partial X^+ \,\} = r\varepsilon_0 \tag{55}$$

for any $r > 0$.

It follows from Lemma 6.3 that there is $r_0 > 0$ such that

$$\|u(T+t;t,r\varphi_{\rm princ}) - rU(T+t,t)\varphi_{\rm princ}\|_X \le \frac{r\varepsilon_0}{3}$$

for all $t \ge 0$ and all $r \in (0,r_0]$.

We claim that there is $T_1 \ge 0$ such that for each $t \ge T_1$ one can find $g \in Z_0$ such that $\|U(T+t,t)\varphi_{\rm princ} - U_{g_0}(T,0)\varphi_{\rm princ}\|_X < \varepsilon_0/3$. Indeed, for each $g \in Z_0$ there is $\delta = \delta(g) > 0$ such that for any $h \in Z$, if $d(g,h) < \delta$ then $\|U_{h_0}(T,0)\varphi_{\rm princ} - U_{g_0}(T,0)\varphi_{\rm princ}\|_X < \varepsilon_0/3$, where $d(\cdot,\cdot)$ stands for the metric in Z. Since Z_0 is compact, there are finitely many $g^{(1)}, \ldots, g^{(n)} \in Z_0$ such that the union of the open balls (in Z) with center $g^{(k)}$ and radius $\delta(g^{(k)})$, $k = 1,\ldots,n$, covers Z_0. Denote this union by B. It suffices now to find $T_1 \ge 0$ such that $\bar{f}\cdot t \in B$ for all $t \ge T_1$, and the existence of such T_1 follows from the fact that Z_0 is, by definition, the ω-limit set (in the compact flow $(Z,\{\zeta_t\})$) of $\bar{f}$.

Fix for the moment $t \ge T_1$, and let $g \in Z_0$ be such that $\|U(T+t,t)\varphi_{\rm princ} - U_{g_0}(T,0)\varphi_{\rm princ}\|_X < \varepsilon_0/3$. We estimate

$$\begin{aligned}
&\|(u(T+t;t,r\varphi_{\rm princ}) - 2r\varphi_{\rm princ}) - (rU_{g_0}(T,0)\varphi_{\rm princ} - 2r\varphi_{\rm princ})\|_X \\
&\quad = \|u(T+t;t,r\varphi_{\rm princ}) - rU_{g_0}(T,0)\varphi_{\rm princ}\|_X \\
&\quad \le \|u(T+t;t,r\varphi_{\rm princ}) - rU(T+t,t)\varphi_{\rm princ}\|_X \\
&\qquad + \|rU(T+t,t)\varphi_{\rm princ} - rU_{g_0}(T,0)\varphi_{\rm princ}\|_X \\
&\quad < \frac{r\varepsilon_0}{3} + \frac{r\varepsilon_0}{3}
\end{aligned}$$

for any $r \in (0,r_0]$. It follows from (55) that $u(t+T;t,r\varphi_{\rm princ}) - 2r\varphi_{\rm princ} \in X^{++}$, that is, $u(t+T;t,r\varphi_{\rm princ}) \gg 2r\varphi_{\rm princ}$, for any $r \in (0,r_0]$ and any $t \ge T_1$.

Fix a nonzero $u_0 \in X^+$. By the comparison principle for parabolic equations, $u(t;t_0,u_0) \gg 0$, that is, $u(t;t_0,u_0)$ belongs to the open subset X^{++} of X, for any $t > t_0$. Since $[T_1, T_1+T] \times \{u_0\} \times \tilde{Z}_0$ is compact, it follows from Proposition 6.2 that the set $\{\, u(t;u_0,g) : t \in [T_1+1, T_1+T+1],\ g \in \tilde{Z}_0 \,\}$ $(\subset X^{++})$ is compact. Consequently, the set $\{\, u(t+t_0;t_0,u_0) : t_0 \ge 0,\ t \in [T_1+1, T_1+T+1] \,\}$ has compact closure contained in X^{++}. By arguments as in the proof of [18, Theorem 7.1.6], there is $\tilde{r} > 0$ such that $u(t+t_0;t_0,u_0) \ge \tilde{r}\varphi_{\rm princ}$ for all $t_0 \ge 0$ and $t \in [T_1+1, T_1+T+1]$.

Assume $\tilde{r} \geq r_0$. Then for each $t \in [T_1+1, T_1+T+1]$ and each $t_0 \geq 0$ we have $u(t+T+t_0;t_0,u_0) = u(t+T+t_0;t+t_0,u(t+t_0;t_0,u_0)) \gg u(t+T+t_0;t+t_0,r_0\varphi_{\mathrm{princ}}) \gg 2r_0\varphi_{\mathrm{princ}}$. By induction, we have $u(t+nT+t_0;t_0,u_0) \gg 2r_0\varphi_{\mathrm{princ}}$ for all $n = 1,2,\dots$. Therefore we can take $\tau(u_0) = T_1+T+1$.

Assume $\tilde{r} < r_0$. Then for each $t \in [T_1+1, T_1+T+1]$ and each $t_0 \geq 0$ such that $u(t+t_0;t_0,u_0) \geq r\varphi_{\mathrm{princ}}$ for some $r < r_0$ we have $u(t+T+t_0;t_0,u_0) = u(t+T+t_0;t+t_0,u(t+t_0;t_0,u_0)) \geq u(t+T+t_0;t+t_0,r\varphi_{\mathrm{princ}}) \gg 2r\varphi_{\mathrm{princ}}$. Repeating this procedure sufficiently many times we obtain that $u(t+nT+t_0;t_0,u_0) \gg 2^n r\varphi_{\mathrm{princ}}$ as long as $2^{n-1}r \leq r_0$. After some calculation we conclude that we can take $\tau(u_0) = \left(\lfloor \frac{\ln r_0 - \ln \tilde{r}}{\ln 2} \rfloor + 2\right)T + T_1 + 1$.

In both cases, $\eta = 2r_0$. □

We finish the section by giving a sufficient condition for the assumptions in Theorem 6.1 to hold.

A function $\hat{f}_0 \in C(\bar{D})$ is called a *time-averaged function* of f_0 if there are subsequences $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$, with $0 < s_n < t_n$ for all $n = 1,2,\dots$, $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty}(t_n - s_n) \to \infty$, such that

$$\hat{f}_0(x) = \lim_{n\to\infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} f(t,x,0)\,dt$$

uniformly for $x \in \bar{D}$.

Let $\hat{Y} := \{\hat{f}_0 : \hat{f}_0$ is a time-averaged function of $f_0\}$. For a given $\hat{f}_0 \in \hat{Y}$, denote by $\lambda_{\mathrm{princ}}(\hat{f}_0)$ the principal eigenvalue of

$$\begin{cases} \Delta u + \hat{f}_0(x)u = \lambda u, & x \in D, \\ \mathcal{B}u = 0, & x \in \partial D. \end{cases} \tag{56}$$

Theorem 6.5. *If* $\lambda_{\mathrm{princ}}(\hat{f}_0) > 0$ *for any* $\hat{f}_0 \in \hat{Y}$*, then* (47)+(48) *is forward uniformly persistent.*

Proof. Observe that the standing assumptions in Sect. 5.2 hold for (49). By Theorem 5.3(1), there is $\hat{f}_0 \in \hat{Y}$ such that $\lambda_{\min} \geq \lambda_{\mathrm{princ}}(\hat{f}_0)$. An application of Theorem 6.1 concludes the proof. □

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A Dynamical Systems Approach to Traveling Wave Solutions for Liquid/Vapor Phase Transition

Haitao Fan and Xiao-Biao Lin

Dedicated to Professor George Sell's 70th birthday

Abstract We study the existence of liquefaction and evaporation waves by the methods derived from dynamical systems theory. A traveling wave solution is a heteroclinic orbit with the wave speed as a parameter. We give sufficient and necessary conditions for the existence of such heteroclinic orbit. After analyzing the local unstable and stable manifolds of two equilibrium points, we show that there exists at least one orbit connecting the local unstable manifold of one equilibrium point to the local stable manifold of another equilibrium point. The method is known as the shooting method in the literature.

Mathematics Subject Classification (2010): Primary; Secondary

1 Introduction

Dynamic flows involving liquid/vapor phase transition is an important phenomenon occurring in many engineering processes. For retrograde fluids, i.e. fluids with high specific heat capacities, such flows can be approximated by assuming that the temperature is constant.

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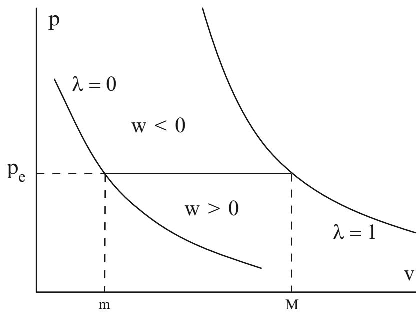

Fig. 1 The pressure function $p = p(\lambda, v)$ for some fixed λ

The one-dimensional case of the system describing such flows in Lagrangian coordinates is

$$\begin{aligned} &v_t - u_x = 0,\\ &u_t + p(\lambda, v)_x = \varepsilon u_{xx},\\ &\lambda_t = \frac{1}{\gamma} w(\lambda, v) + \beta \lambda_{xx}, \end{aligned} \tag{1}$$

where v is the specific volume, u the velocity of the fluid, λ the weight portion of vapor in the liquid/vapor mixture, ε the viscosity, β the diffusion coefficient and $\gamma > 0$ the typical reaction time. The pressure $p(\lambda, v)$ in (1) is a smooth function that satisfies

$$p_v < 0 < p_\lambda, \quad p_{vv} > 0. \tag{2}$$

Although $p_{vv} > 0$ was assumed in many previous papers, we do not use it in the proof of the existence of liquefaction and evaporation waves (Theorems 2.4 and 2.6).

Figure 1 shows the graph of a typical pressure function, where p_e is the equilibrium pressure at which liquid and vapor can coexist and m, M are the Maxwell points.

The function $w(\lambda, v)$ represents the rate of vapor initiation and growth:

$$w(v, \lambda) = (p(\lambda, v) - p_e)\lambda(\lambda - 1). \tag{3}$$

A traveling wave of (1.1) is a solution of the form $(u, v, \lambda)(s)$, where $s = \frac{x-ct}{\varepsilon}$ and c is the speed of the wave. With $u' = du/ds$, we have

$$\begin{aligned} &-cv' - u' = 0,\\ &-cu' + p' = u'',\\ &-c\lambda' = aw(\lambda, v) + b\lambda'',\\ &(u, v, \lambda)(\pm\infty) = (u_\pm, v_\pm, \lambda_\pm), \end{aligned} \tag{4}$$

where $a = \varepsilon/\gamma$, $b = \beta/\varepsilon$. Because λ is the weight portion of the vapor in the liquid/vapor mixture, we only admit solutions with $0 \le \lambda \le 1$.

From the third equation of (4), equilibrium points $(\lambda_\pm, v_\pm)$ must satisfy $w(\lambda, v) = 0$, which has three branches of solutions: $\lambda = 0$ or $\lambda = 1$ or $p(\lambda, v) = p_e$.

From the first two equations of (4), $c^2 v' + p' = u'' = -cv''$. Integrating from $-\infty$ to s we have:

$$-cdv/ds = c^2(v - v_-) + (p - p_-).$$

Let $s = b\xi$ and $u' = du/d\xi$. We have

$$\begin{aligned} \lambda'' &= -c\lambda' - abw(\lambda, v), \\ -\frac{c}{b}v' &= c^2(v - v_-) + (p(\lambda, v) - p_-). \end{aligned} \tag{5}$$

If the traveling wave connects $E_\pm$ with $(\lambda, v) = (\lambda_\pm, v_\pm)$, then

$$c^2(v_+ - v_-) + p(v_+, \lambda_+) - p(\lambda_-, v_-) = 0. \tag{6}$$

Definition 1.1. A liquefaction wave is a solution of (1.4) with

$$\lambda_- = 0,\ 0 < \lambda_+ \le 1,$$
$$p(\lambda_\pm, v_\pm) \ge p_e,\ c^2 + p_v(\lambda_\pm, v_\pm) < 0,$$

while an evaporation wave is that with

$$\lambda_- = 1,\ 0 \le \lambda_+ < 1,$$
$$p(\lambda_\pm, v_\pm) \le p_e,\ c^2 + p_v(\lambda_\pm, v_\pm) < 0.$$

A collapsing wave is a solution of (1.4) with

$$0 \le \lambda_- < 1,\ \ \lambda_+ = 1,$$
$$p(\lambda_\pm, v_\pm) \ge p_e,\ \ c^2 + p_v(\lambda_\pm, v_\pm) > 0,$$

while an explosion wave is that with

$$0 < \lambda_- \le 1,\ \ \lambda_+ = 0,$$
$$p(\lambda_\pm, v_\pm) \le p_e,\ \ c^2 + p_v(\lambda_\pm, v_\pm) > 0.$$

Recall that $\sqrt{-p_v}$ is the speed of sound. The definitions can be summarized in the following table:

	$p \ge p_e$	$p \le p_e$
subsonic $c^2 + p_v < 0$	liquefaction	evaporation
supersonic $c^2 + p_v > 0$	collapsing	explosion

Shearer [12–14], Fan [4] and Slemrod [15] studied the Liquid/vapor phase transition through the p-system of conservation laws of hyperbolic-elliptic mixed type. Fan [5–7] proved the existence of liquefaction and evaporation waves. He also studied the stability of a simplified system consisting of a system of two conservation laws and a KPP equation. See also [1–3, 8] for further discussions of the model.

The following methods were used in [5, 7] in proving the existence of traveling waves:

(1) The Leray–Schauder degree theory.
(2) The theory of monotone systems of parabolic PDEs.

A common feature to methods (1) and (2) is the adding of a small diffusion term ηv_{xx} to the system,

$$v_t = \eta v_{xx} + c^2(v - v_-) + p - p_-,$$
$$\lambda_t = b\lambda_{xx} + a(p(\lambda, v) - p_e)\lambda(\lambda - 1).$$

First one finds traveling waves for the system with small $\eta > 0$. Then one shows that there exists a sequence $\eta_n \to 0$ such that the corresponding solutions $(\lambda^{\eta_n}, v^{\eta_n}) \to (\lambda, v)$. The limit is a traveling wave corresponding to $\eta = 0$.

We briefly describe the use of the Leray–Schauder degree theorem to our system. Consider the modified system:

$$\eta v'' + cv' = -\theta(c^2(v - v_-) + p - p_-),$$
$$b\lambda'' + c\lambda' = -\theta\, a w(v, \lambda), \ -L < \xi < M,$$
$$(v, \lambda)(-L) = (v_-, \lambda_-), \quad (v, \lambda)(M) = (\bar{v}_+, \bar{\lambda}_+).$$

By choosing $(\bar{v}_+, \bar{\lambda}_+)$ properly, once can show that there is a strictly monotone solution for all $\theta \in [0, 1]$.

Write the system as an integral equation $T(x, \theta) = x$. Then $T : \bar{\Omega} \times [0, 1] \to X$ is a compact operator in a real normed space. Moreover the solution exists if $\theta = 0$. From the Leray–Schauder degree, if we assume:

(i) $T(x, \theta) \neq x$ for $x \in \partial\Omega,\ \theta \in [0, 1]$.
(ii) The Leray–Schauder degree $D_I(T(\cdot, 0) - I, \Omega) \neq 0$,

Then for any $0 \leq \theta \leq 1$, $T(x, \theta) = x$ has at least one solution in Ω.

We then find convergent subsequences of monotone solutions such that

(i) $L(n) \to \infty, M(n) \to \infty$
(ii) $\bar{v}_n \to v_+, \bar{\lambda}_n \to \lambda_+$
(iii) $\eta_n \to 0$.

The limit of solutions is the traveling wave solution to the system with $\eta = 0$. Next, we briefly describe the use of the "Method for Monotone Systems of PDES" to our system. Consider

$$v_t = \eta v_{xx} + c^2(v - v_-) + p - p_-,$$
$$\lambda_t = b\lambda_{xx} + a(p - p_e)\lambda(\lambda - 1).$$

We can rewrite the system as

$$U_t = AU_{xx} + F(U,c),$$

where

$$F(U,c) = \begin{pmatrix} c^2(v - v_-) + p - p_- \\ a(p - p_e)\lambda(\lambda - 1) \end{pmatrix},$$
$$\nabla F = \begin{pmatrix} c^2 + p_v & p_\lambda \\ aP_v\lambda(\lambda - 1) & a(p - p_e)(2\lambda - 1) + aP_\lambda\lambda(\lambda - 1) \end{pmatrix}.$$

Under the sufficient conditions, we can verify that

(1) The system is monotone: off-diagonal terms of ∇F are positive.
(2) The eigenvalues of ∇F at U_- are negative.
(3) The wave speed c is sufficiently large.
(4) Other conditions for a monotone system are satisfied.

Then there exists a monotone solution U for small η. By letting $\eta \to 0$ we find the limit of solutions which corresponds to solutions for the system with $\eta = 0$.

Using a geometric/dynamical system's method (shooting method), Fan and Lin simplified the proof of the existence of evaporation and liquefaction waves obtained in [5,7]. We also rigorously proved the existence of collapsing and explosion waves that were only verified numerically before. Define $h(\lambda, v) := c^2(v - v_-) + p(\lambda, v) - p_-$. Then

$$\mathcal{C} := \{(\lambda, v) : h(\lambda, v) = 0\}$$

is the isocline for v due to (5). Based on (6),

$$h(\lambda, v) := c^2(v - v_\pm) + p(\lambda, v) - p_\pm.$$

In this paper we summarize our results obtained by the shooting method from 2005 to 2008 as follows:

Theorem 1.1. *(1) The sufficient and necessary conditions for the existence of collapsing waves are:*

$$c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_\pm, v_\pm) \geq 0.$$

(2) The sufficient conditions for the existence of explosion waves are:

$$c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_-, v_-) \geq 0.$$

The necessary conditions for the existence of explosion waves are:

$$c^2 \geq 4ab|p(\lambda_+,v_+) - p_e|, \quad c^2 + p_v(\lambda_+,v_+) \geq 0.$$

(3) *If* $\lambda_+ = 0,1$, *then the sufficient conditions for the existence of liquefaction or evaporation waves are:*

$$c^2 + p_v(\lambda,v) < 0, \text{ if } v_- \leq v \leq v_+, \; \lambda = 0,1,$$

and

$$c^2 \geq 4ab|p(\lambda_-,v_-) - p_e|.$$

(4) *If* $p_+ = p_e, 0 < \lambda_+ < 1$, *then the sufficient conditions for the existence of liquefaction or evaporation waves are:*

$$c^2 \geq 4ab|p(\lambda_-,v_-) - p_e|,$$
$$c^2 + p_v(\lambda,v) < 0, \text{ if } \lambda = 0 \leq \lambda \leq \lambda_+, \; v_- \leq v \leq v_+,$$

and along the isocline $\mathcal{C}$ *for* v,

$$\sup_{\lambda} \left\{ \frac{ab p_\lambda(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|} \right\} < 1.$$

As proofs of the existence of collapsing and explosion waves were presented in a separate paper [9], in the rest of this paper we will study the existence of liquefaction and evaporation waves only. The existence of liquefaction waves for $\lambda_- = 0$, $\lambda_+ = 1$ will be proved in Theorem 2.4 while the existence of liquefaction waves for $\lambda_- = 0$, $0 < \lambda_+ < 1$, $p_+ = p_e$ will be proved in Theorem 2.6. Similar proofs apply to the evaporation waves and will not be presented in this paper.

For liquefaction and evaporation waves, it will be shown in Lemma 2.1 that the wave speed c is positive. Therefore, from (6),

$$c = \sqrt{-\frac{p(\lambda_+,v_+) - p(\lambda_-,v_-)}{v_+ - v_-}}. \tag{7}$$

In [5, 7], Fan proved that liquefaction and evaporation waves exist if the wave speed $c > 0$ satisfies $c \geq 2\sqrt{ab|p(\lambda_-,v_-) - p_e|}$. On the other hand, if the speeds satisfy $c \leq 2\sqrt{ab|p(\lambda_+,v_+) - p_e|}$, then there is no liquefaction or evaporation waves.

The locations of $(v_\pm, \lambda_\pm)$ for both waves are depicted in Fig. 2.

Recall that the pressure $p(\lambda, v)$ satisfies

$$p_v < 0 < p_\lambda,$$

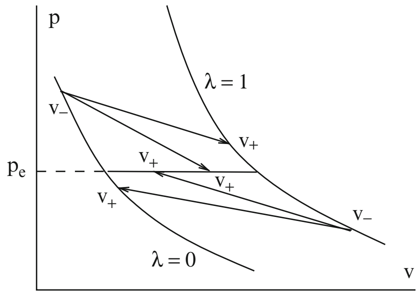

Fig. 2 The points $(v_\pm, \lambda_\pm)$ of liquefaction waves ($p_\pm \geq p_e$) and evaporation waves ($p_\pm \leq p_e$). The *arrows* point to the fronts of the waves

and the growth rate w is

$$w(\lambda, v) = (p - p_e)\lambda(\lambda - 1).$$

Since $p_\lambda > 0$, the function $p = p(\lambda, v)$ can be solved for $\lambda = \lambda^*(v, p)$. For each $v = v_0$, with $v_m < v_0 < v_M$, $w = w(\lambda, v_0)$ has three zeros: $\lambda = 0$, $\lambda_e(v_0) = \lambda^*(v_0, p_e)$, $\lambda = 1$. However, for $v_0 < v_m$ or $v_0 > v_M$, $w = w(\lambda, v_0)$ has two zeros: $\lambda = 0$, $\lambda = 1$.

Let $\mathcal{P} \subset \mathbb{R}^3$ be an open set bounded by finitely many smooth surfaces.

Definition 1.2. For each $P \in \mathcal{P}$ such that the $\Phi(\xi_0, P) \in \partial\mathcal{P}$ for some $\xi_0 > 0$, there exists the first touch time ξ_1 such that

$$\Phi(\xi_1, P) \in \partial\mathcal{P}, \text{ while } \Phi(\xi, P) \in \mathcal{P} \text{ for } 0 \leq \xi < \xi_1.$$

If the first touch time for P exists, then define the first touch point as $B(P) := \Phi(\xi_1, P)$.

The following lemma is related to the Wazewski's principle [10, 11]. It is not as general but works well on our system.

Lemma 1.2. *Assume that there exit two mutually disjoint open subsets of $\partial\mathcal{P}$: S_1 and S_2 such that,*

(1) For any $P \in S_j$, $j = 1, 2$, there exists a small $\varepsilon > 0$ such that $\Phi(\xi, P) \in \mathcal{P}$ for $-\varepsilon < \xi < 0$. Moreover, the flow $\Phi(\xi, \cdot)$ is transverse to S_1 or S_2.
(2) For any $P \in \mathcal{P}$ such that $\Phi(\xi, P) \in \partial\mathcal{P}$ for some $\xi > 0$, we have $B(P) \in S_1 \cup S_2$.
(3) There exists a smooth curve segment $\overline{P_1P_2}$ in $\mathcal{P}$ such that $B(P_1) \in S_1$, $B(P_2) \in S_2$.

Under these conditions, there exits a $P_0 \in \overline{P_1P_2}$ such that $\Phi(\xi, P_0)$ remains in $\mathcal{P}$ for all $\xi > 0$.

The shooting method alone does not provide information on the uniqueness of the traveling waves for each fixed wave speed c. In a work-in-progress paper by Fan and Lin, numerical computation combined with the shooting method has been performed on a similar system. For a given wave speed, these results suggest that each type of traveling wave for liquid/vapor phase transition may be unique.

2 Existence of Liquefaction and Evaporation Waves

In this section we present the proof of the existence of liquefaction waves for the case $\{\lambda_- = 0\} \to \{\lambda_+ = 1\}$, and the case $\{\lambda_- = 0\} \to \{p_+ = p_e\}$. The same proof applies to the evaporation waves with some minor changes.

System (5) can be written as a first order system of three variables (λ, μ, v):

$$\begin{aligned} \lambda' &= \mu, \\ \mu' &= -c\mu - abw(\lambda, v), \\ -\frac{c}{b}v' &= c^2(v - v_-) + (p(\lambda, v) - p_-). \end{aligned} \tag{8}$$

We look for a heteroclinic solution of (8) connecting the equilibrium points $E_\pm := \{(\lambda_\pm, \mu_\pm, v_\pm)\}$.

Equilibrium states are the zeros of the right hand side of (8).

$$\begin{aligned} \mu &= 0, \\ w(\lambda, v) &= 0, \\ c^2(v - v_-) + p(\lambda, v) - p_- &= 0. \end{aligned} \tag{9}$$

The solutions of $w = 0$ form three branches: $\lambda = 0,\ 1$ and $p(\lambda, v) = p_e$. The graph of (9) with a given c is a straight line in the (v, p) plane, see Fig. 2. However, the graph of (9) in the (λ, v) plane is the isocline $\mathcal{C}$ for v.

For each v with $v_m < v < v_M$, by solving $p(\lambda, v) = p_e$ for λ, we have

$$\lambda = \lambda_e(v).$$

The equilibrium E_- is on $\lambda = 0$ with $p_- > p_e$ or on $\lambda = 1$ with $p_- < p_e$. The equilibrium E_+ is on the line $\lambda = 1, p_+ > p_e$ or on $p = p_e, 0 < \lambda < 1$ (liquefaction wave). The equilibrium E_+ is on $\lambda = 0, p < p_e$ or on $p = p_e, 0 < \lambda < 1$ (evaporation wave).

Let $p_+ = p(\lambda_+, v_+)$. The wave speed c and $v_\pm$ are now related by (6): $c^2(v_+ - v_-) + (p_+ - p_-) = 0$.

From (6), for the liquefaction wave $p_+ < p_-$, we must have $v_+ > v_-$; while for the evaporation wave $p_+ > p_-$, we must have $v_+ < v_-$.

2.1 Eigenvalues and Eigenvectors at Equilibrium Points

In this section, we first show that if $c > 0$, then the equilibrium E_- corresponding to $\lambda = 0, 1$ is a saddle with exactly two positive eigenvalues and one negative eigenvalue, while the equilibrium E_+ has one positive eigenvalues and two negative eigenvalues. The traveling wave solution we look for is a heteroclinic solution connecting saddle to saddle. Moreover, as $\xi \to \pm\infty$, the orbit of the traveling wave starts at the two dimensional local unstable manifold $W^u_{loc}(E_-)$ and ends at the two dimensional local stable manifold $W^s_{loc}(E_+)$.

The linear variational system at equilibrium point is

$$\begin{pmatrix}\Lambda\\ M\\ V\end{pmatrix}' = A\begin{pmatrix}\Lambda\\ M\\ V\end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 0 & 1 & 0\\ -abw_\lambda & -c & -abw_v\\ -\frac{b}{c}p_\lambda & 0 & -bc-\frac{b}{c}p_v\end{pmatrix}.$$

Eigenvalues r are determined by

$$\det(rI-A) = \begin{vmatrix} r & -1 & 0\\ abw_\lambda & r+c & abw_v\\ \frac{b}{c}p_\lambda & 0 & r+\frac{b}{c}(p_v+c^2)\end{vmatrix} = 0.$$

We first study eigenvalues at the equilibrium $E_\pm$ with $\lambda = 0$ or 1,

$$w_\lambda(\lambda,v) = (2\lambda-1)(p-p_e), \quad w_v(\lambda,v) = 0,$$

$$\det(rI-A) = (r^2+cr+w_\lambda)(r+\frac{b}{c}(p_v+c^2)).$$

Eigenvalues at $\lambda = 0,1$ are

$$r_1 = -c/2 - \sqrt{(c/2)^2 - ab(2\lambda-1)(p(\lambda,v)-p_e)},$$

$$r_2 = -c/2 + \sqrt{(c/2)^2 - ab(2\lambda-1)(p(\lambda,v)-p_e)},$$

$$r_3 = -\frac{b}{c}(p_v(\lambda,v)+c^2).$$

Lemma 2.1. *(1) Assume that* $c > 0$*. At* E_-*, assume that* $c^2 + p_v(\lambda,v) < 0$ *and* $(2\lambda-1)(p-p_e) < 0$*. Then the equilibrium* E_- *has two positive eigenvalues and one negative eigenvalue. At* E_+*, assume that* $c^2 \geq 4ab|p(\lambda,v)-p_e|$ *and* $c^2 + p_v(\lambda,v) < 0$*. Then if* E_+ *is on the line* $\lambda = 0,1$*, it has two negative eigenvalues and one positive eigenvalue.*

(2) Assume that $c < 0$*. At* E_-*, assume that* $c^2 + p_v(\lambda,v) < 0$ *and* $(2\lambda-1)(p-p_e) < 0$*. Then the equilibrium* E_- *has two negative eigenvalues and one positive eigenvalue. At* E_+*, assume that* $c^2 \geq 4ab|p(\lambda,v)-p_e|$ *and* $c^2 + p_v(\lambda,v) < 0$*. Then if* E_+ *is on the line* $\lambda = 0,1$*, it has two positive eigenvalues and one negative eigenvalue.*

Proof. Proof of (1): We always have $r_3 > 0$ for $\lambda = 0$ or 1. If $\lambda = 0,\ p > p_e$ or if $\lambda = 1,\ p < p_e$, we have

$$ab(2\lambda-1)(p(\lambda,v)-p_e) < 0,$$

and hence $r_1 < 0$ and $r_2 > 0$. Thus E_- has two unstable eigenvalues and one stable eigenvalue. If $\lambda = 1,\ p > p_e$ or if $\lambda = 0,\ p < p_e$, we use $c^2 \geq 4ab|p(\lambda,v) - p_e|$ to show r_1, r_2 are real and $r_1, r_2 < 0$. Thus E_+ has two stable eigenvalues and one unstable eigenvalue.

The proof of (2) is completely similar and shall be omitted. □

Since in the case (2), a heteroclinic connection from E_- to E_+ usually does not happen, we shall assume $c > 0$.

2.2 Existence of Liquefaction Waves for $\lambda_- = 0,\ \lambda_+ = 1$

In this section, we consider the liquefaction wave connecting $\lambda_- = 0$ to $\lambda_+ = 1$. The liquefaction wave that connects $\lambda_- = 0$ to $p_+ = p_e$ shall be constructed later. Since liquefaction and evaporation waves are subsonic waves, cf. Definition 1.1, we assume that the waves satisfy the following assumption in this section:

(H1) $c^2 + p_v(\lambda,v) < 0$, if $v_- \leq v \leq v_+$ and $\lambda = 0,1$.

The traveling waves satisfy the following system of equations:

$$\lambda' = \mu, \quad \mu' = -c\mu - abw(\lambda,v), \tag{10}$$

$$\frac{-c}{b}v' = c^2(v - v_-) + (p(\lambda,v) - p_-). \tag{11}$$

As from Lemma 2.1, we assume that $c > 0$.

The isocline for v means $\mathcal{C} := \{(\lambda,v) : v' = 0\}$. Clearly $(\lambda,v) \in \mathcal{C}$ if $h(\lambda,v) = 0$. It is easy to see that on the two equilibrium points, $(\lambda_\pm, v_\pm) \in \mathcal{C}$.

Due to (H1), on the line $\lambda_- = 0$ we have $c^2 + p_v < 0$. If $v > v_-$, then $h(0,v) < 0$. Therefore $v' > 0$ if $v_- < v < v_+$ and $\lambda = 0$. Similarly, due to (H1) again, if $\lambda = 1$, we can show that $v' < 0$ if $v_- < v < v_+$. Now for each $v \in (v_-, v_+)$, there exists a unique $\lambda \in (0, \lambda_+)$ such that $h(\lambda,v) = 0$, denoted by

$$\lambda = \lambda_c(v).$$

Due to the fact $p_\lambda > 0$, $\lambda_c(v)$ is a smooth function of $v \in (v_-, v_+)$.

In general $\lambda_c(v)$ may not be a monotone function as depicted in Fig. 3.

The isocline for v divides the rectangle $(\lambda_-, \lambda_+) \times (v_-, v_+)$ into two parts. Let

$$\mathcal{N} := \{(\lambda,v) : v_- < v < v_+,\ 0 < \lambda < \lambda_c(v)\}.$$

If $(\lambda,v) \in \mathcal{N}$, then $v'(\xi) > 0$. Let EF be the curve on which $p = p_e$ and $v_- < v < v_+, \lambda_- < \lambda < \lambda_+$, see Fig. 3. Then $EF \subset \mathcal{N}$ where $v' > 0$. This can be shown as follows. Since on EF, $p = p_e < p_+$ and $v < v_+$, we have

$$h(\lambda,v) = c^2(v - v_+) + (p - p_+) < 0.$$

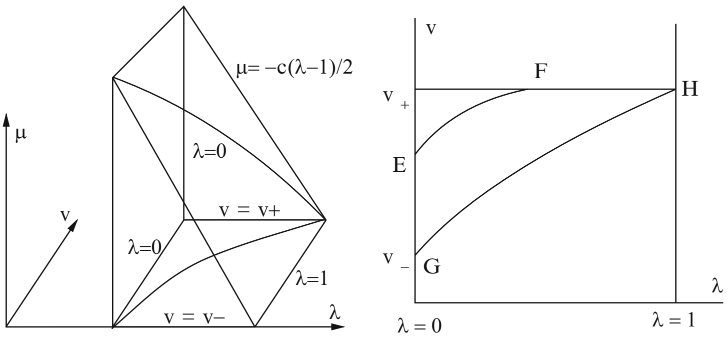

Fig. 3 The pentahedron and the top view of its base $\mathcal{F}_b$. On the curve EF, $p(\lambda,v) = p_e$. On the isocline GH, $v' = 0$

Consider a pentahedron shaped solid $\mathcal{P}$ in (λ,μ,v) space bounded by the five surfaces:

$$\text{Left side } \mathcal{F}_\ell := \{\lambda = 0\};$$
$$\text{Back side } \mathcal{F}_k := \{v = v_+,\ 0 < \lambda < 1,\ 0 < \mu < -c(\lambda-1)/2\};$$
$$\text{Front side } \mathcal{F}_f := \{v_- < v < v_+,\ \lambda = \lambda_c(v),\ 0 < \mu < -c(\lambda-1)/2\};$$
$$\text{Slant side } \mathcal{F}_s := \{c(\lambda-1)/2 + \mu = 0,\ 0 < \lambda < \lambda_c(v), v_- < v < v_+\};$$
$$\text{Bottom side } \mathcal{F}_b := \{\mu = 0,\ 0 < \lambda < 1\}.$$

The bottom side is further divided into $\mathcal{F}_b = \mathcal{F}_{b1} \cup \mathcal{F}_{b2}$, with

$$\mathcal{F}_{b1} := \mathcal{F}_b \cap \{p(\lambda,v) \geq p_e\},$$
$$\mathcal{F}_{b2} := \mathcal{F}_b \cap \{p(\lambda,v) < p_e\}.$$

For each interior point P of $\mathcal{P}$, let $B(P)$ be the first touch point on $\partial\mathcal{P}$ as in Definition 1.2.

(1) Since $d\lambda/d\xi = \mu > 0$ inside $\mathcal{P}$, $B(P) \notin \mathcal{F}_\ell$.
(2) On $\mathcal{F}_k$, we have $v' = -\frac{b}{c}(p(v_+,\lambda) - p_+)$. Since $p_\lambda > 0$ and $\lambda < 1$, we have $p(v_+,\lambda) < p(v_+,1) = p_+$. Thus, $v' > 0$. It is possible $B(P) \in \mathcal{F}_k$.
(3) On $\mathcal{F}_f$, we have $\lambda' > 0$ and $v' = 0$. Let the outward normal of $\mathcal{F}_f$ be $\mathbf{n} = \{(\lambda,\mu,v) = (1,0,-d\lambda_c(v)/dv)$, and let the vector field be $\mathbf{f}$. Then $\mathbf{n}\cdot\mathbf{f} > 0$. The flow starts on $\mathcal{F}_f$ must leave $\mathcal{P}$ transversely. It is possible that $B(P) \in \mathcal{F}_f$ for some $P \in \mathcal{P}$.
(4) On the interior of $\mathcal{F}_{b2}$ we have $d\mu/d\xi < 0$ due to $w > 0$ for $0 < \lambda < 1$ and $p < p_e$. It is possible that $B(P) \in \mathcal{F}_{b2}$.
(5) On the interior of $\mathcal{F}_{b1}$, we have $d\mu/d\xi = -abw(\lambda,v) > 0$ due to $w < 0$ for $0 < \lambda < 1$ and $p > p_e$. Thus $B(P)$ is not in the interior of $\mathcal{F}_{b1}$.

If $B(P) \in \{p = p_e\} \cap \mathcal{F}_{b1}$ at the first touch time $\xi_1 > 0$, then from (10) it is easy to verify that $\mu(\xi_1) = \mu'(\xi_1) = 0$ and $\mu''(\xi_1) < 0$. Therefore, there exists $\delta > 0$ such that $\mu(\xi) < 0$ if $\xi_1 - \delta \leq \xi < \xi_1$, contradicting to ξ_1 being the first touch time. So $B(P)$ cannot be on the line $\{p = p_e\} \cap \mathcal{F}_{b1}$.

The following lemma shows that $B(P) \notin \mathcal{F}_s$.

Lemma 2.2. *The first touch point with the boundary $B(P)$ is not on the slant side $\mathcal{F}_s$.*

Proof. The inward normal of the slant side $\mathcal{F}_s := \{c(\lambda - 1)/2 + \mu = 0\}$ is

$$\mathbf{n} = (\mathbf{n}_\lambda, \mathbf{n}_\mu, \mathbf{n}_v) = (-c/2, -1, 0).$$

The vector field is

$$\mathbf{f} = (\mathbf{f}_\lambda, \mathbf{f}_\mu, \mathbf{f}_v) = (\mu, -c\mu - abw(\lambda, v), v').$$

We want to show that on $\mathcal{F}_s$,

$$\mathbf{n} \cdot \mathbf{f} = -c\mu/2 + c\mu + abw(\lambda, v) > 0.$$

Using $\mu = -c(\lambda - 1)/2$, we have

$$\mathbf{n} \cdot \mathbf{f} = (1 - \lambda)((c/2)^2 - ab\lambda(p - p_e)). \tag{12}$$

Since (λ, v) satisfies

$$c^2(v - v_-) + (p - p_-) \leq 0, \qquad \text{and } v \geq v_-,$$

we have $p \leq p_-$. Therefore,

$$(c/2)^2 > ab(p_- - p_e) \geq ab\lambda(p(\lambda, v) - p_e).$$

It follows that $\mathbf{n} \cdot \mathbf{f} > 0$, see (12). Therefore $B(P) \notin \mathcal{F}_s$. □

Let us check the edges of $\mathcal{P}$ (not including E_+). The four edges that lie on $\lambda = 0$ cannot contain $B(P)$ as shown by (1).

Among the other four edges, two of them bound $\mathcal{F}_s$, so they cannot contain $B(P)$ due to $\mathbf{n} \cdot \mathbf{f} > 0$ as in Lemma 2.2. What left are the two more edges that bound $\mathcal{F}_{b1}$ (not including $\{p = p_e\}$). They cannot contain $B(P)$ due to $\mu' > 0$.

We have shown that if $B(P)$ is the point where $\Phi(\xi, P)$ first hits the boundary of $\mathcal{P}$, either it lies on $S_1 := \mathcal{F}_f$ or it lies on $S_2 := \mathcal{F}_{b2} \cup \mathcal{F}_k \cup \{\mu = 0, v = v_+, 0 < \lambda < \lambda_e(v_+)\}$.

Moreover, $B(P)$ cannot belong to the four boundaries of $\mathcal{F}_\ell$, the three boundaries of $\mathcal{F}_s$ and the three boundaries of $\mathcal{F}_f$. The point $B(P)$ can belong to the common boundary of $\mathcal{F}_k$ and $\mathcal{F}_{b2}$ but not the common boundaries of $\mathcal{F}_k$ and $\mathcal{F}_{b1}$.

Lemma 2.3. *There exist* $P_1, P_2 \in W^u_{loc}(E_-) \cap \mathcal{P}$ *such that* $B(P_1) \in S_1$ *and* $B(P_2) \in S_2$.

Proof. To start the shooting method, we list facts about $W^u_{loc}(E_-)$: Let $r_1 < 0 < r_2$ be the two eigenvalues for (10). Let $r_3 > 0$ be the eigenvalue with the eigenvector $(0,0,1)$.

(1) The line $\{\lambda = 0\}$ is on $W^u_{loc}(E_-)$. On this line, we have

$$v' > 0 \text{ if } v > v_-, \quad v' < 0 \text{ if } v < v_-.$$

(2) $W^u_{loc}(E_-)$ is two dimensional with two linearly independent tangent vectors

$$(\Lambda, M, V) = (0,0,1) \text{ and } (\Lambda, M, V) = (1, r_2, 0).$$

Based on this, we can express the local unstable manifold as

$$W^u_{loc}(E_-) = \{(\lambda, v, \mu) : -\varepsilon_1 < \lambda < \varepsilon_1,\ v_- - \varepsilon_2 < v < v_- + \varepsilon_2,\ \mu = \mu^*(\lambda, v)\}.$$

Moreover, if $\lambda > 0$, then $\mu^* > 0$.

Recall that the isocline $\mathcal{C}$ for v can be expressed as $\lambda = \lambda_c(v)$ with $\lambda_c(v_-) = 0$, and $d\lambda_c(v_-)/dv > 0$. Choose $\bar{v}$ with $v_- < \bar{v} < v_- + \varepsilon_2$ so that $0 < \lambda_c(\bar{v}) < \varepsilon_1$.

For each $0 < \lambda_1 < \lambda_c(\bar{v})$, define a line segment $\overline{P_1P_2}$ on $W^u_{loc}(E_-)$:

$$\overline{P_1P_2} := \{(\lambda, v, \mu) : \lambda_1 \le \lambda \le \lambda_c(\bar{v}), v = \bar{v}, \mu = \mu^*(\lambda, v)\}.$$

It is parameterized by λ with $\lambda = \lambda_c(\bar{v})$ corresponding to P_1 and $\lambda = \lambda_1$ corresponding to P_2. It is also clear that $\overline{P_1P_2}$ is in $\mathcal{P}$ except for the point P_1.

Since the flow on the v-axis is transversal to the plane $\{v = v^+\}$, assuming that λ_1 is sufficiently small so that P_2 is close to the v-axis on $W^u(E_-)$, then the orbit $\Phi(\xi, P_2)$ stays close to the v-axis until it hits $v = v_+$ transversely. It is easy to show that $B(P_2)$ will hit the boundary of $\mathcal{P}$ in S_2. On the other hand, P_1 is on S_1 and the flow $\Phi(\xi, P_1)$ leaves $\mathcal{P}$ transversely at $P_1 \in S_1$. □

Theorem 2.4. *Consider the liquefaction waves with* $\lambda_- = 0$, $\lambda_+ = 1$. *Assume* $c^2 + p_v(\lambda, v) < 0$ *if* $\lambda = 0, 1$ *and* $v \in [v_-, v_+]$ *and* $c^2 > 4ab|p(\lambda_-, v_-) - p_e|$. *Then there exists a liquefaction wave connecting* E_- *to* E_+. *The* (λ, v) *components of the wave are monotone.*

Proof. There exits an relatively open subset $\mathcal{O}_1$ of $\overline{P_1P_2}$ containing every P such that $B(P) \in S_1$. There exits also an relatively open subset $\mathcal{O}_2$ of $\overline{P_1P_2}$ containing every P such that $B(P) \in S_2$. It is also clear that $\mathcal{O}_1$ and $\mathcal{O}_2$ are mutually disjoint. Since $P_1 \in \mathcal{O}_1$, $P_2 \in \mathcal{O}_2$ and $\overline{P_1P_2}$ is a connected set,

$$\overline{P_1P_2} - (\mathcal{O}_1 \cup \mathcal{O}_2) \tag{13}$$

is nonempty. Let P be a point from (13). Then $\Phi(\xi,P)$ cannot hit the boundary of $\mathcal{P}$. It must stay inside $\mathcal{P}$ and approach the equilibrium E_+. Also $\Phi(\xi,P) \to E_-$ as $\xi \to -\infty$ since $P \in W^u_{loc}(E_-)$. □

2.3 Existence of Liquefaction Waves for $\lambda_- = 0,\ 0 < \lambda_+ < 1,\ p_+ = p_e$

Assume $c > 0$ as before so that the equilibrium E_- corresponding to $\lambda = 0, 1$ is a saddle with exactly two positive eigenvalues and one negative eigenvalue. We do not know exactly what are the eigenvalues for E_+ when $p_+ = p_e, \lambda \neq 0, 1$. However, it s not used in the proof of Theorem 2.6.

Assume that

$$c^2 + p_v(\lambda, v) < 0$$

is satisfied throughout the region $\lambda \in [0, \lambda_e], v \in [v_-, v_+]$. The isocline $\mathcal{C} := \{v' = 0\}$ can be solved from the equation

$$c^2(v - v_+) + p(\lambda, v) - p_+ = 0,$$

and the solution can be expressed as

$$v = v_c(\lambda),\ 0 \le \lambda \le \lambda_+,$$

$$\frac{dv_c(\lambda)}{d\lambda} = \frac{-p_\lambda}{c^2 + p_v} > 0.$$

We look for the liquefaction wave connecting $\lambda_- = 0$ to $p_+ = p_e$. The traveling wave satisfies (10) and (11). As from Lemma 2.1, we assume that $c > 0$.

Consider a pentahedron shaped solid $\mathcal{P}$ in (λ, μ, v) space bounded by the five surfaces (Fig. 4):

$$\begin{aligned}
\text{Left side } \mathcal{F}_\ell &:= \{\lambda = 0\};\\
\text{Back side } \mathcal{F}_k &:= \{v = v_+,\ 0 < \lambda < \lambda_e,\ 0 < \mu < -c(\lambda - \lambda_e)/2\};\\
\text{Front side } \mathcal{F}_f &:= \{0 < \lambda < \lambda_e,\ v = v_c(\lambda),\ 0 < \mu < -c(\lambda - \lambda_e)/2\};\\
\text{Slant side } \mathcal{F}_s &:= \{c(\lambda - \lambda_e)/2 + \mu = 0,\ 0 < \lambda < \lambda_e,\ v_c(\lambda) < v < v_+\};\\
\text{Bottom side } \mathcal{F}_b &:= \{\mu = 0,\ 0 < \lambda < \lambda_e\}.
\end{aligned}$$

The bottom side is further divided into $\mathcal{F}_b = \mathcal{F}_{b1} \cup \mathcal{F}_{b2}$, with

$$\mathcal{F}_{b1} := \mathcal{F}_b \cap \{p(\lambda, v) \ge p_e\},$$

$$\mathcal{F}_{b2} := \mathcal{F}_b \cap \{p(\lambda, v) < p_e\}.$$

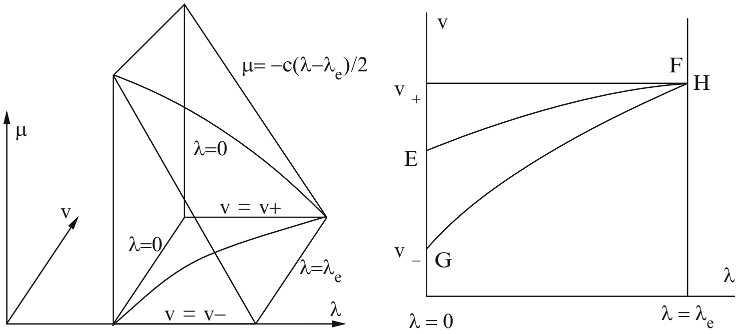

Fig. 4 The pentahedron and the top view of its base $\mathcal{F}_b$. On the curve EF, $p(\lambda,v) = p_e$. On the isocline GH, $v' = 0$

Let P be an interior point of $\mathcal{P}$ and $B(P)$ be the first touch point of the orbit at $\partial\mathcal{P}$. Just as in §2.2, we can show:

(1) $B(P) \notin \mathcal{F}_\ell$.
(2) It is possible that $B(P) \in \mathcal{F}_k$.
(3) It is possible that $B(P) \in \mathcal{F}_f$.
(4) It is possible that $B(P) \in \mathcal{F}_{b2}$.
(5) $B(P)$ is not in the interior of $\mathcal{F}_{b1}$. Also $B(P)$ cannot be on the line $\{p = p_e\} \cap \mathcal{F}_{b1}$.

The following lemma shows that $B(P) \notin \mathcal{F}_s$.

Lemma 2.5. *Assume that along the isocline $\mathcal{C}$, we have*

$$\sup_\lambda \left\{ \frac{abp_\lambda(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|} \right\} < 1.$$

Then $B(P)$ is not on the slant side $\mathcal{F}_s$.

Proof. The inward normal of the slant side $\mathcal{F}_s := \{c(\lambda - \lambda_e)/2 + \mu = 0\}$ is

$$\mathbf{n} = (\mathbf{n}_\lambda, \mathbf{n}_\mu, \mathbf{n}_v) = (-c/2, -1, 0).$$

The vector fields are

$$\mathbf{f} = (\mathbf{f}_\lambda, \mathbf{f}_\mu, \mathbf{f}_v) = (\mu, -c\mu - abw(\lambda, v), v').$$

We want to show that on $\mathcal{F}_s$,

$$\mathbf{n} \cdot \mathbf{f} = -c\mu/2 + c\mu + abw(\lambda, v) > 0.$$

Using $\mu = -c(\lambda - \lambda_e)/2$, we have

$$\mathbf{n} \cdot \mathbf{f} = (\lambda_e - \lambda)\left((c/2)^2 - ab\frac{w(\lambda, v)}{\lambda - \lambda_e}\right). \tag{14}$$

Recall that $p_v < 0$, thus $\partial w/\partial v > 0$. Since on $\mathcal{F}_s$, we have $v_c(\lambda) < v < v_+$, therefore

$$\frac{w(\lambda, v)}{\lambda - \lambda_e} < \frac{w(\lambda, v_c(\lambda))}{\lambda - \lambda_e} \leq \frac{1}{4}\left|\frac{p(\lambda, v_c(\lambda)) - p(\lambda_e, v_c(\lambda_e))}{\lambda - \lambda_e}\right|, \tag{15}$$

by the fact $\lambda(1-\lambda) < 1/4$ and $p(\lambda_e, v_c(\lambda_e)) = 0$.

The difference quotient can be estimated by

$$\begin{aligned}\sup_\lambda \left|\frac{dp(\lambda, v_c(\lambda))}{d\lambda}\right| &= \sup_\lambda |p_\lambda + p_v(dv_c(\lambda)/d\lambda)| \\ &= \sup_\lambda \left|p_\lambda + p_v\frac{-p_\lambda}{c^2 + p_v}\right| = \sup_\lambda \left|\frac{c^2 p_\lambda}{c^2 + p_v}\right|.\end{aligned}$$

If the assumption of the lemma is satisfied, then from from (14) and (15), we have $\mathbf{n} \cdot \mathbf{f} > 0$. □

Theorem 2.6. *Consider the liquefaction waves with* $\lambda_- = 0$, $0 < \lambda_+ < 1$, $p_+ = p_e$. *Assume that* $c^2 + p_v(\lambda, v) < 0$ *throughout the region and*

$$\sup_\lambda \left\{\frac{abp_\lambda(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|}\right\} < 1,$$

along the isocline $\mathcal{C}$ *for* v. *Then there exists a liquefaction wave connecting* E_- *to* E_+ *with* $p_+ = p_e$. *The* (λ, v) *components of the wave are monotone.*

Proof. The rest of the proof of the existence of the liquefaction waves follows exactly like the case where $\lambda_+ = 1$. □

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Instability of Radially-Symmetric Spikes in Systems with a Conserved Quantity

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Dedicated to George Sell on the occasion of his 70th birthday.

Abstract We show that radially symmetric spikes are unstable in a class of reaction-diffusion equations coupled to a conservation law.

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1 Introduction

We consider a class of spatially extended systems that are governed by a scalar reaction-diffusion equation, coupled to a conservation law,

$$\begin{cases} u_t = \nabla \cdot [a(u,v)\nabla u + b(u,v)\nabla v], \\ v_t = \Delta v + f(u,v), \end{cases} \quad t \geq 0,\ x \in \mathbb{R}^k. \tag{1}$$

Here, the functions a, b, and f are of class $C^3(\mathbb{R}^2, \mathbb{R})$. In order to ensure well-posedness on appropriate function spaces, we also assume that $a(u,v) \geq a_0 > 0$ for all $(u,v) \in \mathbb{R}^2$. Equations of the type (1) arise in many physical, biological, and chemical applications. We mention the Keller–Segel model for chemotaxis [6], phase-field models for undercooled liquids [1], and chemical reactions in closed reactors, with stoichiometric conservation laws for chemical species. We refer to [13] for a somewhat more extensive review of the literature and specific examples.

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Many of those systems are known to exhibit patterns in large or unbounded domains. The simplest examples in one space-dimension, $k = 1$, are layers (or interfaces), and possible bound states formed between pairs of such layers. The simplest higher-dimensional patterns are radially symmetric, localized and time-independent patterns $(u,v)(t,x) = (u^*,v^*)(|x|)$. We refer to such solutions as spikes.

In previous work [13], we showed that such localized patterns are always unstable for the dynamics of (1), under quite mild, generic assumptions, in one space-dimension. Our goal here is to extend such instability results to higher space-dimensions. In this introduction, we first briefly characterize the types of radially symmetric solutions that we are interested in, and then state our main results, which provides an instability statement for a large class of spike solutions. Our emphasis is on phenomenological assumptions, related to the spike solution, rather than assumptions on specific shapes and monotonicity properties of a, b, and f.

The following two assumptions characterize spikes as exponentially localized with a stable background.

(rs1) We assume that spikes are nonconstant, exponentially localized, that is

$$|(u^* - u^\infty, v^* - v^\infty)(x)| \le C\mathrm{e}^{-\eta|x|}, \quad \text{for all} \quad x \in \mathbb{R}^k, \quad (u^*,v^*) \not\equiv (u^\infty, v^\infty),$$

for some constants $u^\infty, v^\infty \in \mathbb{R}$, $f(u^\infty, v^\infty) = 0$ and $C, \eta > 0$.

(rs2) Spikes are asymptotic to constant states that are stable for the pure kinetics,

$$u' = 0 \quad v' = f(u,v),$$

that is, we assume $f_v(u^\infty, v^\infty) < 0$.

We will now outline how to construct spikes that satisfy (rs1)–(rs2); for a more detailed discussion, see Sect. 2. One readily checks that radially symmetric spikes satisfy the system

$$\begin{cases} \left[r^{k-1}\left(a(u,v)u_r + b(u,v)v_r\right)\right]_r = 0, \\ v_{rr} + \frac{k-1}{r}v_r + f(u,v) = 0. \end{cases} \tag{2}$$

where $r := |x|$ is the radial variable. The first equation in the above system can be integrated and viewed as a differential equation for u in terms of v,

$$\frac{\mathrm{d}u}{\mathrm{d}v} = -\frac{b(u,v)}{a(u,v)}. \tag{3}$$

Solving this differential equation with appropriate initial conditions, one obtains a solution $u^* = \Phi(v^*)$. Substituting this solution into the second equation of the system (2) we obtain the equation for v^*,

$$v_{rr} + \frac{k-1}{r}v_r + f(\Phi(v),v) = 0. \tag{4}$$

This equation is nothing else than the equation for radially symmetric solutions to

$$\Delta v + f(\Phi(v), v) = 0. \tag{5}$$

Radially symmetric solutions to such a stationary nonlinear Schrödinger equation have been studied extensively in the literature, using a variety of techniques, for instance shooting or variational methods.

Different from the one-dimensional case, even positive solutions of (4) that decay to zero need not be unique and may bifurcate as problem parameters are varied. We will focus here on the simplest case, where the linearization of (4) is invertible and possesses an odd Morse index. More precisely, consider the linearization $\mathcal{K}$ of (5) at v^* as a self-adjoint operator on the closed subspace of $L^2(\mathbb{R}^k)$ consisting of radially symmetric functions, defined by

$$\mathcal{K}v = \Delta v + f_u(\Phi(v^*), v^*)\Phi'(v^*)v + f_v(\Phi(v^*), v^*)v.$$

We will assume the following two conditions.

(rs3) The kernel of $\mathcal{K}$ in $L^2_{\mathrm{rad}}(\mathbb{R}^k)$ is trivial.
(rs4) The operator $\mathcal{K}$ on $L^2_{\mathrm{rad}}(\mathbb{R}^k)$ has an odd number of positive eigenvalues.

Condition (rs3) is typical in the sense that it is violated only for exceptional values of parameters; see for instance [2, Sect. 5]. It is also known to hold for several specific examples, see [5, 8, 12].

We note that in a typical situation the steady-states have Morse index one, they are in fact Mountain-Pass type extreme for the functional associated with (5). In this case, (rs4) is satisfied and $\mathcal{K}$ has exactly one positive eigenvalue.

We are now ready to state our main result.

Theorem 1.1. *Suppose (1) possesses an exponentially localized, radially symmetric spike solution (u^*, v^*) satisfying (rs1)–(rs4). Then (u^*, v^*) is unstable as an equilibrium to (1), considered as an evolution equation on the space of bounded uniformly continuous functions $BUC(\mathbb{R}, \mathbb{C}^2)$ or on $BUC_{\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2)$.*

The formal linearization of (1) along the spike (u^*, v^*) is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{6}$$

where

$$\mathcal{L} = \begin{bmatrix} \nabla\cdot(a^*\nabla + a_u^*\nabla u^* + b_u^*\nabla v^*) & \nabla\cdot(b^*\nabla + a_v^*\nabla u^* + b_v^*\nabla v^*) \\ f_u^* & \Delta + f_v^* \end{bmatrix}. \tag{7}$$

Here "$*$" next to any of the functions a, b, f and their partial derivatives represents the composition of the respective function with the spike (u^*, v^*).

We can view the differential expression $\mathcal{L}$ as a densely defined, closed operator on various function spaces such as $BUC(\mathbb{R}^k,\mathbb{C}^2)$, $L^2(\mathbb{R}^k,\mathbb{C}^2)$ or the exponentially weighted $L^2_\eta(\mathbb{R}^k,\mathbb{C}^2)$ spaces that we define below. One can check that on these spaces $\mathcal{L}$ generates an analytic semigroup, see for instance [7].

We will show in this paper that the spectrum of $\mathcal{L}$ intersects $\operatorname{Re}\lambda > 0$. This readily implies that the spectral radius of the semigroup generated by $\mathcal{L}$ is larger than 1, and, using a result of Henry [3, Theorem 5.1.5], that the spike is actually unstable for the nonlinear evolution. We refer to [13, Sect. 3] for more details on how spectral instability implies nonlinear instability in this context.

The remainder of the paper is organized as follows. We show that spikes naturally come in families parameterized by the asymptotic state, Sect. 2. We then recall some results on the essential spectrum of $\mathcal{L}$ in Sect. 3. The heart of our analysis is contained in Sect. 4, where we show that there exists at least one real unstable eigenvalue for $\mathcal{L}$ whenever the essential spectrum is stable. The main idea is similar to the construction in [13]. We perform a homotopy to a system where the linearization is known to exhibit an odd number of unstable eigenvalues and show that, during the homotopy, eigenvalues may not cross the origin. The crucial difficulty is the presence of essential spectrum at the origin, which makes it difficult to control multiplicities of eigenvalues. We therefore monitor eigenvalues near the origin by using a carefully crafted Lyapunov–Schmidt type reduction procedure, mimicking the extension of Evans functions at the essential spectrum. The procedure here is somewhat more subtle than in [13] since one would not expect Evans functions to be analytic in the presence of terms with decay $1/r$, generated by the Laplacian in higher space-dimension [15].

Notations: We collect some notation that we will use throughout this paper. For an operator T on a Hilbert space X we use T^*, $\operatorname{dom}(T)$, $\ker T$, $\operatorname{im} T$, $\sigma(T)$, $\rho(T)$ and $T_{|Y}$ to denote the adjoint, domain, kernel, range, spectrum, resolvent set and the restriction of T on a subspace Y of X. If $g:\mathbb{R}^2\to\mathbb{R}$ is a smooth function, e.g., a, b, f or one of its partial derivatives, we write $g^* := g(u^*,v^*)$ and $g^\infty := g(u^\infty,v^\infty)$. If u is a radially symmetric function, we write $u(r) = u(x)$ if $|x| = r$, slightly abusing the notation. We denote by $L^2_{\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^m)$ the space of radially symmetric functions that belong to $L^2(\mathbb{R}^k,\mathbb{C}^m)$. For $\eta\in\mathbb{R}$, $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^m)$ denotes the weighted space of vector-valued functions defined via the weighted L^2-norm

$$\|\underline{w}\|^2_\eta = \int_{\mathbb{R}^k} |\underline{w}(x)\mathrm{e}^{\eta|x|}|^2 \mathrm{d}x.$$

2 Families of Spikes

In this section we show that the existence of a spike satisfying conditions (rs1)–(rs4) implies the existence of a family of radially symmetric spikes. Recall that a radially symmetric spike (u^*,v^*) satisfies the system (2)

$$\begin{cases} \left[r^{k-1}\left(a(u,v)u_r + b(u,v)v_r\right)\right]_r = 0, \\ v_{rr} + \frac{k-1}{r} v_r + f(u,v) = 0. \end{cases}$$

Integrating the first equation we obtain that the spike (u^*, v^*) satisfies the equation $a(u,v)u_r + b(u,v)v_r = mr^{1-k}$ for some $m \in \mathbb{R}$. Since u^* and v^* converge exponentially as $r \to \infty$, we obtain that $v_r^* \to 0$ exponentially as $r \to \infty$ by standard ODE arguments. Therefore, u^* satisfies an equation of the form $u_r = mr^{1-k} + \mathcal{O}(\mathrm{e}^{-\varepsilon r})$ for some $\varepsilon > 0$. Since, by (rs1), u^* converges exponentially as $r \to \infty$, we conclude that $m = 0$, which implies that a spike necessarily satisfies the ODE (3),

$$\frac{\mathrm{d}u}{\mathrm{d}v} = -\frac{b(u,v)}{a(u,v)}, \quad u(v_0) = u_0.$$

Since a spike is a bounded solution, we may assume without loss of generality that b is bounded. Moreover, a is bounded away from zero, so that this ODE possesses a global, smooth solution. We denote the solution to initial conditions u_0 at $v = v_0$ by

$$u(v) = \Phi(v, v_0; u_0), \quad \Phi(v_0, v_0; u_0) = u_0. \tag{8}$$

For our particular spike (u^*, v^*), we note that $u^* = \Phi(v^*, v^\infty; u^\infty) =: \varphi_0(v^*)$. Hence, v^* satisfies the equation

$$v_{rr} + \frac{k-1}{r} v_r + H(v) = 0, \quad H(v) := f(\varphi_0(v), v). \tag{9}$$

We recall that according to conditions (rs3)–(rs4), $\mathcal{K} = \partial_r^2 + \frac{k-1}{r}\partial_r + H'(v^*)$ has trivial kernel and the number of its positive eigenvalues is odd. Condition (rs1) asserts that $v^*(r) \to v^\infty$ exponentially for $r \to \infty$. This implies that $H'(v^\infty) < 0$. Indeed, v^* is a solution to the linear equation

$$v_{rr} + \frac{k-1}{r} v_r + \tilde{H}(r)v = 0, \quad \tilde{H}(r) := H(v^*(r))/v^*(r) = \mathcal{O}(\mathrm{e}^{-\delta r}),$$

for some $\delta > 0$. Exponential decay ensures that $v^*(r) = v_1 V_1(r) + v_2 V_2(r) + \mathcal{O}((|v_1| + |v_2|)\mathrm{e}^{-\delta r}$, where V_j are linearly independent solutions to $v_{rr} + \frac{k-1}{r} v_r = 0$. Since neither V_1 nor V_2 decay exponentially, we conclude that $v_1 = v_2 = 0$ and therefore $v^* = 0$, contradicting our assumption that $v^* \not\equiv 0$. We will refer to this assumption specifically later on as ODE-hyperbolicity:

ODE-Hyperbolicity:

$$H'(v^\infty) = f_v^\infty - \frac{b^\infty}{a^\infty} f_u^\infty < 0. \tag{10}$$

In the next lemma, we prove the existence of a smooth family of spikes.

Lemma 2.1. *Under the assumptions of Theorem 1.1, there is* $\varepsilon > 0$ *and a family of spikes* $(u^*(\cdot,\mu), v^*(\cdot,\mu))$ *for* $\mu \in (-\varepsilon,\varepsilon)$*, such that*

(i) the asymptotic values $(u^\infty(\mu), v^\infty(\mu))$ *are smooth functions of* μ *and* $0 \neq \partial_\mu u^\infty(\mu)$*;*
(ii) the spikes $(u^*(\cdot,\mu) - u^\infty(\mu), v^*(\cdot,\mu) - v^\infty(\mu))$ *are given as smooth maps from* $(-\varepsilon,\varepsilon)$ *into* $H^2_{\mathrm{rad}}(\mathbb{R}^k,\mathbb{R}^2)$*; moreover,* $(u^*(\cdot,0), v^*(\cdot,0)) = (u^*(\cdot), v^*(\cdot))$*.*

Proof. The proof, in most of its parts, is similar to the proof of [13, Lemma 2.1]. For completeness we give the main arguments. We first construct a family of asymptotic states that satisfy (i) by solving $f(u,v) = 0$ locally near (u^∞, v^∞) with the implicit function theorem, using (rs2), and denote the solution by $(u^\infty(\mu), v^\infty(\mu))$, $\mu \in (-\varepsilon,\varepsilon)$, with

$$\partial_\mu(u^\infty(0), v^\infty(0)) = (-f_v^\infty, f_u^\infty). \tag{11}$$

We now define

$$\Phi^\mu : \mathbb{R} \to \mathbb{R}, \quad \Phi^\mu(v) = \Phi(v, v^\infty(\mu); u^\infty(\mu))$$

and

$$\tilde{H} : \mathbb{R} \times (-\varepsilon,\varepsilon) \to \mathbb{R}, \quad \tilde{H}(v,\mu) = f(\Phi^\mu(v), v).$$

Note that $\tilde{H}(v^\infty(\mu),\mu) = 0$. Next, we seek radially symmetric spikes that are solutions to

$$v_{rr} + \frac{k-1}{r} v_r + \tilde{H}(v,\mu) = 0, \tag{12}$$

with $u = \Phi^\mu(v)$. Using ODE-Hyperbolicity (10) again, we infer $\tilde{H}_v(v^\infty, 0) = H'(v^\infty) < 0$. It follows that $v^\infty(\mu)$ is the locally unique equilibrium to (12), and v^∞ is hyperbolic.

To solve (12) for $\mu \approx 0$ we make the change of variables $w = v - v^\infty(\mu)$, which yields a nonlinear equation

$$G(w,\mu) = w_{rr} + \frac{k-1}{r} w_r + \tilde{H}(w(\cdot) + v^\infty(\mu), \mu) = 0, \tag{13}$$

where the nonlinearity vanishes at the origin so that $G : H^2_{\mathrm{rad}}(\mathbb{R}^k) \times (-\varepsilon,\varepsilon) \to L^2_{\mathrm{rad}}(\mathbb{R}^k)$ is a smooth map. Since v^* satisfies (9), it follows that $G(v^* - v^\infty, 0) = v^*_{rr} + \frac{k-1}{r} v^*_r + H(v^*) = 0$. The w-derivative of G is given by:

$$\big(G_w(v^* - v^\infty, 0)\big) w = w_{rr} + \frac{k-1}{r} w + \tilde{H}_v(v^*(\cdot), 0) w = \mathcal{K} w.$$

Next, we will show that $\mathcal{K}$ is invertible. Since the kernel of $\mathcal{K}$ is assumed to be trivial, (rs3), it is sufficient to show that $\mathcal{K}$ is Fredholm of index zero. To see this, we first consider $\mathcal{K}$ as an operator from $H^2_{\mathrm{rad}}(\mathbb{R}^k)$ into $L^2_{\mathrm{rad}}(\mathbb{R}^k)$. Since the spike (u^*, v^*) is exponentially localized, $v^*(r) \to v^\infty$, *exponentially* for $r \to \infty$. Thus, the operator $\mathcal{K}$ is a relatively compact (actually, even in the Schatten-von Neumann B_p ideal, for the right choice of p, see [17, Theorem 4.1]) perturbation of $\mathcal{K}_\infty = \Delta + H'(v_\infty)$. By ODE-Hyperbolicity (10), we have $H'(v_\infty) < 0$ so that $\mathcal{K}_\infty$ is invertible and $\mathcal{K}$ is Fredholm with index 0.

Using the Implicit Function Theorem, we now find a local smooth solution $w(\cdot,\mu) \in H^2_{\text{rad}}(\mathbb{R}^k)$. One readily concludes that $w(r;\mu) \to 0$ for $r \to \infty$, which gives the asymptotics of the spike solution $v = w + v^\infty$ as claimed. □

We note that one can verify that all spikes in the family satisfy the conditions (rs1)–(rs4).

3 Essential Spectrum

In this section we compute the essential spectrum and show that it coincides with the essential spectrum of the linearization at (u^∞, v^∞). We say that the essential spectrum is stable if $\operatorname{Re}\sigma_{\text{ess}} \le 0$.

First, we define the limiting operator $\mathcal{L}_\infty$ through

$$\mathcal{L}_\infty = \begin{bmatrix} a^\infty\Delta & b^\infty\Delta \\ f_u^\infty & \Delta + f_v^\infty \end{bmatrix}. \tag{14}$$

Just like the operator $\mathcal{L}$, we will consider the operator $\mathcal{L}_\infty$ on various function spaces, or merely as a differential expression, slightly abusing notation. We recall the definition of essential spectrum that we use in this paper. For a given choice of function space, we say λ is in the essential spectrum $\sigma_{\text{ess}}(\mathcal{L})$ if $\mathcal{L} - \lambda$ is not Fredholm index zero. We refer to the complement of the essential spectrum in the spectrum as the point spectrum $\sigma_{\text{point}}(\mathcal{L})$.

One can compute the essential spectrum using arguments similar to [3, Sect. 5, Appendix]. In the next proposition we collect some results on the essential spectrum. For the proof we refer to [13, Sect. 4].

Proposition 3.1. *Under the assumption (rs1), the following hold true:*

(i) The essential spectra of the operators $\mathcal{L}$ and $\mathcal{L}_\infty$ coincide, and are equal for the choices of function space $X = L^2(\mathbb{R}^k, \mathbb{C}^2)$ and $X = BUC(\mathbb{R}^k, \mathbb{C}^2)$.

(ii) The essential spectrum of $\mathcal{L}$ is given by

$$\sigma_{\text{ess}}(\mathcal{L}) = \left\{\lambda_\pm(\xi) : \xi \in \mathbb{R}^k\right\}, \quad \textit{where} \quad \lambda_\pm(\xi) = \frac{\operatorname{tr}(\xi) \pm \sqrt{\operatorname{tr}(\xi)^2 - 4\det(\xi)}}{2},$$

with

$$\operatorname{tr}(\xi) = -(a^\infty + 1)|\xi|^2 + f_v^\infty, \quad \det(\xi) = a^\infty|\xi|^4 + (f_u^\infty b^\infty - f_v^\infty a^\infty)|\xi|^2.$$

(iii) The essential spectral radius of $e^{\mathcal{L}}$ is larger than 1 if $f_v^\infty > 0$.

(iv) The essential spectral radius of $e^{\mathcal{L}}$ *is 1 if* $f_v^\infty < 0$.

On the radially symmetric subspace, the differential expression for $\mathcal{L}$ is

$$\mathcal{L}_{\text{rad}} = \begin{bmatrix} \frac{1}{r^{k-1}}\partial_r\left[r^{k-1}\left(a^*\partial_r + l_1\right)\right] & \frac{1}{r^{k-1}}\partial_r\left[r^{k-1}\left(b^*\partial_r + l_2\right)\right] \\ f_u^*(r) & \frac{1}{r^{k-1}}\partial_r(r^{k-1}\partial_r) + f_v^*(r) \end{bmatrix}, \tag{15}$$

where

$$l_1 = a_u^* u_r^* + b_u^* v_r^*, \qquad \text{and } l_2 = a_v^* u_r^* + b_v^* v_r^*. \tag{16}$$

Lemma 3.2. *Under the assumptions of Theorem 1.1, the essential spectrum of* $\mathcal{L}$ *and* $\mathcal{L}_{\text{rad}}$ *coincide.*

Proof. Using the same compact perturbation argument given in the proof of Proposition 3.1(i), see also [13, Proposition 4.1], one can show that the essential spectrum of $\mathcal{L}_{\text{rad}}$ and of $\mathcal{L}_{\text{rad}}^\infty$, the restriction of $\mathcal{L}_\infty$ to the set of radially symmetric functions, coincide. From Proposition 3.1(i) it follows that to finish the proof of lemma it is enough to show that $\sigma_{\text{ess}}(\mathcal{L}_{\text{rad}}^\infty) = \sigma_{\text{ess}}(\mathcal{L}_\infty)$.

Simply restricting Fredholm properties to the closed radially symmetric subspace, we find that $\sigma_{\text{ess}}(\mathcal{L}_{\text{rad}}^\infty) \subseteq \sigma_{\text{ess}}(\mathcal{L}_\infty)$. Let $\xi \in \mathbb{R}^k$ and $\lambda \in \{\lambda_-(\xi), \lambda_+(\xi)\}$. From the definition of $\lambda_\pm$ we infer that there is vector $z \neq 0$ such that

$$\left(-D_\infty|\xi|^2 + N_\infty - \lambda\right)z = 0,$$

where

$$D_\infty = \begin{bmatrix} a^\infty & b^\infty \\ 0 & 1 \end{bmatrix}, \; N_\infty = \begin{bmatrix} 0 & 0 \\ f_u^\infty & f_v^\infty \end{bmatrix}. \tag{17}$$

Let $Z = L^2_{\text{rad}}(\mathbb{R}^k) \otimes z = \{f \otimes z : f \in L^2_{\text{rad}}(\mathbb{R}^k)\}$ and let $\mathcal{L}_Z^\infty$ be the restriction of $\mathcal{L}_{\text{rad}}^\infty$ to $Z \cap H^2_{\text{rad}}(\mathbb{R}^k)$. One can easily check that

$$D_\infty^{-1}(\mathcal{L}_Z^\infty - \lambda)(f \otimes z) = \Delta_{\text{rad}} f \otimes z + D_\infty^{-1}(N_\infty - \lambda) f \otimes z = (\Delta_{\text{rad}} + |\xi|^2) f \otimes z.$$

Since $\sigma_{\text{ess}}(\Delta_{\text{rad}}) = (-\infty, 0]$ (see for instance [4, Theorem 2]), we have that $\Delta_{\text{rad}} + |\xi|^2$ is not Fredholm, which implies that $D_\infty^{-1}(\mathcal{L}_Z^\infty - \lambda)$ is not Fredholm, as an operator from $H^2_{\text{rad}}(\mathbb{R}^k) \otimes z$ to Z. Thus, $\mathcal{L}_{\text{rad}}^\infty - \lambda$ is not Fredholm. Hence, $\sigma_{\text{ess}}(\mathcal{L}_{\text{rad}}^\infty) = \sigma_{\text{ess}}(\mathcal{L}_\infty)$, proving the lemma. □

4 Tracing the Point Spectrum

This section presents the core of our arguments which yield the existence of an unstable eigenvalue provided that the essential spectrum is stable. We first construct a homotopy of our equation to a simpler equation, Sect. 4.1 and show that spikes are unstable at the end of the homotopy, Sect. 4.2. We discuss the kernel of the linearization, Fredholm properties in weighted spaces, and far-field asymptotics of eigenfunctions in Sects. 4.3–4.5. The crucial step is carried out in Sect. 4.6, where we control for eigenvalues in a neighborhood of $\lambda = 0$. The discussion in Sects. 4.3–4.6 is valid during the entire homotopy and will allow us to prove our main result in Sect. 4.7.

4.1 Homotopy

In this section we make use of the homotopy constructed in [13, Sect. 5.1] in order to relate (1) to a "simpler" system. For this simpler system, we can easily compute the point spectrum. The homotopy is constructed so that it does not modify the structure of the spikes and does not change the stability of $\sigma_{\text{ess}}(\mathcal{L})$. To be precise, we introduce the homotopy parameter $\tau \in [0,1]$ and consider the system

$$\begin{cases} u_t = \nabla \cdot [a_\tau(u,v)\nabla u + b_\tau(u,v)\nabla v], \\ v_t = \Delta v + \tilde{f}(u,v,\tau), \end{cases} \quad t \geq 0,\ x \in \mathbb{R}^k. \tag{18}$$

The functions $a_\tau, b_\tau : \mathbb{R}^2 \to \mathbb{R}$ and $\tilde{f} : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ are defined by

$$a_\tau(u,v) = (1-\tau)a(u,v) + \tau, \quad b_\tau(u,v) = (1-\tau)b(u,v),$$
$$\tilde{f}(u,v,\tau) = f(u,v) - f(\varphi_\tau(v),v) + f(\varphi_0(v),v),$$

where φ_τ is the solution of the Cauchy problem

$$\frac{\mathrm{d}u}{\mathrm{d}v} = -\frac{b_\tau(u,v)}{a_\tau(u,v)}, \quad u(v^\infty) = u^\infty. \tag{19}$$

We collect some aspects of this homotopy in the following remark.

Remark 4.1. The homotopy satisfies the following properties.

- (i) The homotopy originates at our (1), $\tilde{f}(u,v,0) = f(u,v)$;
- (ii) $H_\tau(v) := \tilde{f}(\varphi_\tau(v),v,\tau) = H(v)$;
- (iii) If we define $u^*_\tau := \varphi_\tau(v^*)$ and $v^*_\tau := v^*$ then (u^*_τ, v^*_τ) is a spike for (18) satisfying conditions (rs1)–(rs4);
- (iv) The background states for the system (18), $\lim_{|x|\to\infty} u^*_\tau(x) = u^\infty$ and $\lim_{|x|\to\infty} v^*_\tau(x) = v^\infty$, do not depend on τ.

The linearization of (18) along the spike (u^*_τ, v^*_τ) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}_\tau \begin{pmatrix} u \\ v \end{pmatrix}, \tag{20}$$

where $\mathcal{L}_\tau$ is defined by replacing a, b and f in the definition (7) of $\mathcal{L}$ by a_τ, b_τ and $\tilde{f}(\cdot,\cdot,\tau)$, respectively.

Lemma 4.2. *Assume (rs1) and (rs2). Then, the essential spectrum of* $\mathcal{L}_\tau$ *is stable, that is,* $\operatorname{Re}\sigma_{\text{ess}}(\mathcal{L}_\tau) \le 0$.

Proof. From Proposition 3.1(iv) it follows that it suffices to show that $\tilde{f}_v^\infty < 0$. The latter was proved in [13, Lemma 5.2]. □

4.2 Instability at $\tau = 1$

At the end of the homotopy, the system possesses a lower triangular structure and we can readily infer instability.

Lemma 4.3. *Assume (rs1) and (rs2). Then, the spikes in the system (18) with* $\tau = 1$ *are unstable. Moreover,* $\mathcal{L}_1$ *has an odd number of positive eigenvalues.*

Proof. We note that if $\tau = 1$ the operator $\mathcal{L}_1$ has lower triangular block structure

$$\mathcal{L}_1 = \begin{bmatrix} \Delta & 0 \\ f_u^* & \mathcal{R}_1 \end{bmatrix},$$

where $\mathcal{R}_1 = \Delta + H_1'(v^*) = \Delta + H'(v^*)$, using Remark 4.1(ii) . Therefore, the spectrum of $\mathcal{L}_1$ is the union of the spectra of Δ and $\mathcal{R}_1$. We note that the restriction of $\mathcal{R}_1$ to the space of radially symmetric functions is $\mathcal{K}$, which according to (rs4) has an odd number of unstable eigenvalues. Hence, $\mathcal{L}_1$ has an odd number of unstable eigenvalue, proving the lemma. □

4.3 The Kernel of $\mathcal{L}_{\text{rad}}$

From our assumptions on a, b, f, one can see that the kernel of $\mathcal{L}_{\text{rad}}$ (and of $\mathcal{L}_\tau$ for all τ) consists of smooth functions for all functions spaces in consideration here. In fact, functions in the kernel of $\mathcal{L}_{\text{rad}}$ solve the system of ODEs

$$a^* u_r + (a_u^* u_r^* + b_u^* v_r^*)u + b^* v_r + (a_v^* u_r^* + b_v^* v_r^*)v = 0; \tag{21}$$

$$v_{rr} + \frac{k-1}{r} v_r + f_u^* u + f_v^* v = 0. \tag{22}$$

To solve the system (21)–(22), we first solve the first equation for u in terms of v.

Lemma 4.4. *If* $u, v \in BUC(\mathbb{R}_+)$ *satisfy* (21) *then* $u = \alpha(\partial_\mu u^*_{|\mu=0} + \frac{b^*}{a^*}\partial_\mu v^*_{|\mu=0}) - \frac{b^*}{a^*} v$, *for some constant* $\alpha \in \mathbb{C}$. *Here,* $(u^*(\cdot,\mu), v^*(\cdot,\mu))$ *refers to the family of spikes as constructed in Lemma 2.1.*

Proof. The proof is a straightforward adaptation of the argument given in [13, Lemma 5.4] and will be omitted here. □

Lemma 4.5. *A pair (u,v) belongs to the kernel of $\mathcal{L}_{\text{rad}}$ in $BUC(\mathbb{R}_+,\mathbb{C}^2)$ if and only if for some $\alpha \in \mathbb{C}$,*

$$u = \alpha \partial_\mu u^*_{|\mu=0} \quad v = \alpha \partial_\mu v^*_{|\mu=0}. \tag{23}$$

Proof. From Lemma 4.4 we have that $u = \alpha(\partial_\mu u^*_{|\mu=0} + \frac{b^*}{a^*}\partial_\mu v^*_{|\mu=0}) - \frac{b^*}{a^*}v$, for some constant $\alpha \in \mathbb{C}$. Substituting this expression into (22) we obtain that a function v from the kernel of $\mathcal{L}$ in BUC satisfies the following equation

$$v_{rr} + \frac{k-1}{r}v_r + \left(f_v^* - f_u^*\frac{b^*}{a^*}\right)v = -\alpha f_u^*\left(\partial_\mu u^*_{|\mu=0} + \frac{b^*}{a^*}\partial_\mu v^*_{|\mu=0}\right). \tag{24}$$

Once again using the fact that $(u^*(\cdot,\mu), v^*(\cdot,\mu))$ is a family of spikes, we know that $v^*(\cdot,\mu)$ satisfies the equation

$$v^*_{rr}(r,\mu) + \frac{k-1}{r}v^*_r(r,\mu) + f(u^*(r,\mu), v^*(r,\mu)) = 0.$$

Differentiating with respect to μ in this equation and setting $\mu = 0$, we infer that $\alpha\partial_\mu v^*_{|\mu=0}$ is a particular solution of (24). Hence, the general solution of (24) is of the form $v = \alpha\partial_\mu v^*_{|\mu=0} + \tilde{v}$, where $\tilde{v}$ is a solution of the equation

$$\tilde{v}_{rr} + \frac{k-1}{r}\tilde{v}_r + \left(f_v^* - f_u^*\frac{b^*}{a^*}\right)\tilde{v} = 0, \tag{25}$$

which is equivalent to $\mathcal{K}\tilde{v} = 0$. From (rs3) we obtain that $\tilde{v} = 0$, proving the lemma. □

4.4 Fredholm Properties of $\mathcal{L}_{\text{rad}}$ on Exponentially Weighted Spaces

From now on, we will study the operator $\mathcal{L}_{\text{rad}}$. Since the structure of the equation and (rs1)–(rs2) are preserved under the homotopy, the discussion applies equally to the operators $\mathcal{L}_\tau$ for all $0 \le \tau \le 1$.

We will set up a perturbation problem for eigenvalues near $\lambda = 0$ using exponentially weighted spaces, introduced at the end of Sect. 1. It turns out that the linearized operator is Fredholm in spaces with small nonzero exponential weight, so that one can attempt to use regular Fredholm perturbation theory for eigenvalues.

Lemma 4.6. *There exists $\eta^* > 0$ such that the operator $\mathcal{L}_{\text{rad}}$ is Fredholm with index -1 in $L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ for all $\eta \in (0,\eta^*)$.*

Proof. The proof is a somewhat non-standard generalization of Palmer's theorem; see [10, 11], and [16] for generalizations and applications to perturbation theory. The details of the proof will be presented elsewhere [14]. □

Next, we consider the adjoint of $\mathcal{L}_{\text{rad}}$ with respect to the non-weighted L^2-scalar product, so that $\mathcal{L}^*_{\text{rad}}$ is a closed operator on $L^2_{-\eta}(\mathbb{R},\mathbb{C}^2)$.

Lemma 4.7. *The kernel of* $\mathcal{L}^*_{\text{rad}}$ *in* $L^2_{-\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ *is spanned by the constant vector-valued function* $(1,0)^{\text{T}}$.

Proof. From Lemma 4.5 it follows that the kernel of $\mathcal{L}_{\text{rad}}$ on $L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ is trivial, hence its cokernel is one-dimensional by Lemma 4.6. We conclude that the kernel of $\mathcal{L}^*_{\text{rad}}$ in $L^2_{-\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ is one-dimensional. A short explicit calculation shows that the vector $(1,0)^{\text{T}}$ belongs to the kernel of $\mathcal{L}^*_{\text{rad}}$, which proves the lemma. □

4.5 Asymptotics of Eigenfunctions

We recall that $\mathcal{L}^\infty_{\text{rad}}$ is the restriction of $\mathcal{L}_\infty$ to the set of radially symmetric functions. The eigenvalue problem associated with the operator $\mathcal{L}^\infty_{\text{rad}}$, is given by

$$\begin{cases} a^\infty(u_{rr}+\frac{k-1}{r}u_r)+b^\infty(v_{rr}+\frac{k-1}{r}v_r)=\lambda u \\ v_{rr}+\frac{k-1}{r}v_r+f_u^\infty u+f_v^\infty v=\lambda v. \end{cases} \tag{26}$$

This system can be rewritten in the form

$$D_\infty\left(\partial_{rr}+\frac{k-1}{r}\partial_r\right)\begin{pmatrix}u\\v\end{pmatrix}+\left(N_\infty-\lambda\right)\begin{pmatrix}u\\v\end{pmatrix}=0, \tag{27}$$

where D_∞ and N_∞ were defined in (17). We therefore define the linear dispersion relation

$$\Lambda(\lambda,\nu)=\begin{bmatrix}a^\infty\nu^2-\lambda & b^\infty\nu^2\\ f_u^\infty & \nu^2+f_v^\infty-\lambda\end{bmatrix}\quad\text{and}\quad d(\lambda,\nu)=\det\Lambda(\lambda,\nu).$$

Remark 4.8. Next, we collect some results proved in [13, Lemma 5.10].

(*i*) In case Re $\lambda>0$, there are two roots ν_j, $j=1,2$ of the equation $d(\lambda,\nu)=0$ with Re $\nu_j>0$.
(*ii*) Setting $\lambda=\gamma^2$, the ν_j, $j=1,2$, can be considered as analytic functions of γ with expansion

$$\nu_1(\gamma)=\sqrt{\frac{f_v^\infty}{a^\infty f_v^\infty-b^\infty f_u^\infty}}\gamma+\mathcal{O}(\gamma^2)\quad \nu_2(\gamma)=\sqrt{-H'(v^\infty)}+\mathcal{O}(\gamma^2),$$

where H was defined in (9), and $H'(v^\infty)<0$ is guaranteed by (10).

We can now state the main result of this subsection.

Lemma 4.9. *The solutions to (26) for λ in a complex neighborhood of the origin can be characterized as follows.*

(i) *In case* $\operatorname{Re}\lambda > 0$*, the solutions of the system* (26) *are of the form*

$$(u(r), v(r))^{\mathrm{T}} = C_1 r^{1-k/2} K_{k/2-1}(\nu r) + C_2 r^{1-k/2} I_{k/2-1}(\nu r),$$

where, C_j*,* $j = 1,2$ *are vectors in the kernel of* $\Lambda(\lambda, \nu_j)$ *and* $\nu = \nu_j$*,* $j = 1,2$.[1]

(ii) *Setting* $\lambda = \gamma^2$*, the solution* $r^{1-k/2}K_{k/2-1}(\nu_1 r)$ *is bounded at* $+\infty$ *and the constant* C_1 *can be chosen as an analytic function* $C_1 = \alpha(\gamma)$ *with expansion*

$$\alpha(\gamma) = (-f_v^\infty, f_u^\infty)^{\mathrm{T}} + \mathcal{O}(\gamma)$$

such that the function defined by $\alpha(\gamma) r^{1-k/2} K_{k/2-1}(\nu_1(\gamma) r)$ *satisfies* (26).

Proof. The proof of (i) follows immediately from the definition of the modified Bessel functions and Remark 4.8. To prove (ii) one can argue in a similar way to the proof of [13, Lemma 5.10(ii)]. □

4.6 The Eigenvalue Problem Near Zero

In this section we discuss the eigenvalue problem $\mathcal{L}_{\mathrm{rad}} u = \lambda u$, near $\lambda = 0$ using Lyapunov-Schmidt reduction. The following proposition states that the eigenvalue problem can be reduced to finding the roots of a single scalar function.

Proposition 4.10. *Under the assumptions of Theorem 1.1, there exists* $\delta > 0$ *and function* $E : [0, \delta] \to \mathbb{C}$*, such that for any* $\gamma > 0$*,*

$$\gamma^2 \in \sigma_{\mathrm{point}}(\mathcal{L}_{\mathrm{rad}}) \quad \textit{if and only if} \quad E(\gamma) = 0. \tag{28}$$

Moreover, we have that

(i) *if* $k = 2$ *the function* E *is continuous on* $[0, \delta]$*, differentiable on* $(0, \delta]$*,* $E(0) = 0$ *and*

$$E(\gamma) = \frac{a^\infty f_v^\infty - b^\infty f_u^\infty}{\ln \gamma} + \frac{1}{\ln \gamma}\mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2).$$

It can be extended analytically to $B(0, \delta) \setminus [-\delta, 0]$*;*

(ii) *if* $k \geq 3$ *the function* E *can be extended analytically to* $B(0, \delta)$ *and* $E(0) \neq 0$.

Remark 4.11. The function E is constructed only near $\lambda = 0$ in our method. One can construct (but generally not compute) such a function so that it extends to

[1] Here, I_α and K_α are the modified Bessel functions satisfying the equation $r^2 h'' + rh - (r^2 + \alpha^2)h = 0$, such that K_α is bounded at ∞, I_α is bounded at 0, see [9].

a complement of the essential spectrum using Evans function constructions. This is however not necessary, here, since we are only interested in tracking possible eigenvalue crossings at $\lambda = 0$.

Proof. **Step 1: The ansatz.** We are interested in solutions to

$$\left(\mathcal{L}_{\text{rad}} - \gamma^2\right)\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{29}$$

for $\gamma \sim 0$. We use the ansatz

$$(u,v)^{\text{T}} = w + \beta\alpha(\gamma)h_k(\gamma), \tag{30}$$

where $w \in L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ and $\beta \in \mathbb{C}$, $\alpha(\cdot)$ are defined in Remark 4.8. The function h_k is defined as

$$[h_k(\gamma)](r) = \chi(r)j_k(\gamma)r^{1-k/2}K_{k/2-1}(\nu_1(\gamma)r) \tag{31}$$

for $r > 0$, $\gamma \in B(0,\delta)\setminus[-\delta,0]$ and $\chi \in C^\infty(\mathbb{R}_+)$ is a smooth function satisfying $\chi(r) = 0$ for all $r \leq 1$ and $\chi(r) = 1$ for all $r \geq 2$. The function j_k is defined as follows: $j_2(\gamma) = (\ln\gamma)^{-1}$ and $j_k(\gamma) = \gamma^{k-2}(\nu_1(\gamma))^{1-k/2}$ for $k \geq 3$. Again, $\nu_1(\cdot)$ is as in Remark 4.8. In the sequel we will show that with this choice of j_k, the function h_k can be extended smoothly up to $\gamma = 0$ in an appropriate sense. Clearly, a function $(u,v)^{\text{T}}$ of the form (30) that solves the eigenvalue (29) for $\gamma > 0$ belongs to the kernel of $\mathcal{L}_{\text{rad}}$. We will see in Step 4.6 that any eigenfunction is actually of the form (30).

Summarizing, the ansatz allows us to consider the eigenvalue problem in the smaller space $L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$, only, instead of $L^2_{\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$, at the expense of adding a free parameter β.

Since spikes are exponentially localized, we obtain that

$$\begin{aligned} w_0 :=& (\partial_\mu u^*_{|\mu=0} - \partial_\mu u^\infty(0), \partial_\mu v^*_{|\mu=0} - \partial_\mu v^\infty(0))^{\text{T}} \\ =& (\partial_\mu u^*_{|\mu=0}, \partial_\mu v^*_{|\mu=0})^{\text{T}} - \alpha(0) \in L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2). \end{aligned} \tag{32}$$

Step 2: Setup of the bifurcation problem. As shown in Sect. 4.4 , we can choose $\eta > 0$ small enough, but fixed, so that $\mathcal{L}_{\text{rad}}$ is Fredholm on $L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ and $\ker\mathcal{L}^*_{\text{rad}} = \text{Span}\{(1,0)^{\text{T}}\}$ on $L^2_{-\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$. Here $\mathcal{L}^*_{\text{rad}}$ refers to the L^2 adjoint. Thus, we have the following characterization of the image:

$$\operatorname{im}\mathcal{L}_{\text{rad}} = \left\{(u,v)^{\text{T}} \in L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2) : \int_0^\infty u(r)r^{k-1}\mathrm{d}r = 0\right\}.$$

It follows that (29) with ansatz (30) is equivalent to the following system,

$$\begin{cases} F(w,\beta,\gamma) = 0 \\ \left\langle \left(\mathcal{L}_{\mathrm{rad}} - \gamma^2\right)\left(w + \beta\alpha(\gamma)h_k(\gamma)\right), (1,0)^{\mathrm{T}} \right\rangle_{L^2(0,\infty;r^{k-1}\mathrm{d}r)} = 0. \end{cases} \tag{33}$$

Here, the function $F : H^2_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^2) \times \mathbb{C}^2 \to \operatorname{im}\mathcal{L}_{\mathrm{rad}}$ is defined by

$$F(w,\beta,\gamma) = P_0\left(\mathcal{L}_{\mathrm{rad}} - \gamma^2\right)\left(w + \beta\alpha(\gamma)h_k(\gamma)\right)$$

and P_0 is the orthogonal projection in $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ onto $\operatorname{im}\mathcal{L}_{\mathrm{rad}}$. We view (33) as an equation $\mathcal{F}(w,\beta,\gamma) = 0$ for the variables $(w,\beta,\gamma) \in H^2_{\eta,\mathrm{rad}}(\mathbb{R},\mathbb{C}^2) \times \mathbb{C}^2$, with values $\mathcal{F} \in \operatorname{im}\mathcal{L}_{\mathrm{rad}} \times \mathbb{C}$.

Step 3: Smoothness of the bifurcation problem. For our perturbation analysis, we will rely on expansions of F. For this it is essential to establish smoothness properties. We will see that we can extend the function

$$\mathcal{F}_0 : \Omega_k(\delta) \to \left(\mathcal{L}_{\mathrm{rad}} - \gamma^2\right)(\alpha(\gamma)h_k(\gamma)) \in L^2_\eta(\mathbb{R},\mathbb{C}^2)$$

to a domain $\Omega_k(\delta)$,

$$\begin{aligned} \Omega_2(\delta) =& B(0,\delta) \setminus [0,\delta], \\ \Omega_k(\delta) =& B(0,\delta), \text{ for } k \geq 3. \end{aligned}$$

which is in fact given by the domain of analyticity of h_k; see Lemma A.4.

Now, let $\psi, \phi_0 \in C^\infty(\mathbb{R}_+)$ be smooth functions satisfying $\psi(r) = 1$ for all $r \in [1,2]$ and $\psi(r) = 0$ for all $r \geq 3$ and all $r \in [0,\frac{1}{2}]$. We choose the function ψ_0 such that it satisfies the conditions $\psi_0(r) = 0$ for all $r \in [0,\frac{1}{2}]$ and $\psi_0(r) = 1$ for all $r \geq 1$. We note that $\psi_0\chi = \chi$, and thus, from (31) we obtain that

$$\psi_0 h_k(\gamma) = h_k(\gamma) \quad \text{for all} \quad \gamma \in \Omega_k(\delta). \tag{34}$$

Next, we will show that

$$\left(\mathcal{L}^\infty_{\mathrm{rad}} - \gamma^2\right)(\alpha(\gamma)h_k(\gamma)) = \chi_{[1,2]}\left(\mathcal{L}^\infty_{\mathrm{rad}} - \gamma^2\right)(\alpha(\gamma)\psi h_k(\gamma)), \tag{35}$$

where $\chi_{[1,2]}$ is the characteristic function of the interval $[1,2]$. Since we know from Lemma 4.9(ii) that $\alpha(\gamma)[h_k(\gamma)](r)$ satisfies (26) and $\chi(r) = 1$ for all $r \geq 2$ and $\chi(r) = 0$ for all $r \in [0,1]$, we obtain that

$$\left(\mathcal{L}^\infty_{\mathrm{rad}} - \gamma^2\right)(\alpha(\gamma)h_k(\gamma))(r) = 0 \quad \text{for all} \quad r \in [0,1] \quad \text{and all} \quad r \geq 2. \tag{36}$$

Also, since $\mathcal{L}^\infty_{\mathrm{rad}}$ is a differential operator and $\psi(r) = 1$ and $\psi'(r) = \psi''(r) = 0$ for all $r \in [1,2]$ one can verify that

$$\chi_{[1,2]}\mathcal{L}^\infty_{\mathrm{rad}}(\psi u) = \chi_{[1,2]}\mathcal{L}^\infty_{\mathrm{rad}} u \quad \text{for all} \quad u \in H^2_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{C}^2).$$

This, together with (36), proves (35) which together with (34) implies the representation for $\mathcal{F}_0$

$$\mathcal{F}_0(\gamma) = (\mathcal{L}_{\text{rad}} - \mathcal{L}_{\text{rad}}^\infty)\left(\psi_0\alpha(\gamma)h_k(\gamma)\right) + \chi_{[1,2]}\left(\mathcal{L}_{\text{rad}}^\infty - \gamma^2\right)\left(\alpha(\gamma)\psi h_k(\gamma)\right). \tag{37}$$

Since $\mathcal{L}_{\text{rad}} - \mathcal{L}_{\text{rad}}^\infty$ is a second order differential operator whose matrix-valued coefficients decay exponentially at ∞ and ψ_0 is a bounded C^∞ function and $\text{supp}(\psi_0) \subseteq [\frac{1}{2}, \infty)$, we have that the linear operator defined by $w \to (\mathcal{L}_{\text{rad}} - \mathcal{L}_{\text{rad}}^\infty)(\psi_0 w)$ is bounded from $H^2_{-\eta,\text{rad}}(\mathbb{R}, \mathbb{C}^2)$ to $L^2_{\eta,\text{rad}}(\mathbb{R}, \mathbb{C}^2)$. Moreover, since $\psi \in C^\infty(\mathbb{R}_+)$ has compact support, $\text{supp}(\psi) \subseteq [\frac{1}{2}, 3]$, the operator of multiplication by ψ is bounded from $H^2_{-\eta,\text{rad}}(\mathbb{R}^k)$ to $H^2_{\eta,\text{rad}}(\mathbb{R}^k)$. Recall that v_1 is analytic on $B(0,\delta)$ by Remark 4.8.

It remains to investigate the smoothness of h_k. Lemma A.4 in the appendix states that h_k is analytic from $\Omega_k(\delta)$ to $H^2_{-\eta,\text{rad}}(\mathbb{R}^k)$. Moreover, in the case $k = 2$, the limit $\lim_{\gamma\to 0, \gamma>0} \mathcal{F}_0(\gamma) = \mathcal{L}_{\text{rad}}(\alpha(0)h_k^0)$ exists in $L^2_{\eta,\text{rad}}(\mathbb{R}^k)$. Now, using the representation (37), we conclude that $\mathcal{F}_0$ is well defined and analytic on $\Omega_k(\delta)$, and continuous on $[0, \delta]$ in the case $k = 2$.

Step 4: Construction of the Evans function. From the definition of w_0 in (32) it follows that $F(w_0, 1, 0) = 0$. Since $\mathcal{F}$ is bounded linear in w and β, and since $\mathcal{F}_0$ is analytic on $\Omega_k(\delta)$, we conclude that $\mathcal{F}$ is analytic on $H^2_{\eta,\text{rad}}(\mathbb{R}^k) \times \mathbb{C} \times \Omega_k(\delta)$. Differentiating F in w, we obtain that

$$F_w(w_0, 1, 0) = P_0\mathcal{L}_{\text{rad}}.$$

Since $\mathcal{L}_{\text{rad}}$ is Fredholm of index -1 with trivial kernel, and P_0 projects onto its range, we infer that the linearization in w, $F_w(w_0, 1, 0)$, is boundedly invertible. From the Implicit Function Theorem it follows that we can solve locally the first equation in (33), and find a unique smooth solution

$$w^* : B(1,\delta) \times \Omega_k(\delta) \subset \mathbb{C}^2 \to H^2_{\eta,\text{rad}}(\mathbb{R}^k, \mathbb{C}^2), \quad w^*(1,0) = w_0,$$

so that locally

$$F(w, \beta, \gamma) = 0 \Longleftrightarrow w = w^*(\beta, \gamma).$$

From this local uniqueness we conclude that $w^*(\beta, \gamma) = \beta w^*(1, \gamma)$ for all $(\beta, \gamma) \in B(1, \delta) \times \Omega_k(\delta)$, so that we may restrict to $\beta = 1$ in the sequel. By continuity of the solution, $w^*(1, \gamma) + \alpha(\gamma)h(\gamma) \neq 0$ for small γ.

Substituting w^* into the second equation of (33) completes the Lyapunov–Schmidt reduction and gives us a bifurcation equation $E(\gamma) = 0$, where $E : \Omega_k(\delta) \to \mathbb{C}$ is defined as

$$E(\gamma) = \left\langle \left(\mathcal{L}_{\text{rad}} - \gamma^2\right)\left(w^*(1,\gamma) + \alpha(\gamma)h_k(\gamma)\right), (1,0)^{\text{T}} \right\rangle_{L^2}. \tag{38}$$

Now, w^* is an analytic function on $\Omega_k(\delta)$, continuous on $[0,\delta]$ for $k=2$ by the implicit function theorem. Also, the map $\mathcal{F}_0 : \gamma \to \left(\mathcal{L}-\gamma^2\right)(\alpha(\gamma)h_k(\gamma))$ is an analytic functions on $\Omega_k(\delta)$ and continuous on $[0,\delta]$ by Step 4.6. We can conclude that E is analytic and continuous on $\gamma \in [0,\delta]$ in the case $k=2$.

Step 5: Invertibility for $E(\gamma) \neq 0$. In this subsection we show that the eigenfunctions are necessarily of the form described in the ansatz (30) which is equivalent to prove that $\mathcal{L}_{\text{rad}} - \lambda$ is invertible for $\operatorname{Re}\lambda > 0$ whenever $E(\gamma) \neq 0$, where $\lambda = \gamma^2$. Consider the system

$$\left(\mathcal{L}_{\text{rad}} - \gamma^2\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{39}$$

for right-hand sides $g_1, g_2 \in L^2_{\text{rad}}(\mathbb{R}^k, \mathbb{C}^2)$. Since γ^2 is not in the essential spectrum for $\operatorname{Re}\lambda > 0$ we have that $\mathcal{L}_{\text{rad}} - \gamma^2$ is Fredholm of index 0. It is therefore sufficient to solve this equation for right-hand sides g_j, $j=1,2$ in a dense subset of $L^2_{\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$, for example $L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$. Seeking solutions of (39) in the form of our ansatz (30), which for $\operatorname{Re}\lambda > 0$ clearly provides us with L^2-functions, we obtain the system

$$\begin{cases} F(w,\beta,\gamma) = P_0(g_1,g_2)^{\mathrm{T}} \\ \left\langle \left(\mathcal{L}_{\text{rad}} - \gamma^2\right)\left(w + \beta\alpha(\gamma)h_k(\gamma)\right), (1,0)^{\mathrm{T}} \right\rangle_{L^2} = \left\langle 1, g_1 \right\rangle_{L^2}. \end{cases} \tag{40}$$

This is a linear system in w and β, and the joint linearization is Fredholm of index zero, since the Fredholm index of $\mathcal{L}_{\text{rad}}$ is -1. Thus, we can solve this equation with bounded inverse provided that there is no kernel, which is equivalent to $E(\gamma) \neq 0$.

Step 6: Estimating $E(\gamma)$ for $\gamma > 0$. We will split the argument for the two cases that are significantly different, that is , when $k=2$ or $k \geq 3$.

The case $k=2$. Since $w^*(1,\gamma) \in H^2_{\eta,\text{rad}}(\mathbb{R}^2,\mathbb{C}^2)$ we have that

$$E(\gamma) = \left\langle \mathcal{L}_{\text{rad}}\left(\alpha(\gamma)h_2(\gamma)\right), (1,0)^{\mathrm{T}} \right\rangle_{L^2} + \frac{1}{\ln\gamma}\mathcal{O}(\gamma^2).$$

Recall the definition of the functions $l_j(r)$ from (16). Since $l_j(r) \to 0$ exponentially, as $r \to \infty$, $j=1,2$, one has that

$$\int_0^\infty \frac{1}{r}\partial_r[r(l_j(r)v(r))]r\mathrm{d}r = 0$$

for any function $v \in H^2_{-\eta,\text{rad}}(\mathbb{R}^2)$ whose support does not include $r=0$. This implies that

$$E(\gamma) = \int_0^\infty \frac{1}{r}\partial_r\Big(r((b^*(r)f_u^\infty - a^*(r)f_v^\infty)[h_2(\gamma)]'(r)\Big)r\mathrm{d}r. \tag{41}$$

We expand $h_2(\gamma)$, next. Therefore recall that the modified Bessel function can be written in the form

$$K_0(z) = (\ln z) f_0(z) + g_0(z), \qquad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}_-, \tag{42}$$

with two entire function f_0 and g_0.

Since ν_1 is analytic, $\nu_1(0) = 0$ and $\nu_1'(0) > 0$ the function $\rho : B(0,\delta) \to \mathbb{C}$ defined by $\rho(\gamma) = \ln\left(\frac{\nu_1(\gamma)}{\gamma}\right)$ is analytic, for some small $\delta > 0$. Next we note that

$$\ln(\nu_1(\gamma) r) = \ln\gamma + \rho(\gamma) + \ln r \quad \text{for all} \quad \gamma \in B(0,\delta), r > 0. \tag{43}$$

From (43) and since the functions f_0 and g_0 are entire functions, we can expand

$$[h_2(\gamma)](r) = \frac{\chi(r)}{\ln\gamma}\left(g_0(0) - \ln\gamma - \rho(\gamma) + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2)\ln\gamma\right),$$

and

$$\begin{aligned}[h_2(\gamma)]'(r) = {} & \frac{\chi'(r)}{\ln\gamma}\left(g_0(0) - \ln\gamma - \rho(\gamma) + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2)\ln\gamma\right) \\ & + \frac{\chi(r)}{\ln\gamma}(-r^{-1} + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2)\ln\gamma).\end{aligned}$$

Substituting these expansions for h_2 into (41), we arrive at the expansion for E

$$E(\gamma) = \frac{a^\infty f_v^\infty - b^\infty f_u^\infty}{\ln\gamma} + \frac{1}{\ln\gamma}\mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2). \tag{44}$$

The case $k \geq 3$**.** By Lemma A.4, the limit $\lim_{\gamma\to 0,\gamma>0}(h_k)(r) = c_k\chi(r)r^{2-k}$ exists for any fixed r, and convergence holds in the $H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$-norm. Moreover, we have $c_k \neq 0$. Using the definition of the function E, we therefore obtain

$$\begin{aligned} E(0) &= \left\langle \mathcal{L}_{\mathrm{rad}}(\alpha(0)h_k(0)), (1,0)^{\mathrm{T}}\right\rangle_{L^2} \\ &= \int_0^\infty \frac{1}{r^{k-1}}\partial_r\left(r^{k-1}((b^*(r)f_u^\infty - a^*(r)f_v^\infty)c_k\partial_r(\chi(r)r^{2-k})\right)r^{k-1}\mathrm{d}r \\ &\quad + \int_0^\infty \frac{1}{r^{k-1}}\partial_r\left(r^{k-1}(l_1(r)+l_2(r))\chi(r)r^{2-k}\right)r^{k-1}\mathrm{d}r \\ &= (k-2)c_k(a^\infty f_v^\infty - b^\infty f_u^\infty), \end{aligned}$$

where we used $l_j(r) \to 0$ as $r \to \infty$, exponentially, $j = 1,2$, in the last equality. □

4.7 Proof of Theorem 1.1

Proposition 4.10 implies that for any $\tau \in [0,1]$, there exists $\delta_\tau > 0$ and an analytic function $E_\tau : \Omega_k(\delta_\tau) \to \mathbb{C}$ that detects the eigenvalues of $\mathcal{L}_{\tau,\mathrm{rad}}$ according to (28). Since E is smooth in τ, there is a constant c^* independent of τ such that $|E_\tau(\gamma)| \geq \frac{c^*}{\ln \gamma}$, uniformly in τ, so that we can exclude eigenvalues of $\mathcal{L}_{\tau,\mathrm{rad}}$ in $(0,\delta^*)$. In addition, (6) is well-posed for all τ, which implies $\sup \mathrm{Re}\, \sigma(\mathcal{L}_{\tau,\mathrm{rad}}) < \infty$. Thus, the number of real unstable eigenvalues of $\mathcal{L}_{\tau,\mathrm{rad}}$ is finite,

$$N(\tau) = \#\{\lambda(\tau) \in \sigma_{\mathrm{point}}(\mathcal{L}_{\tau,\mathrm{rad}}) : \lambda(\tau) > 0\} < \infty,$$

for all $\tau \in [0,1]$. This fact allows us to define a parity index as follows,

$$i_{\mathrm{p}}(\tau) = (-1)^{N(\tau)}. \tag{45}$$

Since eigenvalues are uniformly bounded away from 0 and ∞ on the positive real axis, they can only leave the positive axis in complex pairs. Therefore, i_{p} is constant, independent of τ. Also, by Lemma 4.3 $i_{\mathrm{p}}(1) = -1$, so that $N(\tau) \neq 0$. This proves the linear instability of spikes and Theorem 1.1.

Appendix A

In this appendix we discuss the analyticity of h_k and the possibility of extending the function analytically to a neighborhood of 0. We start with two abstract lemmas.

Lemma A.1. *Given $\eta > 0$, $p : \mathbb{R}_+ \to \mathbb{R}$ a C^∞ function, and f an entire function such that*

(i) $p(r) = 0$ for all $r \in [0,1]$;
(ii) $|p(r)| \leq cr^m$ for all $r > 0$, for some constants $c > 0$ and $m \in \mathbb{N}$;
(iii) $|f(z)| \leq c\mathrm{e}^{\omega|z|}$ for all $z \in \mathbb{C}$, for some constants $c > 0$ and $\omega \in \mathbb{R}_+$.

Then, there exists $\delta > 0$ such that the function $F_p : B(0,\delta) \to L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$, given by $[F_p(\gamma)](r) = p(r)f(\nu_1(\gamma)r)$, is well defined and analytic on $B(0,\delta)$.

Proof. Since ν_1 is analytic and $\nu_1(0) = 0$ we can choose $\delta > 0$ such that $\omega|\nu_1(\gamma)| \leq \eta/2$. Using the hypothesis (i)–(iii) we estimate

$$|[F_p(\gamma)](r)| \leq c|p(r)|\mathrm{e}^{\omega|\nu_1(\gamma)r|} \leq c\chi_{[1,\infty)}(r)r^m\mathrm{e}^{\eta/2r} \quad \text{for all} \quad \gamma \in B(0,\delta), r > 0. \tag{46}$$

It follows that $F_p(\gamma) \in L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ for all $\gamma \in B(0,\delta)$ and so, F_p is well-defined. From Lebesgue's Dominated Convergence Theorem and the estimate (46) we infer that F_p is continuous. Next, we prove that F_p is weakly analytic, that is, the function defined by $F_p^v(\gamma) := \langle F_p(\gamma), v\rangle_{L^2}$ is analytic for all $v \in L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k)$, which will then imply that F_p is analytic. To check this, we integrate F_p^v on the boundary of a rectangle $\mathrm{R} \subset B(0,\delta)$ using Fubini's Theorem. Since ν_1 and f are analytic we have

$$\oint_{\partial \mathrm{R}} F_p^v(\gamma)\mathrm{d}\gamma = \int_0^\infty \left(\oint_{\partial \mathrm{R}} (p(r)f(\nu_1(\gamma)r))\,\mathrm{d}\gamma\right) v(r)r^{k-1}\mathrm{d}r = 0. \qquad \square$$

Lemma A.2. *Given $\eta > 0$, $p : \mathbb{R}_+ \to \mathbb{R}$ a C^∞ function, and f an entire function such that*

(i) $p(r) = 0$ *for all* $r \in [0,1]$*;*
(ii) $\max(|p(r)|,|p'(r)|,|p''(r)|) \le cr^m$ *for all* $r > 0$*, for some constants* $c > 0$ *and* $m \in \mathbb{N}$*;*
(iii) $|f(z)| \le c\mathrm{e}^{\omega|z|}$ *for all* $z \in \mathbb{C}$*, for some constants* $c > 0$ *and* $\omega \in \mathbb{R}_+$*.*

Then, there exists $\delta > 0$ such that the function $F_p : B(0,\delta) \to H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$, defined in Lemma A.1, is well defined and analytic on $B(0,\delta)$.

Proof. First we note that using standard complex analysis arguments one can show that the functions f' and f'' are entire functions and they satisfy condition (iii) from Lemma A.1. Also, we note that by our assumption the functions p, p' and p'' satisfy conditions (i)–(ii) from Lemma A.1. We have

$$[F_p(\gamma)]'(r) = p'(r)f(\nu_1(\gamma)r) + \nu_1(\gamma)p(r)f'(\nu_1(\gamma)r)$$

$$[F_p(\gamma)]''(r) = p''(r)f(\nu_1(\gamma)r) + 2\nu_1(\gamma)p'(r)f'(\nu_1(\gamma)r) + \nu_1(\gamma)^2p(r)f''(\nu_1(\gamma)r),$$

for all $\gamma \in B(0,\delta)$ and all $r > 0$. Now, using analyticity of ν_1, we conclude from Lemma A.1 that the maps $\gamma \to [F_p(\gamma)]^{(j)} : B(0,\delta) \to L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ are well defined and analytic. This proves the lemma. $\square$

Applying these two lemmas to our particular situation we obtain the following result.

Lemma A.3. *For each $\eta > 0$ there exists $\delta > 0$ such that*

(i) There exists $F_2, G_2 : B(0,\delta) \to H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ two analytic functions such that

$$h_2(\gamma) = \frac{1}{\ln\gamma}F_2(\gamma) + G_2(\gamma) \quad \textit{for all} \quad \gamma \in B(0,\delta) \setminus [0,\delta]; \tag{47}$$

(ii) For $k \ge 3$, the function h_k can be extended as an analytic function from $B(0,\delta)$ into $H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$.

Proof. (i) Recall the decomposition of K_0, the definitions of f_0 and g_0 (42), and the expansion of $\ln(\nu_1(\gamma)r)$, (43).

Using this representation and the definition of h_2 in (31) we calculate

$$[h_2(\gamma)](r) = \frac{1}{\ln\gamma}\chi(r)K_0(\nu_1(\gamma)r) = \frac{1}{\ln\gamma}\chi(r)\Big[(\ln\gamma+\rho(\gamma)+\ln r)f_0(\nu_1(\gamma)r)$$
$$+g_0(\nu_1(\gamma)r)\Big]$$
$$= \frac{1}{\ln\gamma}\Big[\rho(\gamma)\chi(r)f_0(\nu_1(\gamma)r)+\chi(r)g_0(\nu_1(\gamma)r)+\chi(r)(\ln r)f_0(\nu_1(\gamma)r)\Big]$$
$$+\chi(r)f_0(\nu_1(\gamma)r).$$

We note that the functions $p(r)=\chi(r)$ and $q(r)=\chi(r)\ln r$ satisfy conditions (i)–(ii) from Lemma A.2, the functions f_0 and g_0 have exponential growth, that is they satisfy condition (iii) from Lemma A.2. Since, in addition, ρ is analytic, we obtain that there is a $\delta>0$ such that the functions $F_2, G_2 : B(0,\delta) \to H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ defined by

$$[F_2(\gamma)](r) = \chi(r)\left[(\rho(\gamma)+\ln r)f_0(\nu_1(\gamma)r)+g_0(\nu_1(\gamma)r)\right],$$

$$[G_2(\gamma)](r) = \chi(r)f_0(\nu_1(\gamma)r),$$

are analytic, proving (i).

(ii) The proof of (ii) is similar to the proof of (i). Indeed, if $k\geq 3$, then there are two entire functions $f_{k/2}$ and $g_{k/2}$ such that $K_{k/2-1}(z) = z^{k/2-1}f_{k/2}(z)+z^{1-k/2}g_{k/2}(z)$. From the definition of h_k in (31) we calculate

$$[h_k(\gamma)](r) = \gamma^{k-2}\chi(r)(\nu_1(\gamma)r)^{1-k/2}K_{k/2-1}(\nu_1(\gamma)r)$$
$$= \gamma^{k-2}\chi(r)\left[f_{k/2}(\nu_1(\gamma)r)+(\nu_1(\gamma)r)^{2-k}g_{k/2}(\nu_1(\gamma)r)\right]$$
$$= \gamma^{k-2}\chi(r)f_{k/2}(\nu_1(\gamma)r)+\left(\frac{\gamma}{\nu_1(\gamma)}\right)^{k-2}\chi(r)r^{2-k}g_{k/2}(\nu_1(\gamma)r)$$
$$= \gamma^{k-2}\chi(r)f_{k/2}(\nu_1(\gamma)r)+\mathrm{e}^{-(k-2)\rho(\gamma)}\chi(r)r^{2-k}g_{k/2}(\nu_1(\gamma)r)$$

Again, we note $p(r)=\chi(r)$ and $q(r)=\chi(r)r^{2-k}$ satisfy conditions (i)–(ii) from Lemma A.2, the functions $f_{k/2}$ and $g_{k/2}$ satisfy condition (iii) from Lemma A.2. Since ρ is analytic, from Lemma A.2 we conclude that h_k can be extended as an analytic function from $B(0,\delta)$ into $H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$. □

We collect the main conclusions of this appendix in the following lemma:

Lemma A.4. *Let $\Omega_2(\delta) = B(0,\delta)\setminus[0,\delta]$ and $\Omega_k(\delta) = B(0,\delta)$ for $k\geq 3$. The function $h_k : \Omega_k(\delta) \to H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ is well-defined and analytic. For all k, the limit*

$\lim_{\gamma \to 0, \gamma > 0} h_k = h_k^0$ *exists in* $H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$, *and* $h_k^0(r) = c_k \chi(r) r^{2-k}$, *for some non-zero constant* c_k.

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Global Hopf Bifurcation Analysis of a Neuron Network Model with Time Delays

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Dedicated to Professor G. R. Sell for the occasion of his 70th birthday

Abstract For a two-neuron network with self-connection and time delays, we carry out stability and bifurcation analysis. We establish that a Hopf bifurcation occurs when the total delay passes a sequence of critical values. The stability and direction of the local Hopf bifurcation are determined using the normal form method and center manifold theorem. To show that periodic solutions exist away from the bifurcation points, we establish that local Hopf branches globally extend for arbitrarily large delays.

Mathematics Subject Classification (2010): Primary 34C25, 34K18; Secondary

1 Introduction

Research has been devoted to rigorous stability and bifurcation analysis of small neural network models with time delays [1–6, 8, 10, 11, 15, 16, 18, 20, 21]. Shayer and Campbell [19] studied bifurcation and multistability in the following two-neuron network with self-connection and time delays:

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$$\begin{cases} \dot{x}_1(t) = -kx_1(t) + \beta \tanh(x_1(t)) + a_{12} \tanh(x_2(t-\tau_2)), \\ \dot{x}_2(t) = -kx_2(t) + \beta \tanh(x_2(t)) + a_{21} \tanh(x_1(t-\tau_1)), \end{cases} \tag{1}$$

where $k > 0$, β, a_{12}, a_{21} are all constants. Their numerical investigation shows that the model possesses very rich dynamics. For a more general class of neural network model

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + a_1 f_1(u_1(t)) + b_1 g_1(u_2(t-\tau_1)), \\ \dot{u}_2(t) = -\mu_2 u_2(t) + a_2 f_2(u_2(t)) + b_2 g_2(u_1(t-\tau_2)), \end{cases} \tag{2}$$

Wei et al. [21] carried out bifurcation analysis for the case $\mu_1 = \mu_2$ and $a_1 f_1 = a_2 f_2$. In this chapter, we carry out complete and detailed analysis on the stability, the bifurcation, and the global existence of periodic solutions for the general two-neuron network (2).

In Sect. 2, we investigate stability and local Hopf bifurcation as we vary the total delay $\tau = \tau_1 + \tau_2$, by analyzing the related characteristic equation for system (2). We show that a sequence of Hopf bifurcations occur at the origin as the total delay increases. In Sect. 3, we establish the direction and stability of the first Hopf bifurcation branch using the center manifold theorem and normal form method. Global extensions of the local Hopf branch are established in Sect. 4, where we apply a global Hopf bifurcation theorem of Wu [22] and higher-dimensional Bendixson–Dulac criteria for ordinary differential equations of Li and Muldowney [14]. Numerical simulations are carried out to support our theoretical results.

2 Stability and Local Hopf Bifurcation

In this section, we investigate the effect of delay on the dynamic behaviors of the two-neuron network model (2).

Let $x_1(t) = u_1(t-\tau_2)$, $x_2(t) = u_2(t)$, and $\tau = \tau_1 + \tau_2$. Then system (2) becomes

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + a_1 f_1(x_1(t)) + b_1 g_1(x_2(t-\tau)), \\ \dot{x}_2(t) = -\mu_2 x_2(t) + a_2 f_2(x_2(t)) + b_2 g_2(x_1(t)). \end{cases} \tag{3}$$

We make the following assumptions.

$$(\mathbf{H}_1)\ f_i, g_i \in C^4,\ x f_i(x) > 0,\ \text{and}\ x g_i(x) > 0\ \text{ for }\ x \neq 0,\ i = 1,2.$$

Under $(\mathbf{H}_1)$, the origin $(0,0)$ is an equilibrium of system (3). Without loss of generality, we assume that $f_i'(0) = 1$ and $g_i'(0) = 1$, $i = 1,2$. Then the linearization of system (3) at the origin is

$$\begin{cases} \dot{y}_1(t) = -\mu_1 y_1(t) + a_1 y_1(t) + b_1 y_2(t-\tau), \\ \dot{y}_2(t) = -\mu_2 y_2(t) + a_2 y_2(t) + b_2 y_1(t). \end{cases}$$

Its characteristic equation is

$$\lambda^2 + [(\mu_1 - a_1) + (\mu_2 - a_2)]\lambda + (\mu_1 - a_1)(\mu_2 - a_2) - b_1 b_2 \mathrm{e}^{-\lambda\tau} = 0. \tag{4}$$

Lemma 2.1. *Suppose that there exists a* $\tau_0 > 0$ *such that (4) with* τ_0 *has a pair of purely imaginary roots* $\pm i\omega_0$*, and the root of (4)*

$$\lambda(\tau) = \alpha(\tau) + \mathrm{i}\omega(\tau)$$

satisfies $\alpha(\tau_0) = 0$ *and* $\omega(\tau_0) = \omega_0$*. Then* $\alpha'(\tau_0) > 0$.

Proof. Substituting $\lambda(\tau)$ into (4) and differentiating with respect to τ, we have

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = -\frac{2\lambda + [(\mu_1 - a_1) + (\mu_2 - a_2)]}{b_1 b_2 \lambda \mathrm{e}^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Note that

$$b_1 b_2 \mathrm{e}^{-\lambda\tau} = \lambda^2 + [(\mu_1 - a_1) + (\mu_2 - a_2)]\lambda + (\mu_1 - a_1)(\mu_2 - a_2),$$

and $\lambda(\tau_0) = \mathrm{i}\omega_0$. We have

$$\left(\frac{\mathrm{d}\lambda(\tau_0)}{\mathrm{d}\tau}\right)^{-1} = \frac{[(\mu_1-a_1)+(\mu_2-a_2)]+2\mathrm{i}\omega_0}{[(\mu_1-a_1)+(\mu_2-a_2)]\omega_0^2 - \mathrm{i}\omega_0[(\mu_1-a_1)(\mu_2-a_2)-\omega_0^2]} - \mathrm{i}\frac{\tau_0}{\omega_0},$$

and thus,

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda(\tau_0)}{\mathrm{d}\tau}\right)^{-1} = \frac{\omega_0^2}{\triangle}[(\mu_1 - a_1)^2 + (\mu_2 - a_2)^2 + 2\omega_0^3],$$

where

$$\triangle = [(\mu_1 - a_1) + (\mu_2 - a_2)]^2 \omega_0^4 + \omega_0^2[(\mu_1 - a_1)(\mu_2 - a_2) - \omega_0^2]^2.$$

The conclusion follows. □

We make the following further assumption.
$(\mathbf{H}_2)$ $(\mu_1 + \mu_2) - (a_1 + a_2) > 0$.

Lemma 2.2. *Suppose that assumption* $(\mathbf{H}_2)$ *is satisfied.*

(i) *If*

$$|b_1 b_2| \le |(\mu_1 - a_1)(\mu_2 - a_2)| \quad \textit{and} \quad 0 < (\mu_1 - a_1)(\mu_2 - a_2) \ne b_1 b_2,$$

then all the roots of (4) have negative real parts for all $\tau \ge 0$.

(ii) *If*

$$b_1 b_2 > (\mu_1 - a_1)(\mu_2 - a_2),$$

then (4) has at least one root with positive real part for all $\tau \geq 0$. *If, in addition,*

$$b_1 b_2 > |(\mu_1 - a_1)(\mu_2 - a_2)|,$$

then there exist a sequence values of τ, $\bar{\tau}_0 < \bar{\tau}_1 < \cdots$, *such that (4) has a pair of purely imaginary roots* $\pm \mathrm{i}\omega_0$ *when* $\tau = \bar{\tau}_j,\ j = 0, 1, 2, \ldots$.

(iii) *If*

$$b_1 b_2 < -|(\mu_1 - a_1)(\mu_2 - a_2)|,$$

then, for the same sequence $\bar{\tau}_0 < \bar{\tau}_1 < \bar{\tau}_2 < \cdots$ *as in* (ii), *all the roots of (4) have negative real parts when* $\tau \in [0, \bar{\tau}_0)$; *(4) has at least a pair of roots with positive real parts when* $\tau > \bar{\tau}_0$; *and (4) has a pair of purely imaginary root* $\pm \mathrm{i}\omega_0$ *when* $\tau = \bar{\tau}_j,\ j = 0, 1, 2, \ldots$. *Furthermore, all the roots of (4) with* $\tau = \bar{\tau}_0$ *have negative real parts except* $\pm \mathrm{i}\omega_0$.

Proof. When $\tau = 0$, the roots of (4) are

$$\lambda_{1,2} = \frac{1}{2}\Big\{ -[(\mu_1 - a_1) + (\mu_2 - a_2)] \\ \pm \sqrt{[(\mu_1 - a_1) + (\mu_2 - a_2)]^2 - 4[(\mu_1 - a_1)(\mu_2 - a_2) - b_1 b_2]} \Big\}.$$

This leads to

$$\mathrm{Re}\lambda_{1,2} < 0 \quad \text{when} \quad (\mu_1 - a_1)(\mu_2 - a_2) > b_1 b_2,$$

and

$$\mathrm{Re}\lambda_1 > 0 \ \text{ and } \ \mathrm{Re}\lambda_2 < 0 \ \text{ when } \ (\mu_1 - a_1)(\mu_2 - a_2) < b_1 b_2.$$

Equation (4) has a pair of purely imaginary roots $i\omega$ $(\omega > 0)$ if and only if ω satisfies

$$\begin{cases} (\mu_1 - a_1)(\mu_2 - a_2) - \omega^2 = b_1 b_2 \cos \omega\tau, \\ ((\mu_1 - a_1) + (\mu_2 - a_2))\omega = -b_1 b_2 \sin \omega\tau. \end{cases} \tag{5}$$

It follows from (5) that

$$\omega^4 + ((\mu_1 - a_1)^2 + (\mu_2 - a_2)^2)\omega^2 + [(\mu_1 - a_1)^2(\mu_2 - a_2)^2 - b_1^2 b_2^2] = 0,$$

and thus,

$$\omega^2 = \frac{1}{2}\Big\{ -[(\mu_1 - a_1)^2 + (\mu_2 - a_2)^2] \\ \pm \sqrt{[(\mu_1 - a_1)^2 + (\mu_2 - a_2)^2]^2 - 4[(\mu_1 - a_1)^2(\mu_2 - a_2)^2 - b_1^2 b_2^2]} \Big\}. \tag{6}$$

Clearly, a real number ω does not exist when $|(\mu_1 - a_1)(\mu_2 - a_2)| \geq |b_1 b_2|$. This shows that (4) has no root on the imaginary axis. The conclusion (i) follows.

A real number ω satisfies (6) when $|(\mu_1 - a_1)(\mu_2 - a_2)| < |b_1 b_2|$. In this case, define

$$\omega_0 = \frac{1}{\sqrt{2}}\Big[-[(\mu_1 - a_1)^2 + (\mu_2 - a_2)^2] + \sqrt{[(\mu_1 - a_1)^2 - (\mu_2 - a_2)^2]^2 + 4b_1^2 b_2^2}\Big]^{\frac{1}{2}} \tag{7}$$

and

$$\bar{\tau}_j = \frac{1}{\omega_0}\Big[\arccos\frac{(\mu_1 - a_1)(\mu_2 - a_2) - \omega_0^2}{b_1 b_2} + 2j\pi\Big], \quad j = 0, 1, 2, \ldots. \tag{8}$$

Then $\pm i\omega_0$ is a pair of purely imaginary roots of (4) with $\tau = \bar{\tau}_j$. Since (4) with $\tau = 0$ has a root with positive real part when $b_1 b_2 > (\mu_1 - a_1)(\mu_2 - a_2)$, conclusion (ii) follows from Lemma 2.1.

Similarly, since the roots of (4) with $\tau = 0$ have negative real parts when $b_1 b_2 < (\mu_1 - a_1)(\mu_2 - a_2)$, and $\bar{\tau}_0$ is the first value of $\tau \geq 0$ such that (4) has a root on the imaginary axis, we know that conclusion (iii) follows from Lemma 2.1. □

Applying Lemmas 2.1, 2.2, and a result in Hale [12, Theorem 1.1, p. 147] , we have the following result.

Theorem 2.3. *Suppose that assumptions* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *are satisfied.*

(i) *If*

$$|b_1 b_2| \leq |(\mu_1 - a_1)(\mu_2 - a_2)| \quad \text{and} \quad 0 < (\mu_1 - a_1)(\mu_2 - a_2) \neq b_1 b_2,$$

then the zero solution of system (3) is absolutely stable, that is, the zero solution is asymptotically stable for all $\tau \geq 0$.

(ii) *If*

$$b_1 b_2 > (\mu_1 - a_1)(\mu_2 - a_2),$$

then the zero solution is unstable for all $\tau \geq 0$. *If*

$$b_1 b_2 > |(\mu_1 - a_1)(\mu_2 - a_2)|,$$

then there exist a sequence of values of τ, $\bar{\tau}_0 < \bar{\tau}_1 < \bar{\tau}_2 < \cdots$ *defined in (8), such that system (3) undergoes a Hopf bifurcation at the origin when* $\tau = \bar{\tau}_j$, $j = 0, 1, 2, \ldots$, *where* $\bar{\tau}_j$ *is defined in (8).*

(iii) *If*

$$b_1 b_2 < -|(\mu_1 - a_1)(\mu_2 - a_2)|,$$

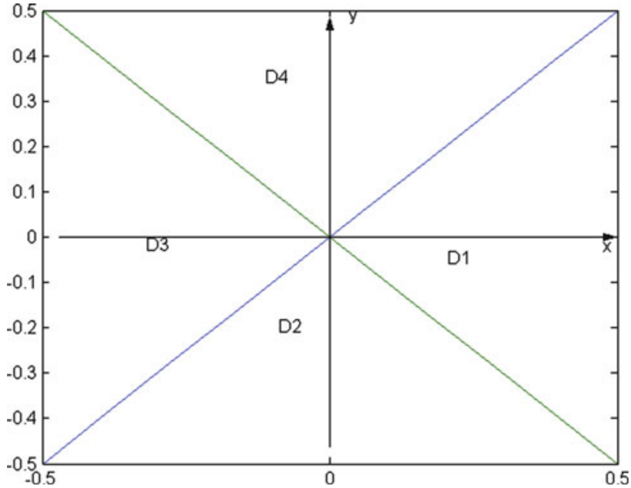

Fig. 1 Illustration of bifurcation sets. The horizontal axis is for values of $x=(\mu_1-a_1)(\mu_2-a_2)$, and vertical axis is for $y=b_1b_2$. The lines $b_1b_2=\pm(\mu_1-a_1)(\mu_2-a_2)$ divide the plane into four regions, D_1, D_2, D_3, and D_4. D_1 is an absolutely stable region, D_2 is a conditionally stable region, and $D_3\cup D_4$ is an unstable region

then, for the same sequence, $\bar\tau_0<\bar\tau_1<\bar\tau_2<\cdots$ defined in (8), the zero solution of system (3) is asymptotically stable when $\tau\in[0,\bar\tau_0)$ and unstable when $\tau>\bar\tau_0$, and system (3) undergoes a Hopf bifurcation at the origin when $\tau=\bar\tau_j,\ \ j=0,1,2,\ldots$.

The conclusions of Theorem 2.3 are illustrated in Fig. 1.

3 Direction and Stability of the Local Hopf Bifurcation at $\bar\tau_0$

In this section, we derive explicit formula for determining the direction and stability of the Hopf bifurcation at the first critical value $\bar\tau_0$, using the normal form and center manifold theory as presented in [13]. From Lemmas 2.1 and 2.2 we know that, if assumption $(\mathbf{H}_2)$ and condition

$$b_1b_2<-|(\mu_1-a_1)(\mu_2-a_2)|$$

are satisfied, then, at $\tau=\bar\tau_0$, all the roots of (4) except $\pm i\omega_0$ have negative real parts, and the transversality condition is satisfied.

We introduce the following change of variables:

$$y_1(t)=x_1(\tau t)\quad\text{and}\quad y_2(t)=x_2(\tau t).$$

Then system (3) becomes

$$\begin{cases} \dot{y}_1(t) = -\mu_1 \tau y_1(t) + a_1 \tau f_1(y_1(t)) + b_1 \tau g_1(y_2(t-1)), \\ \dot{y}_2(t) = -\mu_2 \tau y_2(t) + a_2 \tau f_2(y_2(t)) + b_2 \tau g_2(y_1(t)). \end{cases} \tag{9}$$

The characteristic equation associated with the linearization of system (9) at $(0,0)$ is

$$v^2 + \tau[(\mu_1 - a_1) + (\mu_2 - a_2)]v + \tau^2(\mu_1 - a_1)(\mu_2 - a_2) + \tau^2 b_1 b_2 e^{-v} = 0. \tag{10}$$

Comparing (10) and (4), we see that $v = \lambda\tau$. All the roots of (10) at $\tau = \bar{\tau}_0$ except $\pm i\bar{\tau}_0\omega_0$ have negative real parts, and the root of (10)

$$v(\tau) = \beta(\tau) + i\gamma(\tau)$$

with $\beta(\bar{\tau}_0) = 0$ and $\gamma(\bar{\tau}_0) = \bar{\tau}_0\omega_0$ satisfies

$$\beta'(\bar{\tau}_0) = \bar{\tau}_0\alpha'(\bar{\tau}_0).$$

For convenience of notation, we drop the bar in $\bar{\tau}_0$ and let $\tau = \tau_0 + \nu, \quad \nu \in \mathbb{R}$. Then $\nu = 0$ is a Hopf bifurcation value for system (9). Choose the phase space as $C = C([-1,0], \mathbb{R}^2)$. Under the assumption $(\mathbf{H}_1)$, system (9) can be rewritten as

$$\begin{cases} \dot{y}_1(t) = -(\tau_0 + \nu)(\mu_1 - a_1)y_1(t) \\ \qquad + (\tau_0 + \nu)a_1\left[\frac{f_1''(0)}{2}y_1^2(t) + \frac{f_1'''(0)}{6}y_1^3(t) + \cdots\right] \\ \qquad + (\tau_0 + \nu)b_1\left[y_2(t-1) + \frac{g_1''(0)}{2}y_2^2(t-1) + \frac{g_1'''(0)}{6}y_2^3(t-1) + \cdots\right], \\ \dot{y}_2(t) = -(\tau_0 + \nu)(\mu_2 - a_2)y_2(t) \\ \qquad + (\tau_0 + \nu)a_2\left[\frac{f_2''(0)}{2}y_2^2(t) + \frac{f_2'''(0)}{6}y_1^3(t) + \cdots\right] \\ \qquad + (\tau_0 + \nu)b_2\left[y_1(t) + \frac{g_2''(0)}{2}y_1^2(t) + \frac{g_2'''(0)}{6}y_1^3(t) + \cdots\right]. \end{cases} \tag{11}$$

For $\varphi \in C$, let

$$L_\nu\varphi = -B_0\varphi(0) + B_1\varphi(-1),$$

where

$$B_0 = \begin{bmatrix} (\tau_0 + \nu)(\mu_1 - a_1) & 0 \\ -(\tau_0 + \nu)b_2 & (\tau_0 + \nu)(\mu_1 - a_1) \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & -(\tau_0 + \nu)b_1 \\ 0 & 0 \end{bmatrix},$$

and

$$F(\nu,\phi)=\frac{\tau_0+\nu}{2}\begin{bmatrix}a_1f_1''(0)\varphi_1^2(0)+b_1g_1''(0)\varphi_2^2(-1)\\ a_2f_2''(0)\varphi_2^2(0)+b_2g_2''(0)\varphi_1^2(-1)\end{bmatrix}$$
$$+\frac{\tau_0+\nu}{6}\begin{bmatrix}a_1f_1'''(0)\varphi_1^3(0)+b_1g_1'''(0)\varphi_2^3(-1)\\ a_2f_2'''(0)\varphi_2^3(0)+b_2g_2'''(0)\varphi_1^3(-1)\end{bmatrix}+O(|\varphi|^4).$$

By the Riesz representation theorem, there exists matrix $\eta(\theta,\nu)$, whose components are functions of bounded variation in $\theta\in[-1,0]$, such that

$$L_\nu\varphi=\int_{-1}^{0}\mathrm{d}\eta(\theta,\nu)\varphi(\theta),\ \text{ for }\ \varphi\in C.$$

In fact, we can show

$$\eta(\theta,\nu)=\begin{cases}-B_0, & \theta=0,\\ -B_1\delta(\theta+1), & \theta\in[-1,0).\end{cases}$$

For $\varphi\in C^1([-1,0],\mathbb{R}^2)$, define

$$A(\nu)\varphi=\begin{cases}\dfrac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta(t,\nu)\varphi(t), & \theta=0,\end{cases}$$

and

$$R\varphi=\eta(\theta,\nu)=\begin{cases}0, & \theta\in[-1,0),\\ F(\nu,\varphi), & \theta=0.\end{cases}$$

We can rewrite (11) in the following form:

$$\dot{y}_t=A(\nu)y_t+R(\nu)y_t, \tag{12}$$

where $y=(y_1,y_2)^T,\ y_t=y(t+\theta)$ for $\theta\in[-1,0]$. For $\psi\in C^1([0,1],\mathbb{R}^2)$, we define

$$A^*\psi=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s\in(0,1],\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta^T(t,0)\psi(-t), & s=0.\end{cases}$$

For $\varphi\in C[-1,0]$ and $\psi\in[0,1]$, consider the bilinear form

$$\langle\psi,\varphi\rangle\ =\ \bar{\psi}(0)\varphi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\varphi(\xi)\mathrm{d}\xi,$$

where $\eta(\theta) = \eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. Since $\pm i\tau_0\omega_0$ are eigenvalues of $A(0)$, they are also eigenvalues of A^*.

By direct computation, we obtain that

$$q(\theta) = \begin{bmatrix} 1 \\ \frac{\mu_1 - a_1 + \mathrm{i}\omega_0}{b_1}\mathrm{e}^{\mathrm{i}\tau_0\omega_0} \end{bmatrix} \mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta}$$

is the eigenvector of $A(0)$ for $\mathrm{i}\tau_0\omega_0$, and

$$q^*(s) = D\begin{bmatrix} 1 \\ \frac{b_1}{\mu_2 - a_2 - \mathrm{i}\omega_0}\mathrm{e}^{\mathrm{i}\tau_0\omega_0} \end{bmatrix} \mathrm{e}^{\mathrm{i}\tau_0\omega_0 s}$$

is the eigenvector of A^* for $-\mathrm{i}\tau_0\omega_0$, where

$$D = \left[1 + \frac{\mu_1 - a_1 - \mathrm{i}\omega_0}{\mu_2 - a_2 - \mathrm{i}\omega_0} + \tau_0(\mu_1 - a_1 - \mathrm{i}\omega_0)\right]^{-1}.$$

Moreover, $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*, \bar{q}\rangle = 0$.

Using the same notations as in [13], we first compute the center manifold $\mathcal{C}_0$ at $\nu = 0$. Let y_t be the solution of system (11) with $\nu = 0$. Define

$$z(t) = \langle q^*, y_t\rangle \quad \text{and} \quad W(t,\theta) = y_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta).$$

Let z and $\bar{z}$ be the local coordinates for the center manifold $\mathcal{C}_0$ in the direction of $q^*$and $\bar{q^*}$. Then, on $\mathcal{C}_0$, we have $W(t,\theta) = W(z,\bar{z},\theta)$, where $W(z,\bar{z},\theta) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots$.

Note that W is real if y_t is real. We only consider real-valued solutions. For solution $y_t \in \mathcal{C}_0$ of system (11), since $\nu = 0$,

$$\begin{aligned} \dot{z}(t) &= \mathrm{i}\tau_0\omega_0 z + \langle q^*(s), F(W + 2\mathrm{Re}\{z(t)q(0)\})\rangle \\ &= \mathrm{i}\tau_0\omega_0 z + \bar{q^*}(0)F(W(z,\bar{z},0) + 2\mathrm{Re}\{zq(0)\}) \\ &\stackrel{def}{=} \mathrm{i}\tau_0\omega_0 z + \bar{q^*}(0)F_0(z,\bar{z}). \end{aligned}$$

We rewrite this as

$$\dot{z}(t) = \mathrm{i}\tau_0\omega_0 z + g(z,\bar{z}), \tag{13}$$

where

$$\begin{aligned} g(z,\bar{z}) &= \bar{q^*}(0)F(W(z,\bar{z},0) + 2\mathrm{Re}\{z(t)q(0)\}) \\ &= g_{20} + \frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \end{aligned}$$

From (12) and (13) we have

$$\dot{W} = \dot{y}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2\mathrm{Re}\{\bar{q^*}(0)F_0 q(\theta)\}, & \theta \in [-1,0), \\ AW - 2\mathrm{Re}\{\bar{q^*}(0)F_0 q(\theta)\} + F_0, & \theta = 0. \end{cases}$$

$$\stackrel{def}{=} AW + H(z,\bar{z},\theta),$$

where

$$H(z,\bar{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{14}$$

Expanding the above series and comparing the coefficients, we get

$$(A - 2i\tau_0\omega_0 I)W_{20} = -H_{20}(\theta),\ \ AW_{11}(\theta) = -H_{11}(\theta), \ldots. \tag{15}$$

Note

$$y_1(t) = W^{(1)}(0) + z + \bar{z},$$

$$y_2(t) = W^{(2)}(0) + \frac{\mu_1 - a_1 + \mathrm{i}\omega_0}{b_1}\mathrm{e}^{\mathrm{i}\tau_0\omega_0}z + \frac{\mu_1 - a_1 - \mathrm{i}\omega_0}{b_1}\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}\bar{z},$$

$$y_2(t-1) = W^{(2)}(-1) + \frac{\mu_1 - a_1 + \mathrm{i}\omega_0}{b_1}z + \frac{\mu_1 - a_1 - \mathrm{i}\omega_0}{b_1}\bar{z},$$

where

$$W^{(1)}(0) = W^{(1)}_{20}(0)\frac{z^2}{2} + W^{(1)}_{11}(0)z\bar{z} + W^{(1)}_{02}(0)\frac{\bar{z}^2}{2} + \cdots,$$

$$W^{(2)}(0) = W^{(2)}_{20}(0)\frac{z^2}{2} + W^{(2)}_{11}(0)z\bar{z} + W^{(2)}_{02}(0)\frac{\bar{z}^2}{2} + \cdots,$$

$$W^{(2)}(-1) = W^{(2)}_{20}(-1)\frac{z^2}{2} + W^{(2)}_{11}(-1)z\bar{z} + W^{(2)}_{02}(-1)\frac{\bar{z}^2}{2} + \cdots,$$

and

$$F_0 = \frac{\tau_0}{2}\begin{bmatrix} a_1 f_1''(0)y_1^2(t) + b_1 g_1''(0)y_2^2(t-1) \\ a_2 f_2''(0)y_2^2(t) + b_2 g_2''(0)y_1^2(t) \end{bmatrix} + \frac{\tau_0}{6}\begin{bmatrix} a_1 f_1'''(0)y_1^3(t) + b_1 g_1'''(0)y_2^3(t-1) \\ a_2 f_2'''(0)y_2^3(t) + b_2 g_2'''(0)y_1^3(t), \end{bmatrix} + \cdots.$$

Denote

$$M_1 = \frac{\mu_1 - a_1 + \mathrm{i}\omega_0}{b_1} \quad \text{and} \quad M_2 = \frac{b_1}{\mu_1 - a_2 + \mathrm{i}\omega_0}.$$

Then

$$q^*(0) = D\begin{bmatrix} 1 \\ \bar{M}_2 e^{i\tau_0\omega_0} \end{bmatrix},$$

$$y_2(t) = W^2(0) + M_1 e^{i\tau_0\omega_0} z + \bar{M}_1 e^{-i\tau_0\omega_0}\bar{z},$$

$$y_2(t-1) = W^2(-1) + M_1 z + \bar{M}_1\bar{z},$$

and

$$F_0 = \tau^0 \begin{bmatrix} \frac{a_1}{2} f_1''(0)(W^{(1)}(0)+z+\bar{z})^2 + \frac{b_1}{2} g_1''(0)(W^{(2)}(-1)+M_1 z+\bar{M}_1\bar{z})^2 \\ +\frac{a_1}{6} f_1'''(0)(W^{(1)}(0)+z+\bar{z})^3 + \frac{b_1}{6} g_1'''(0)(W^{(2)}(-1)+M_1 z+\bar{M}_1\bar{z})^3 \\ \frac{a_2}{2} f_2''(0)(W^{(2)}(0)+M_1 e^{i\tau_0\omega_0} z+\bar{M}_1 e^{-i\tau_0\omega_0}\bar{z})^2 \\ +\frac{b_1}{2} g_2''(0)(W^{(1)}(0)+z+\bar{z})^2 \\ +\frac{a_2}{6} f_2'''(0)(W^{(2)}(0)+M_1 e^{i\tau_0\omega_0} z+\bar{M}_1 e^{-i\tau_0\omega_0}\bar{z})^3 \\ +\frac{b_2}{6} g_2'''(0)(W^{(1)}(0)+z+\bar{z})^3 \end{bmatrix}$$

$$+\cdots\cdots$$

$$= \tau^0 \begin{bmatrix} a_1 f_1''(0) + b_1 g_1''(0) M_1^2 \\ a_2 f_2''(0) M_1^2 e^{2i\tau_0\omega_0} + b_2 g_2''(0) \end{bmatrix} \frac{z^2}{2} + \tau_0 \begin{bmatrix} a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2 \\ a_2 f_2''(0)|M_1|^2 + b_2 g_2''(0) \end{bmatrix} z\bar{z}$$

$$+ \tau_0 \begin{bmatrix} a_1 f_1''(0) + b_1 g_1''(0)\bar{M}_1^2 \\ a_2 f_2''(0)\bar{M}_1^2 e^{-2i\tau_0\omega_0} + b_2 g_2''(0) \end{bmatrix} \frac{\bar{z}^2}{2}$$

$$+ \tau_0 \begin{bmatrix} a_1 f_1''(0)(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + b_1 g_1''(0)(2W_{11}^{(2)}(-1)M_1 \\ +W_{20}^{(2)}(-1)\bar{M}_1) + a_1 f_1'''(0) + b_1 g_1'''(0)|M_1|^2 M_1 \\ a_2 f_2''(0)(2W_{11}^{(2)}(0)M_1 e^{i\tau_0\omega_0} + W_{20}^{(2)}(0)\bar{M}_1 e^{-i\tau_0\omega_0}) + b_2 g_2''(0)(2W_{11}^{(1)}(0) \\ +W_{20}^{(1)}(0)) + a_2 f_2'''(0)|M_1|^2 M_1 e^{i\tau_0\omega_0} + b_2 g_2'''(0) \end{bmatrix} \frac{z^2\bar{z}}{2}$$

$$+\cdots\cdots.$$

Here

$$g(z,\bar{z}) = \bar{q}^*(0)F_0 = \bar{D}(1, M_2 e^{-i\tau_0\omega_0})F_0$$

$$= \bar{D}\tau_0\Big[(a_1 f_1''(0) + b_1 g_1''(0)M_1^2 + a_2 f_2''(0)M_1^2 M_2 e^{i\tau_0\omega_0} + b_2 M_2 g_2''(0)e^{-i\tau_0\omega_0})\frac{z^2}{2}$$

$$+ (a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2 + a_2 f_2''(0)|M_1|^2 M_2 e^{-i\tau_0\omega_0} + b_2 M_2 g_2''(0)e^{-i\tau_0\omega_0})z\bar{z}$$

$$+ (a_1 f_1''(0) + b_1 g_1''(0)\bar{M}_1^2 + a_2 f_2''(0)\bar{M}_1^2 M_2 e^{-3i\tau_0\omega_0} + b_2 M_2 g_2''(0)e^{-i\tau_0\omega_0})\frac{\bar{z}^2}{2}$$

$$+(a_1 f_1''(0)(2W_{11}^{(1)}(0)+W_{20}^{(1)}(0))+b_1 g_1''(0)(2W_{11}^{(2)}(-1)M_1+W_{20}^{(2)}(-1)\bar{M}_1)$$
$$+a_1 f_1'''(0)+b_1 g_1'''(0)|M_1|^2 M_1+a_2 f_2''(0)M_2(2W_{11}^{(2)}(0)M_1+W_{20}^{(2)}(0)\bar{M}_1 \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0})$$
$$+b_2 g_2''(0)M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}(2W_{11}^{(1)}(0)+W_{20}^{(1)}(0))+a_2 f_2'''(0)|M_1|^2 M_1 M_2$$
$$+b_2 g_2'''(0)M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0})\frac{z^2\bar{z}}{2}\Big]+\cdots\cdots.$$

This gives that

$$\begin{aligned}
g_{20} &= \tau_0\bar{D}[a_1 f_1''(0)+b_1 g_1''(0)M_1^2+a_2 f_2''(0)M_1^2 M_2 \mathrm{e}^{\mathrm{i}\tau_0\omega_0}+b_2 M_2 g_2''(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}];\\
g_{11} &= \tau_0\bar{D}[a_1 f_1''(0)+b_1 g_1''(0)|M_1|^2+a_2 f_2''(0)|M_1|^2 M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}+b_2 M_2 g_2''(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}];\\
g_{02} &= \tau_0\bar{D}[a_1 f_1''(0)+b_1 g_1''(0)\bar{M}_1^2+a_2 f_2''(0)\bar{M}_1^2 M_2 \mathrm{e}^{-3\mathrm{i}\tau_0\omega_0}+b_2 M_2 g_2''(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}];\\
g_{21} &= \tau_0\bar{D}[a_1 f_1''(0)(2W_{11}^{(1)}(0)+W_{20}^{(1)}(0))+b_1 g_1''(0)(2W_{11}^{(2)}(-1)M_1+W_{20}^{(2)}(-1)\bar{M}_1)\\
&\quad +a_1 f_1'''(0)+b_1 g_1'''(0)|M_1|^2 M_1\\
&\quad +a_2 f_2''(0)M_2(2W_{11}^{(2)}(0)M_1+W_{20}^{(2)}(0)\bar{M}_1 \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0})\\
&\quad +b_2 g_2''(0)M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}(2W_{11}^{(1)}(0)+W_{20}^{(1)}(0))+a_2 f_2'''(0)|M_1|^2 M_1 M_2\\
&\quad +b_2 g_2'''(0)M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}].
\end{aligned} \tag{16}$$

We need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}. Comparing the coefficients of

$$\begin{aligned}
H(z,\bar{z},\theta) &= -2\mathrm{Re}\{\bar{q}^*(0)F_0 q(\theta)\} = -g(z,\bar{z})q(\theta)-\bar{g}(z,\bar{z})\bar{q}(\theta)\\
&= -\left(g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+\cdots\right)q(\theta)\\
&\quad -\left(\bar{g}_{20}\frac{\bar{z}}{2}+\bar{g}_{11}z\bar{z}+\bar{g}_{02}\frac{z^2}{2}+\cdots\right)\bar{q}(\theta),
\end{aligned}$$

with those in (14), we obtain

$$H_{20}\theta = -g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta) \ \text{ and } \ H_{11}\theta = -g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta).$$

It follows from (15) that

$$\dot{W}_{20}(\theta) = 2\mathrm{i}\tau_0\omega_0 W_{20}(\theta)-g_{20}q(0)\mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta}-\overline{g}_{02}\overline{q}(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0\theta}.$$

Solving for $W_{20}(\theta)$ we obtain

$$W_{20}(\theta) = -\frac{\mathrm{i}g_{20}}{\tau_0\omega_0}q(0)\mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta}+\frac{\mathrm{i}\overline{g}_{02}}{3\tau_0\omega_0}\overline{q}(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0\theta}+E_1\mathrm{e}^{2\mathrm{i}\tau_0\omega_0\theta}. \tag{17}$$

Similarly,

$$W_{11}(\theta) = -\frac{\mathrm{i}g_{11}}{\tau_0\omega_0}q(0)\mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta} + \frac{\mathrm{i}\overline{g}_{11}}{\tau_0\omega_0}\overline{q}(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0\theta} + E_2,$$

where E_1 and E_2 are both two-dimensional constant vectors and can be determined by setting $\theta = 0$ in H. In fact, since

$$H(z,\bar{z},0) = -2\mathrm{Re}\{\bar{q}^*(0)F_0q(0)\} + F_0,$$

we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + \tau_0\begin{bmatrix} a_1f_1''(0) + b_1g_1''(0)M_1^2 \\ a_2f_2''(0)M_1^2\mathrm{e}^{2\mathrm{i}\tau_0\omega_0} + b_2g_2''(0)\end{bmatrix} \tag{18}$$

and

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + \tau_0\begin{bmatrix} a_1f_1''(0) + b_1g_1''(0)|M_1|^2 \\ a_2f_2''(0)M_1^2 + b_2g_2''(0)\end{bmatrix}.$$

From (15) and the definition of A, we have

$$\tau_0\begin{bmatrix} -\mu_1 + a_1 & 0 \\ b_1 & -\mu_2 + a_2\end{bmatrix}W_{20}(0) + \tau_0\begin{bmatrix} 0 & b_1 \\ 0 & 0\end{bmatrix}W_{20}(-1) = 2\mathrm{i}\tau_0\omega_0W_{20}(0) - H_{20}(0), \tag{19}$$

and

$$\tau_0\begin{bmatrix} -\mu_1 + a_1 & 0 \\ b_1 & -\mu_2 + a_2\end{bmatrix}W_{11}(0) + \tau_0\begin{bmatrix} 0 & b_1 \\ 0 & 0\end{bmatrix}W_{11}(-1) = -H_{11}(0).$$

Substituting (17) into (19) and noticing that

$$\tau_0\begin{bmatrix} -\mu_1 + a_1 - \mathrm{i}\omega_0 & b_1\mathrm{e}^{-\mathrm{i}\tau_0\omega_0} \\ b_2 & -\mu_2 + a_2 - \mathrm{i}\omega_0\end{bmatrix}q(0) = 0,$$

we have

$$\tau_0\begin{bmatrix} -\mu_1 + a_1 - 2\mathrm{i}\omega_0 & b_1\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0} \\ b_2 & -\mu_2 + a_2 - 2\mathrm{i}\omega_0\end{bmatrix}E_1 = -g_{20}q(0) - \bar{g}_{20}\bar{q}(0) - H_{20}(0).$$

Substituting (18) into this relation we get

$$\begin{bmatrix} -\mu_1 + a_1 - 2\mathrm{i}\omega_0 & b_1\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0} \\ b_2 & -\mu_2 + a_2 - 2\mathrm{i}\omega_0\end{bmatrix}E_1 = -\begin{bmatrix} a_1f_1''(0) + b_1g_1''(0)M_1^2 \\ a_2f_2''(0)M_1^2\mathrm{e}^{2\mathrm{i}\tau_0\omega_0} + b_2g_2''(0)\end{bmatrix}.$$

Solving the equation for $E_1 = (E_1^{(1)}, E_2^{(2)})$ we get $E_1 = (\frac{\triangle_1^{(1)}}{\triangle_1}, \frac{\triangle_1^{(2)}}{\triangle_1})$, where

$$\begin{aligned}
\triangle_1 &= (\mu_1 - a_1 + 2\mathrm{i}\omega_0)(\mu_2 - a_2 + 2\mathrm{i}\omega_0) - b_1 b_2 \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0},\\
\triangle_1^{(1)} &= (\mu_2 - a_2 + 2\mathrm{i}\omega_0)(a_1 f_1''(0) + b_1 g_1''(0) M_1^2)\\
&\quad + b_1 \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}(a_2 f_2''(0) M_1^2 \mathrm{e}^{2\mathrm{i}\tau_0\omega_0} + b_2 g_2''(0)),\\
\triangle_1^{(2)} &= (\mu_1 - a_1 + 2\mathrm{i}\omega_0)(a_2 f_2''(0) M_1^2 \mathrm{e}^{2\mathrm{i}\tau_0\omega_0} + b_2 g_2''(0))\\
&\quad + b_2(a_1 f_1''(0) + b_1 g_1''(0) M_1^2).
\end{aligned}$$

Similarly, we can get

$$\begin{bmatrix} -\mu_1 + a_1 & b_1 \\ b_2 & -\mu_2 + a_2 \end{bmatrix} E_2 = -\begin{bmatrix} a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2 \\ a_2 f_2''(0)|M_1|^2 + b_2 g_2''(0) \end{bmatrix},$$

and thus $E_2 = \left(\frac{\triangle_2^{(1)}}{\triangle_2}, \frac{\triangle_2^{(2)}}{\triangle_2}\right)$, where

$$\begin{aligned}
\triangle_2 &= (\mu_1 - a_1)(\mu_2 - a_2) - b_1 b_2,\\
\triangle_2^{(1)} &= (\mu_2 - a_2)(a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2)\\
&\quad + b_1(a_2 f_2''(0)|M_1|^2 + b_2 g_2''(0)),\\
\triangle_2^{(2)} &= (\mu_1 - a_1)(a_2 f_2''(0)|M_1|^2 + b_2 g_2''(0))\\
&\quad + b_2(a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2).
\end{aligned}$$

Based on the above analysis, we see that each g_{ij} in (16) can be determined by the parameters and delay in (3). Therefore, we can compute the following quantities:

$$\begin{aligned}
c_1(0) &= \frac{\mathrm{i}}{2\tau_0\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\
\nu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\tau_0\alpha'(\tau_0)},\\
\beta_2 &= 2\mathrm{Re}\{c_1(0)\},\\
T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \tau_0\nu_2\,\mathrm{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0},
\end{aligned}$$

which determine the properties of bifurcating periodic solutions at the critical value $\bar{\tau}_0$. More specifically, parameter ν_2 determines the direction of the Hopf bifurcation: if $\nu_2 > 0$ ($\nu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0$ ($\tau < \tau_0$); parameter β_2 determines the stability of the bifurcating periodic solutions: they are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and parameter T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

Consider a special case for system (2),

$$\begin{cases} \dot{x}_1(t) = -\mu x_1(t) + af(x_1(t)) + b_1 f(x_2(t-\tau_1)), \\ \dot{x}_2(t) = -\mu x_2(t) + af(x_2(t)) + b_2 f(x_1(t-\tau_2)). \end{cases} \tag{20}$$

When $a = 0$ and $b_1 = b_2$, system (20) has been studied in Chen and Wu [4].

We make the following assumptions.

(**P**) $f \in C^3,\ xf(x) > 0$ for $x \neq 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) \neq 0,\ \mu - a > 0$, and $b_1 b_2 < -(\mu - a)^2$.

Let

$$\tau_0 = \frac{1}{\omega_0} \arccos \frac{(\mu - a)^2 - \omega_0^2}{b_1 b_2},$$

and

$$\omega_0 = [-(\mu - a)^2 + |b_1 b_2|]^{\frac{1}{2}}.$$

We have the following result.

Theorem 3.1. *If the hypothesis (P) is satisfied, then there exists $\tau_0 > 0$ such that the zero solution of system (20) is asymptotically stable for $\tau \in (0, \tau_0]$, and unstable for $\tau > \tau_0$, and system (20) undergoes a Hopf bifurcation at the origin when $\tau = \tau_0$. Moreover, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by the sign of $f'''(0)$. In fact, if $f'''(0) < 0$ ($f'''(0) > 0$), then the Hopf bifurcation is supercritical (subcritical), and the bifurcating periodic solutions are orbitally asymptotically stable (unstable).*

The conclusions on stability of the zero solution and the existence of Hopf bifurcation follow from (iii) in Theorem 2.3. Using the fact that $f''(0) = 0$ and relation (16), we have

$$g_{20} = g_{11} = g_{02} = 0,$$

and

$$g_{21} = \tau_0 f'''(0)\bar{D}[a(1 + |M_1|^2 M_1 M_2) + b_1 |M_1| M_2 + b_2 M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}], \tag{21}$$

where

$$M_1 = \frac{\mu - a + \mathrm{i}\omega_0}{b_1}, \quad M_2 = \frac{b_1}{\mu - a + \mathrm{i}\omega_0}, \quad D = (2 + \tau_0(\mu - a - \mathrm{i}\omega_0))^{-1},$$

and

$$\mathrm{e}^{-\mathrm{i}\tau_0\omega_0} = \frac{[\mathrm{i}\omega_0 + (\mu - a)]^2}{b_1 b_2}.$$

Substituting $M_1,\ M_2, D$, and $e^{-\mathrm{i}\tau_0\omega_0}$ into (21), we obtain

$$\mathrm{Re}\{g_{21}\} = \frac{\tau_0}{\triangle}(1 + \frac{(\mu - a)^2 + \omega_0^2}{b_2^2})[\mu(2 + \tau_0(\mu - a)) + \tau_0\omega_0^2]f'''(0),$$

where

$$\triangle = [2+\tau_0(\mu-a)]^2+\omega_0^2.$$

Hence,

$$\beta_2 = 2\mathrm{Re}\{c_1(0)\} < 0\ (>0)\ \text{ when }\ f'''(0) < 0 \quad (>0),$$

and

$$\nu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\tau_0\alpha'(\tau_0)} > 0\ (<0)\ \text{ when }\ f'''(0) < 0 \quad (>0).$$

The conclusions of the theorem follow from the standard Hopf bifurcation results [13].

Example 3.2. Let $f(x) = \tanh(x)$ in (20); we arrive at the neural network model

$$\begin{cases} \dot{u}_1(t) = -\mu u_1(t) + a\tanh(u_1(t)) + b_1\tanh(u_2(t-\tau_1)), \\ \dot{u}_2(t) = -\mu u_2(t) + a\tanh(u_2(t)) + b_2\tanh(u_1(t-\tau_2)), \end{cases} \tag{22}$$

where $\mu, a, b_1, b_2, \tau_1 > 0$, and $\tau_2 > 0$ are all constants. Noting that $f'''(0) = -2$, by Theorems 2.3 and 3.1, we obtain the following result, which generalizes results in [1] and [20] where system (22) was investigated when $a = 0$.

Corollary 3.3. *Suppose $\mu - a > 0$ and $b_1b_2 < -(\mu-a)^2$. Then there exists $\tau_0 > 0$ such that the zero solution of (22) is asymptotically stable when $\tau \in [0,\tau_0)$ and unstable when $\tau > \tau_0$. Furthermore, (22) undergoes a supercritical Hopf bifurcation at the origin when $\tau = \tau_0$, and the bifurcating periodic solutions are orbitally asymptotically stable.*

Theorem 2.3 shows that under the assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, if

$$|b_1b_2| > |(\mu_1-a_1)(\mu_2-a_2)|$$

is satisfied, then there exists sequence

$$0 < \bar{\tau}_0 < \bar{\tau}_1 < \bar{\tau}_2 < \cdots < \bar{\tau}_j < \cdots$$

such that system (3) undergoes a Hopf bifurcation at the origin when $\tau = \bar{\tau}_j,\ j = 0,1,2,\ldots$. We have only investigated properties of the bifurcation at $\tau = \bar{\tau}_0$ when $b_1b_2 < 0$. Using a similar procedure, we can investigate the direction and stability of the Hopf bifurcations occurring at $\tau = \bar{\tau}_j$ for $j > 0$. In fact, for system (20), we can show that the Hopf bifurcations at $\tau = \bar{\tau}_j$ $(j \geq 0)$ are supercritical (resp. subcritical), with nontrivial periodic solution orbits stable (resp. unstable) on the center manifold if $f'''(0) < 0$ (resp. $f'''(0) > 0$).

4 Global Existence of Periodic Solutions

In this section, we show that periodic solutions of system (2) exist when the total delay $\tau = \tau_1 + \tau_2$ is away from the bifurcation points. We apply a global Hopf bifurcation theorem of Wu [7, 22] to establish global extension of local Hopf branches. A key step of the proof is to establish that system (2) has no periodic solutions of period 2τ. This is equivalent to show that a four-dimensional ordinary differential equation has no nonconstant periodic solutions. This will be done by applying high-dimensional Bendixson–Dulac criteria developed by Li and Muldowney [14], which we briefly describe in the following.

Consider an n-dimensional ordinary differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f \in C^1. \tag{23}$$

Let $x = x(t, x_0)$ be the solution to (23) such that $x(0, x_0) = x_0$. The second compound equation of (23) with respect to $x(t, x_0)$

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(x(t,x_0))z(t) \tag{24}$$

is a linear system of dimension $\binom{n}{2}$, where $\frac{\partial f}{\partial x}^{[2]}$ is the second additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$ [9, 17]. System (24) is said to be equi-uniformly asymptotically stable with respect to an open set $D \subset \mathbb{R}^n$, if it is uniformly asymptotically stable for each $x_0 \in D$, and the exponential decay rate is uniform for x_0 in each compact subset of D. The equi-uniform asymptotic stability of (24) implies the exponential decay of the surface area of any compact two-dimensional surface in D. If D is simply connected, this precludes the existence of any invariant simple closed rectifiable curve in D, including periodic orbits. In particular, the following result is proved in [15].

Proposition 4.1. *Let $D \subset \mathbb{R}^n$ be a simply connected open set. Assume that the family of linear systems*

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(x(t,x_0))z(t), \quad x_0 \in D$$

is equi-uniformly asymptotically stable. Then:

(a) *D contains no simple closed invariant curves including periodic orbits, homoclinic orbits, and heteroclinic cycles.*
(b) *Each semi-orbit in D converges to a simple equilibrium.*

In particular, if D is positively invariant and contains a unique equilibrium $\bar{x}$, then $\bar{x}$ is globally asymptotically stable in D.

The uniform asymptotic stability requirement for the family of linear systems (24) can be verified by constructing suitable Lyapunov functions. For instance, (24) is equi-uniformly asymptotically stable if there exists a positive definite function $V(z)$, such that $\frac{\mathrm{d}V(z)}{\mathrm{d}t}|_{(32)}$ is negative definite, and V and $\frac{\mathrm{d}V}{\mathrm{d}t}|_{(32)}$ are both independent of x_0.

For a 4×4 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

its second additive compound matrix $A^{[2]}$ is [14, 17],

$$A^{[2]} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}. \tag{25}$$

Consider the ODE system

$$\begin{cases} \dot{x}_1 = -\mu_1 x_1 + a_1 f_1(x_1) + b_1 g_1(x_4), \\ \dot{x}_2 = -\mu_2 x_2 + a_2 f_2(x_2) + b_2 g_2(x_1), \\ \dot{x}_3 = -\mu_1 x_3 + a_1 f_1(x_3) + b_1 g_1(x_2), \\ \dot{x}_4 = -\mu_2 x_4 + a_2 f_2(x_4) + b_2 g_2(x_3). \end{cases} \tag{26}$$

We make the following assumptions:

($\mathbf{H}_3$) There exists $L>0$ such that $|f_i(x)|\le L$ and $|g_i(x)|\le L$ for $x\in\mathbb{R}$ and $i=1,2$.

($\mathbf{H}_4$) There exist $\alpha_j>0,\ j=1,2,3,4,5$, such that

$$\begin{aligned} \sup_{x\in\mathbb{R}^4}\Big\{ & -(\mu_1+\mu_2-a_1 f_1'(x_1)-a_2 f_2'(x_2)) + \frac{\alpha_1}{\alpha_5}|b_1 g_1'(x_4)|, \\ & -(2\mu_1 - a_1 f_1'(x_1) - a_1 f_1'(x_3)) + \frac{\alpha_2}{\alpha_1}|b_1 g_1'(x_2)| + \alpha_2 |b_1 g_1'(x_4)|, \\ & -(\mu_1+\mu_2-a_1 f_1'(x_1)-a_2 f_2'(x_2)) + \frac{\alpha_3}{\alpha_2}|b_2 g_2'(x_3)|, \\ & -(\mu_1+\mu_2-a_1 f_1'(x_3)-a_2 f_2'(x_2)) + \frac{\alpha_4}{\alpha_2}|b_2 g_2'(x_1)|, \\ & -(2\mu_2 - a_2 f_2'(x_2) - a_2 f_2'(x_4)) + \frac{\alpha_5}{\alpha_3}|b_2 g_2'(x_1)| + \frac{\alpha_5}{\alpha_4}|b_2 g_2'(x_3)|, \\ & -(\mu_1+\mu_2-a_1 f_1'(x_3)-a_2 f_2'(x_4)) + \frac{1}{\alpha_5}|b_1 g_1'(x_2)| \Big\} < 0. \end{aligned} \tag{27}$$

Proposition 4.2. *Suppose that assumptions* $(\mathbf{H}_1)$*,* $(\mathbf{H}_3)$*, and* $(\mathbf{H}_4)$ *are satisfied. Then system (26) has no nonconstant periodic solutions. Furthermore, the unique equilibrium* $(0,0,0,0)$ *is globally asymptotically stable in* $\mathbb{R}^4$.

Proof. First of all, we verify that the solutions of (26) are uniformly ultimately bounded. Let

$$V(x_1,x_2,x_3,x_4)=\frac{1}{2}\left[x_1^2+x_2^2+x_3^2+x_4^2\right].$$

Then the derivative of V along a solution of (26) is

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(34)} &= -\mu_1x_1^2-\mu_2x_2^2-\mu_1x_3^2-\mu_2x_4^2\\ &\quad +a_1x_1f_1(x_1)+b_1x_1g_1(x_4)+a_2x_2f_2(x_2)+b_2x_2g_2(x_1)\\ &\quad +a_1x_3f_1(x_3)+b_1x_3g_1(x_2)+a_2x_4f_2(x_4)+b_2x_4g_2(x_3).\end{aligned}$$

Using $(\mathbf{H}_3)$ we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(34)} \le -\mu\,\Sigma_{i=1}^4 x_i^2+2aL\Sigma_{i=1}^4|x_i|,$$

where $\mu=\min\{\mu_1,\mu_2\}$, and $a=\max_{1\le i\le 2}\{|a_i|,|b_i|\}$. Then there exists $M>1$ such that $\left.\frac{dV}{dt}\right|_{(34)}<0$ for $\Sigma_{i=1}^4x_i^2\ge M^2$. As a consequence, solutions of (26) are uniformly ultimately bounded.

Let $x=(x_1,x_2,x_3,x_4)$ and

$$\begin{aligned}f(x)=(&-\mu_1x_1+a_1f_1(x_1)+b_1g_1(x_4),-\mu_2x_2+a_2f_2(x_2)+b_2g_2(x_1),\\ &-\mu_1x_3+a_1f_1(x_3)+b_1g_1(x_2),-\mu_2x_4+a_2f_2(x_4)+b_2g_2(x_3))^T.\end{aligned}$$

Then $\frac{\partial f}{\partial x}$ is given as follows:

$$\begin{bmatrix}-\mu_1+a_1f_1'(x_1) & 0 & 0 & b_1g_1'(x_4)\\ b_2g_2'(x_1) & -\mu_2+a_2f_2'(x_2) & 0 & 0\\ 0 & b_1g_1'(x_2) & -\mu_1+a_1f_1'(x_3) & 0\\ 0 & 0 & b_2g_2'(x_3) & -\mu_2+a_2f_2'(x_4)\end{bmatrix}.$$

By (25),

$$\frac{\partial f}{\partial x}^{[2]}(x)=(m_{ij})_{6\times 6}$$

with

$$\begin{aligned}m_{11}&=-(\mu_1+\mu_2)+a_1f_1'(x_1)+a_2f_2'(x_2),m_{12}=m_{13}=m_{14}=0,\\ m_{15}&=-b_1g_1'(x_4),\ m_{16}=0;\end{aligned}$$

$$m_{21} = b_1 g_1'(x_2),\ m_{22} = -2\mu_1 + a_1 f_1'(x_1) + a_1 f_1'(x_3),$$
$$m_{23} = m_{24} = m_{25} = 0,\ m_{26} = -b_1 g_1'(x_4);$$
$$m_{31} = 0,\ m_{32} = b_2 g_2'(x_3),\ m_{33} = -(\mu_1 + \mu_2) + a_1 f_1'(x_1) + a_2 f_2'(x_4),$$
$$m_{34} = m_{35} = m_{36} = 0;$$
$$m_{41} = 0,\ m_{42} = b_2 g_2'(x_1),\ m_{43} = 0,\ m_{44} = -(\mu_1 + \mu_2) + a_2 f_2'(x_2) + a_1 f_1'(x_3),$$
$$m_{45} = m_{46} = 0;$$
$$m_{51} = m_{52} = 0,\ m_{53} = b_2 g_2'(x_1),\ m_{54} = b_2 g_2'(x_3),$$
$$m_{55} = -2\mu_2 + a_2 f_2'(x_2) + a_2 f_2'(x_4),\ m_{56} = 0;$$
$$m_{61} = m_{62} = m_{63} = m_{64} = 0,\ m_{65} = b_1 g_1'(x_2),$$
$$m_{66} = -(\mu_1 + \mu_2) + a_1 f_1'(x_3) + a_2 f_2'(x_4).$$

The second compound system

$$\dot{Z} = \frac{\partial f}{\partial x}^{[2]}(x)Z, \qquad Z = (z_1, \ldots, z_6), \tag{28}$$

is

$$\begin{cases} \dot{z}_1 = -(\mu_1 + \mu_2 - a_1 f_1'(x_1(t)) - a_2 f_2'(x_2(t)))z_1 - b_1 g_1'(x_4(t))z_5, \\ \dot{z}_2 = b_1 g_1'(x_2(t))z_1 - (2\mu_1 - a_1 f_1'(x_1(t)) - a_1 f_1'(x_3(t)))z_2 - b_1 g_1'(x_4(t))z_6, \\ \dot{z}_3 = b_2 g_2'(x_3(t))z_2 - (\mu_1 + \mu_2 - a_1 f_1'(x_1(t)) - a_2 f_2'(x_4(t)))z_3, \\ \dot{z}_4 = b_2 g_2'(x_1(t))z_2 - (\mu_1 + \mu_2 - a_2 f_2'(x_2(t)) - a_1 f_1'(x_3(t)))z_4, \\ \dot{z}_5 = b_2 g_2'(x_1(t))z_3 + b_2 g_2'(x_3(t))z_4 - (2\mu_2 - a_2 f_2'(x_2(t)) - a_2 f_2'(x_4(t)))z_5, \\ \dot{z}_6 = b_1 g_1'(x_2(t))z_5 - (\mu_1 + \mu_2 - a_1 f_1'(x_3(t)) - a_2 f_2'(x_4(t)))z_6, \end{cases}$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ is a solution of system (26) with $x(0) = x_0 \in \mathbb{R}^4$. Set

$$W(t) = \max\{\alpha_1|z_1|, \alpha_2|z_2|, \alpha_3|z_3|, \alpha_4|z_4|, \alpha_5|z_5|, |z_6|\}.$$

Then direct calculation leads to the following inequalities:

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\alpha_1|z_1| \le -(\mu_1 + \mu_2 - a_1 f_1'(x_1(t)) - a_2 f_2'(x_2(t)))\alpha_1|z_1| + \frac{\alpha_1}{\alpha_5}|b_1 g_1'(x_4(t))|\alpha_5|z_5|,$$
$$\frac{\mathrm{d}^+}{\mathrm{d}t}\alpha_2|z_2| \le -(2\mu_1 - a_1 f_1'(x_1(t)) - a_1 f_1'(x_3(t)))\alpha_2|z_2|$$
$$+ \frac{\alpha_2}{\alpha_1}|b_1 g_1'(x_2(t))|\alpha_1|z_1| + \alpha_2|b_1 g_1'(x_4(t))||z_6|,$$

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\alpha_3|z_3| \le -(\mu_1+\mu_2-a_1f_1'(x_1(t))-a_2f_2'(x_4(t)))\alpha_3|z_3| + \frac{\alpha_3}{\alpha_2}|b_2g_2'(x_3(t))|\alpha_2|z_2|,$$

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\alpha_4|z_4| \le -(\mu_1+\mu_2-a_1f_1'(x_3(t))-a_2f_2'(x_2(t)))\alpha_4|z_4| + \frac{\alpha_4}{\alpha_2}|b_2g_2'(x_1(t))|\alpha_2|z_2|,$$

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\alpha_5|z_5| \le -(2\mu_2-a_2f_2'(x_2(t))-a_2f_2'(x_4(t)))\alpha_5|z_5| + \frac{\alpha_5}{\alpha_3}|b_2g_2'(x_1(t))|\alpha_3|z_3| + \frac{\alpha_5}{\alpha_4}|b_2g_2'(x_3(t))|\alpha_4|z_4|,$$

$$\frac{\mathrm{d}^+}{\mathrm{d}t}|z_6| \le -(\mu_1+\mu_2-a_1f_1'(x_3(t))-a_2f_2'(x_4(t)))|z_6| + \frac{1}{\alpha_5}|b_1g_1'(x_2(t))|\alpha_5|z_5|,$$

where $\frac{\mathrm{d}^+}{\mathrm{d}t}$ denotes the right-hand derivative. Therefore,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}W(Z(t)) \le \mu(t)W(Z(t)),$$

with

$$\begin{aligned}\mu(t) = \max\Big\{ &-(\mu_1+\mu_2-a_1f_1'(x_1(t))-a_2f_2'(x_2(t))) + \frac{\alpha_1}{\alpha_5}|b_1g_1'(x_4(t))|,\\ &-(2\mu_1-a_1f_1'(x_1(t))-a_1f_1'(x_3(t))) + \frac{\alpha_2}{\alpha_1}|b_1g_1'(x_2(t))| + \alpha_2|b_1g_1'(x_4(t))|,\\ &-(\mu_1+\mu_2-a_1f_1'(x_1(t))-a_2f_2'(x_2(t))) + \frac{\alpha_3}{\alpha_2}|b_2g_2'(x_3(t))|,\\ &-(\mu_1+\mu_2-a_1f_1'(x_3(t))-a_2f_2'(x_2)(t)) + \frac{\alpha_4}{\alpha_2}|b_2g_2'(x_1(t))|,\\ &-(2\mu_2-a_2f_2'(x_2(t))-a_2f_2'(x_4(t))) + \frac{\alpha_5}{\alpha_3}|b_2g_2'(x_1(t))| + \frac{\alpha_5}{\alpha_4}|b_2g_2'(x_3(t))|,\\ &-(\mu_1+\mu_2-a_1f_1'(x_3(t))-a_2f_2'(x_4(t))) + \frac{1}{\alpha_5}|b_1g_1'(x_2(t))|\Big\}.\end{aligned}$$

Thus, under assumption $(\mathbf{H}_4)$, and by the boundedness of solution to (26), there exists a $\delta > 0$ such that $\mu(t) \le -\delta < 0$, and hence

$$W(Z(t)) \le W(Z(s))\mathrm{e}^{-\delta(t-s)}, \quad t \ge s > 0.$$

This establishes the equi-uniform asymptotic stability of the second compound system (28), and hence the conclusions of Proposition 4.2 follow from Proposition 4.1. □

Now we are in the position to state the main result of this section.

Theorem 4.3. *Suppose that assumptions* $(\mathbf{H}_1)$ – $(\mathbf{H}_4)$ *and the condition*

$$|b_1 b_2| > |(\mu_1 - a_1)(\mu_2 - a_2)|$$

are satisfied. Let τ_j *be defined in (8).*

(i) *If* $b_1 b_2 > 0$*, then system (3) has at least* $j+1$ *nonconstant periodic solutions for* $\tau > \bar{\tau}_j,\ j \geq 0$*.*
(ii) *If* $b_1 b_2 < 0$*, then system (3) has at least* j *nonconstant periodic solutions for* $\tau > \bar{\tau}_j,\ j \geq 1$*.*

Proof. We regard (τ, p) as parameters and apply Theorem 3.3 in Wu [22]. By $(\mathbf{H}_1)$ we know that the origin is an equilibrium of system (3). Hence, $(0, \tau, p)$ is a stationary point of (3), and the corresponding characteristic function is

$$\Delta_{(0,\tau,p)}(\lambda) = \lambda^2 + [(\mu_1 - a_1) + (\mu_2 - a_2)]\lambda + (\mu_1 - a_1)(\mu_2 - a_2) - b_1 b_2 \mathrm{e}^{-\lambda\tau}.$$

Clearly, $\Delta_{(0,\tau,p)}(\lambda)$ is continuous in $(\tau, p, \lambda) \in R_+ \times R_+ \times C$. To locate centers, we consider

$$\Delta_{(0,\tau,p)}\left(\mathrm{i}\frac{2m\pi}{p}\right) = -\left(\frac{2m\pi}{p}\right)^2 + \mathrm{i}\,[(\mu_1 - a_1) + (\mu_2 - a_2)]\,\frac{2m\pi}{p}$$
$$+ (\mu_1 - a_1)(\mu_2 - a_2) - b_1 b_2 \mathrm{e}^{-\mathrm{i}\frac{2m\pi}{p}\tau}.$$

Using the conclusion (ii) in Lemma 2.2 we know that $(0, \tau, p)$ is a center if and only if $m = 1, \tau = \bar{\tau}_j$ and $p = \frac{2\tau}{\omega_0}$. In particular, $(0, \bar{\tau}_j, \frac{2\tau}{\omega_0})$ is a center, and all the centers are isolated. In fact, the set of centers is countable and can be expressed as

$$\left\{\left(0, \bar{\tau}_j, \frac{2\tau}{\omega_0}\right) \;:\; j = 0, 1, 2, \ldots\right\},$$

where ω_0 and $\bar{\tau}_j$ are defined in (7) and (8), respectively.

Consider $\Delta_{(0,\tau,p)}(\lambda)$ with $m = 1$. By Lemmas 2.1 and 2.2, for fixed j, there exist $\varepsilon, \delta > 0$ and a smooth curve $\lambda : (\bar{\tau}_j - \delta, \bar{\tau}_j + \delta) \to C$, such that $\Delta_{(0,\tau,p)}(\lambda(\tau)) = 0$, $|\lambda(\tau) - \mathrm{i}\omega_0| < \varepsilon$ for all $\tau \in (\bar{\tau}_j - \delta, \bar{\tau}_j + \delta)$, and

$$\lambda(\bar{\tau}_j) = \mathrm{i}\omega_0, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Re}\lambda(\tau)|_{\tau=\bar{\tau}_j} > 0.$$

Let

$$\Omega_\varepsilon = \left\{(v, p) \;:\; 0 < v < \varepsilon,\ \left|p - \frac{2\tau}{\omega_0}\right| < \varepsilon\right\}.$$

Clearly, if $|\tau - \bar{\tau}_j| < \delta$ and $(v, p) \in \partial\Omega_\varepsilon$ such that $q(v + \mathrm{i}\frac{2\pi}{p}) = 0$, then $\tau = \bar{\tau}_j, v = 0$, and $p = \frac{2\pi}{\omega_0}$. This verifies the hypothesis (A_4) for $m = 1$ in Theorem 3.3 of Wu [22]. Moreover, if we set

$$H_m^{\pm}\left(0,\bar{\tau}_j,\frac{2\tau}{\omega_0}\right)(v,p)=\Delta_{(0,\bar{\tau}_j\pm\delta,p)}\left(v+\mathrm{i}m\frac{2\pi}{p}\right),$$

then, at $m=1$, we have

$$\gamma_m\left(0,\bar{\tau}_j,\frac{2\tau}{\omega_0}\right)=\deg_B\left(H_m^{-}\left(0,\bar{\tau}_j,\frac{2\tau}{\omega_0}\right),\Omega_\varepsilon\right)-\deg_B\left(H_m^{+}\left(0,\bar{\tau}_j,\frac{2\tau}{\omega_0}\right),\Omega_\varepsilon\right)$$
$$=-1. \tag{29}$$

If (3) has another equilibrium, say (x_1^*,x_2^*), then the characteristic equation associated with the linearization of (3) at (x_1^*,x_2^*) is

$$\lambda^2+[(\mu_1-a_1f_1'(x_1^*))+(\mu_2-a_2f_2'(x_2^*))]\lambda$$
$$+(\mu_1-a_1f_1'(x_1^*))(\mu_2-a_2f_2'(x_2^*))-b_1b_2g_1'(x_2^*)g_2'(x_1^*)\mathrm{e}^{-\lambda\tau}=0. \tag{30}$$

Suppose that equation (30) has a pair of purely imaginary roots $\pm\mathrm{i}\omega^*$ when $\tau=\tau^*$. Denote

$$\lambda(\tau)=\alpha^*(\tau)+\mathrm{i}\omega^*(\tau)$$

be the root of (30) satisfying $\alpha^*(\tau^*)=0$ and $\omega^*(\tau^*)=\omega^*$. Similar to Lemma 2.1, we have

$$\left.\frac{\mathrm{d}\alpha^*(\tau)}{\mathrm{d}\tau}\right|_{\tau=\tau^*}>0.$$

Similar to the discussion above, we know $((x_1^*,x_2^*),\tau^*,\frac{2\pi}{\omega^*})$ is an isolate center of (3), and the crossing number, at $m=1$, is

$$\gamma_m\left(\left(x_1^*,x_2^*\right),\tau^*,\frac{2\pi}{\omega^*}\right)=-1.$$

Let

$$\Sigma=\mathrm{cl}\{(x,\tau,p)\ :\ x \text{ is a } p\text{-periodic solution of (3)}\}.$$

By Theorem 3.3 in [22], we conclude that the connected component $\mathcal{C}(0,\bar{\tau}_j,\frac{2\pi}{\omega_0})$ through $(0,\bar{\tau}_j,\frac{2\pi}{\omega_0})$ in Σ is nonempty. Meanwhile, (29) and (30) imply that the first crossing number of each center is always -1. Therefore, we conclude that $\mathcal{C}(0,\bar{\tau}_j,\frac{2\pi}{\omega_0})$ is unbounded by Theorem 3.3 of [22].

Now, we prove that periodic solutions of (3) are uniformly bounded. Let

$$\mu=\min\{\mu_1,\mu_2\},\quad M\geq\max\{1,\ L(|a_1+b_1|+|a_2+b_2|)/\mu\},$$

and

$$r(t)=\sqrt{x_1^2(t)+x_2^2(t)}.$$

Differentiating $r(t)$ along a solution of (3) we have

$$\begin{aligned}\dot{r}(t) &= \frac{1}{r(t)}[x_1(t)\dot{x}_1(t)+x_2(t)\dot{x}_2(t)]\\ &= \frac{1}{r(t)}[-(\mu_1 x_1^2(t)+\mu_2 x_2^2(t))+a_1x_1(t)f_1(x_1(t))+b_1x_1(t)g_1(x_2(t-\tau))\\ &\quad +a_2x_2(t)f_2(x_2(t))+b_2x_2(t)g_2(x_1(t))]\\ &\le \frac{1}{r(t)}[-\mu(x_1^2(t)+x_2^2(t))+L(|a_1+b_1||x_1(t)|+|a_2+b_2||x_2(t)|)].\end{aligned}$$

If there exists $t_0>0$ such that $r(t_0)=A\ge M$, we have

$$\dot{r}(t_0)\le \frac{1}{A}[-\mu A^2+AL(|a_1+b_1|+|a_2+b_2|)]=-\mu A+L(|a_1+b_1|+|a_2+b_2|)<0.$$

It follows that if $x(t)=(x_1(t),x_2(t))^T$ is a periodic solution of (3), then $r(t)<M$ for all t. This shows that the periodic solutions of (3) are uniformly bounded.

Next, we establish that system (3) has no 2τ-periodic solutions. Suppose $x(t)=(x_1(t),x_2(t))^T$ is a 2τ-periodic solution of system (3). Let

$$x_3(t)=x_1(t-\tau),\quad x_4(t)=x_2(t-\tau).$$

Then $(x_1(t),x_2(t),x_3(t),x_4(t)))$ is a nonconstant periodic solution to system (26). This contradicts to the conclusion of Proposition 4.2 and implies that system (3) has no 2τ-periodic solutions.

By the definition of $\bar{\tau}_j$ in (8), we have that $\bar{\tau}_j<\bar{\tau}_{j+1},\ j\ge 0$, and

$$\omega_0\bar{\tau}_0=\arcsin\left(-\frac{[(\mu_1-a_1)+(\mu_2-a_2)]\omega_0}{b_1b_2}\right)\in(\pi,2\pi),$$

when $b_1b_2>0$. Hence, $\frac{2\pi}{\omega_0}<2\bar{\tau}_0$. Thus, there exists an integer m such that $\frac{2\bar{\tau}_0}{m+1}<\frac{2\pi}{\omega_0}<\frac{2\bar{\tau}_0}{m}$. Since system (3) has no 2τ-periodic solutions, it has no $\frac{2\tau}{n}$-periodic solutions for any integer n. This implies that the period p of a periodic solution on the connected component $\mathcal{C}(0,\bar{\tau}_0,\frac{2\pi}{\omega_0})$ satisfies $\frac{2\tau}{m+1}<p<\frac{2\tau}{m}$. Therefore, the periods of the periodic solutions of system (3) on $\mathcal{C}(0,\bar{\tau}_0,\frac{2\pi}{\omega_0})$ are uniformly bounded for $\tau\in[0,\bar{\tau})$, where $\bar{\tau}$ is fixed.

The inequality (27) implies that

$$-[(\mu_1-a_1f_1'(x))+(\mu_2-a_2f_2'(x))]<0 \ \text{ for } \ (x_1,x_2)\in\mathbb{R}^2,$$

and hence

$$\begin{aligned}&\frac{\partial}{\partial x_1}[-\mu_1x_1+a_1f_1(x_1)+b_1g_1(x_2)]+\frac{\partial}{\partial x_2}[-\mu_2x_2+a_2f_2(x_2)+b_2g_2(x_1)]\\ &\quad=-[(\mu_1-a_1f_1'(x_1)+(\mu_2-a_2f_2'(x_2))]<0\end{aligned}$$

for all $(x_1,x_2) \in \mathbb{R}^2$. This shows that system (3) with $\tau = 0$ has no nonconstant periodic solutions, by the classical Bendixson's criterion. Thus, the projection of $\mathcal{C}(0,\bar{\tau}_0,\frac{2\pi}{\omega_0})$ onto the τ-space must be an interval $[T,\infty)$ with $0 < T \le \bar{\tau}_0$. This shows that for any $\tau > \bar{\tau}_0$, system (3) has at least one nonconstant periodic solution on $\mathcal{C}(0,\bar{\tau}_0,\frac{2\pi}{\omega_0})$.

Similarly, we can show that, for any $\tau > \bar{\tau}_j,\ j \ge 1$, system (3) has at least one nonconstant periodic solution on $\mathcal{C}(0,\bar{\tau}_j,\frac{2\pi}{\omega_0})$. Therefore, for any $\tau > \bar{\tau}_j$, system (3) has at least $j+1$ nonconstant periodic solutions in the case of $b_1b_2 > 0$. The proof of (i) is complete.

The proof of (ii) is similar and is omitted. □

Example 4.4. Consider the neural network model

$$\begin{cases} \dot{u}_1(t) = -\mu u_1(t) + a\tanh(u_1(t)) + b_1\tanh(u_2(t-\tau_1)), \\ \dot{u}_2(t) = -\mu u_2(t) + a\tanh(u_2(t)) + b_2\tanh(u_1(t-\tau_2)). \end{cases} \tag{31}$$

For $j = 0,1,2,\ldots$, let

$$\bar{\tau}_j = \frac{1}{\omega_0}\left[\arccos\frac{(\mu-a)^2-\omega_0^2}{b_1b_2} + 2j\pi\right], \quad j = 0,1,2,\ldots,$$

and

$$\omega_0 = [-(\mu-a)^2 + |b_1b_2|]^{\frac{1}{2}}.$$

We have the following result.

Corollary 4.5. *Suppose that $a > 0,\ b_1b_2 > (\mu-a)^2$ and*

$$\mu - a > \max\left\{|b_1|/\sqrt{2},\ |b_2|/\sqrt{2}\right\}. \tag{32}$$

Then for any $\tau > \bar{\tau}_j\ j = 0,1,2,\ldots$, system (31) has at least $j+1$ nonconstant periodic solutions.

It is sufficient to verify that $(\mathbf{H}_4)$ is satisfied. Noting that $f_1 = f_2 = g_1 = g_2 = \tanh$ and $0 < \tanh'(x) \le 1$ and taking $\alpha_1 = \alpha_3 = \alpha_4 = 1$, we have

$$-(2\mu - a\tanh'(x_1) - a\tanh'(x_2)) + \frac{1}{\alpha_5}|b_1\tanh'(x_4)| \le -2(\mu-a) + \frac{1}{\alpha_5}|b_1|,$$

$$-(2\mu - a\tanh'(x_1) - a\tanh'(x_3)) + \alpha_2|b_1\tanh'(x_2)| + \alpha_2|b_1\tanh'(x_4|$$
$$\le -2(\mu-a) + 2\alpha_2|b_1|,$$

$$-(2\mu - a\tanh'(x_1) - a\tanh'(x_2)) + \frac{1}{\alpha_2}|b_2\tanh'(x_3)| \le -2(\mu-a) + \frac{1}{\alpha_2}|b_2|,$$

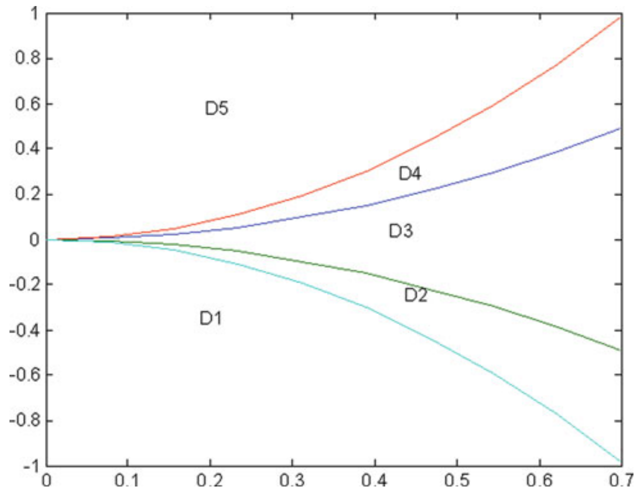

Fig. 2 The curves $b_1b_2 = \pm(\mu-a)^2$ and $b_1b_2 = \pm 2(\mu-a)^2$ divide the right half plane into five regions, D_1, D_2, D_3, D_4, and D_5. D_3 is an absolutely stable region, $D_1 \cup D_2$ is a conditionally stable region, and $D_4 \cup D_5$ is an unstable region. Values of $\mu - a$ are plotted on the horizontal axis and b_1b_2 plotted on the vertical axis. If $(\mu - a, b_1b_2) \in D_4$ (resp. D_2) and (32) is satisfied, then system (31) has at least one nonconstant periodic solution for $\tau > \bar{\tau}_0$ (resp. $\tau > \bar{\tau}_1$)

$$
\begin{aligned}
&-(2\mu - a\tanh'(x_2) - a\tanh'(x_3)) + \frac{1}{\alpha_2}|b_2\tanh'(x_1)| \le -2(\mu-a) + \frac{1}{\alpha_2}|b_2|,\\
&-(2\mu - a\tanh'(x_2) - a\tanh'(x_4)) + \alpha_5|b_2\tanh'(x_1)| + \alpha_5|b_2\tanh'(x_3|\\
&\quad \le -2(\mu-a) + 2\alpha_5|b_2|,\\
&-(2\mu - a\tanh'(x_3) - a\tanh'(x_4)) + \frac{1}{\alpha_5}|b_1\tanh'(x_2)| \le -2(\mu-a) + \frac{1}{\alpha_5}|b_1|.
\end{aligned}
$$

Let $\alpha_2 = \alpha_5 = \frac{1}{\sqrt{2}}$. Then (32) implies that

$$
\begin{aligned}
&-2(\mu-a) + \frac{1}{\alpha_5}|b_1| < 0, \qquad &&-2(\mu-a) + \alpha_2|b_1| < 0,\\
&-2(\mu-a) + \frac{1}{\alpha_2}|b_2| < 0, \qquad &&-2(\mu-a) + \alpha_5|b_2| < 0.
\end{aligned}
$$

Therefore, $(\mathbf{H}_4)$ is satisfied. The conclusion of Corollary 4.5 is illustrated in Fig. 2.

To demonstrate the Hopf bifurcation results in Theorems 3.1 and 4.3, we carry out numerical simulations on system (31). The simulations are done using Mathematica with different values of μ, a, b_i, and τ_i and different initial values for u_i. The simulations consistently show the bifurcating periodic solution being asymptotically stable and global existence of periodic solution: existence of periodic solutions for values $\tau = \tau_1 + \tau_2$ near $\bar{\tau}_0$ and far away from $\bar{\tau}_k$. In Fig. 3, we show one

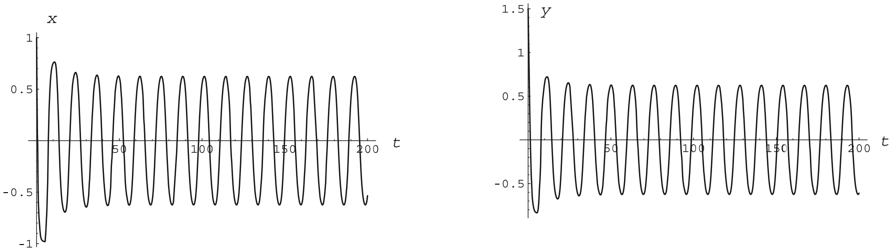

Fig. 3 Mathematical simulations of a periodic solution to system (31) with $\mu = 2$, $a = 0.8$, $b_1 = -1.2$, $b_2 = 1.5$, $\tau_1 = 2.8$, and $\tau_2 = 2.2$. The total delay $\tau = \tau_1 + \tau_2 = 5$ is greater than the first Hopf bifurcation value $\bar{\tau}_0 = 3.6905$

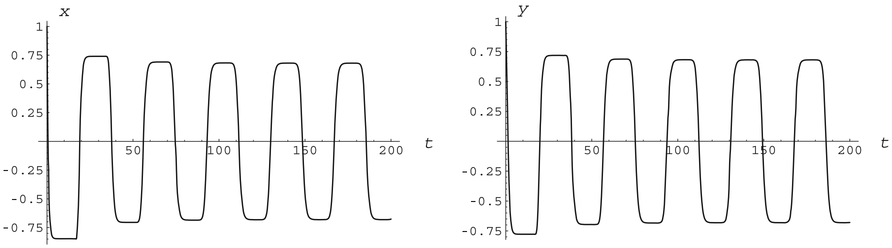

Fig. 4 Mathematical simulations show that an asymptotically stable periodic solution to system (31), with $\mu = 2$, $a = 0.8$, $b_1 = -1.2$, $b_2 = 1.5$, $\tau_1 = 9$, and $\tau_2 = 8$, continues to exist when the total delay $\tau = \tau_1 + \tau_2 = 17$ is between the two consecutive Hopf bifurcation values $\bar{\tau}_1 = 14.1625$ and $\bar{\tau}_2 = 24.6345$

of the simulations using $\mu = 2$, $a = 0.8, b_1 = -1, b_2 = 1.5$ such that (32) is satisfied and $(\mu - a, b_1 b_2) \in D_2$. In this case, it can be calculated that $\omega_0 = 0.6$ and for $k = 0, 1, 2, \ldots,$ $\bar{\tau}_k = 3.6905 + 10.472 \times k$. The delays are chosen as $\tau_1 = 2.8$, $\tau_2 = 2.2$ so that $\tau = \tau_1 + \tau_2 = 5$ is larger than $\bar{\tau}_0 = 3.6905$. An asymptotically stable periodic solution is shown to exist in Fig. 3. Similarly, in Fig. 4, the parameters μ, a, and b_i are chosen as above; the delays are chosen as $\tau_1 = 9$, $\tau_2 = 8$ so that $\tau = \tau_1 + \tau_2 = 17$ is between the two Hopf bifurcation values $\bar{\tau}_1 = 14.1625$ and $\bar{\tau}_2 = 24.6345$. An asymptotically stable periodic solution is shown in Fig. 4.

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Instability of Low Density Supersonic Waves of a Viscous Isentropic Gas Flow Through a Nozzle

Weishi Liu and Myunghyun Oh

This paper is dedicated to Professor George Sell on the occasion of his 70th birthday

Abstract In this work, we examine the stability of stationary non-transonic waves for viscous isentropic compressible flows through a nozzle with varying cross-section areas. The main result in this paper is, for small viscous strength, stationary supersonic waves with sufficiently low density are spectrally unstable; more precisely, we will establish the existence of positive eigenvalues for the linearization along such waves. The result is achieved via a center manifold reduction of the eigenvalue problem. The reduced eigenvalue problem is then studied in the framework of the Sturm–Liouville Theory.

Mathematics Subject Classification (2010): Primary 35B35; Secondary 76N17, 35L65

1 Introduction

In this work, we examine the stability of some stationary solutions for the viscous compressible gas flows through a nozzle. The model system is

$$
\begin{aligned}
(a\rho)_t + (a\rho u)_x &= \varepsilon(a\rho_x)_x \\
(a\rho u)_t + (a\rho u^2)_x + a(P(\rho))_x &= \varepsilon(a(\rho u)_x)_x,
\end{aligned} \tag{1}
$$

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where ρ, u, P and $a = a(x)$ are the density, velocity, pressure of the gas and the area of the cross section at x of the rotationally symmetric tube of the nozzle. The pressure P is assumed to be a given function of the density ρ. Our assumptions on the pressure P is:

$$\begin{aligned}
&\textit{For } \rho > 0,\ P(\rho) > 0 \textit{ and } P'(\rho) > 0;\\
&\lim_{\rho\to 0^+} \frac{P'(\rho)}{\rho} = \infty,\ \lim_{\rho\to 0^+} \rho^2 P'(\rho) = \lim_{\rho\to 0^+} \frac{\rho^8 P'''(\rho)}{P'(\rho)} = \lim_{\rho\to 0^+} \frac{\rho^9 P^{(4)}(\rho)}{P'(\rho)} = 0; \qquad (2)\\
&\textit{There exists } M > 0 \textit{ such that for } \rho > 0,\ \left|\frac{\rho P''(\rho)}{P'(\rho)}\right| \le M.
\end{aligned}$$

For polytropic gas, $P(\rho) = A\rho^\gamma$ with some constant $A > 0$ and $1 \le \gamma \le 5/3$ clearly satisfies the above assumption. Our assumption on $a(x)$ is

$$\lim_{x\to\pm\infty} a(x) = a_\pm > 0 \quad \text{and} \quad \lim_{x\to\pm\infty} a_x(x) = \lim_{x\to\pm\infty} a_{xx}(x) = 0. \qquad (3)$$

The inviscid ($\varepsilon = 0$) system (1) is a well-known one-dimensional Euler equation describing the motion of isentropic compressible fluid through a narrow nozzle with variable cross-section area (see [1–3, 10–14, 16–18] etc.). The specific form of the viscous terms is not completely physical and should be regarded as an artificial one. The study of the problem with the more physical viscosity is an ongoing project.

In [13], Liu studied global solutions of the initial value problem for general quasi-linear strictly hyperbolic systems including the inviscid isentropic compressible gas flow:

$$w_t + f(w)_x = g(x,w).$$

Roughly speaking, it was shown that, for an initial data $w_0(x)$, if all eigenvalues $\lambda_j(w)$ of f_w are nonzero and the L^1-norm of g and g_w are small for w uniformly close to w_0, then a global solution exists and tends pointwise to a steady-state solution. For polytropic gas flow, the main assumptions become the flow at $t = 0$ is not close to transonic and that the total variation of the cross-section area $a(x)$ is sufficiently small.

Liu then focused on *transonic* waves of gas flow in a nozzle of varying area via the inviscid $\varepsilon = 0$ model (1) in [14]. Various types of solutions were shown to exist that demonstrated significant qualitative differences between a contracting nozzle (e.g., $a_x(x) < 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin (0,1)$) and an expanding nozzle (e.g., $a_x(x) > 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin (0,1)$). Asymptotic states along a nozzle that contracts and then expands ($a_x(x) < 0$ for $-1 < x < 0$, $a_x(x) > 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin [-1,1]$) are also examined to exhibit a number of interesting phenomena including the choking phenomenon.

In [10], Hsu and Liu studied a singular Sturm–Liouville problem

$$\varepsilon u_{xx} = f(x,u)_x - h(x)g(u) \qquad (4)$$

and applied the result to the viscous steady-state problem of (1) with more physical viscosity terms. Assuming the nozzle is uniform outside a bounded portion and is either contracting or expanding otherwise, they reformulated the problem as a boundary value problem and gave a detailed analysis on the existence, multiplicity and uniqueness of solutions. Viewing solutions of the boundary value problem as steady-states of the corresponding reaction-diffusion equation

$$u_t = \varepsilon u_{xx} - f(x,u)_x + h(x)g(u),$$

stability results were also obtained.

Recently, in [6,7], Hong, et al. studied the steady-state problem of system (1) with different choices of the viscosity and for expanding-contracting ($a_x > 0$ for $x < 0$ and $a_x < 0$ for $x > 0$) and contracting-expanding ($a_x < 0$ for $x < 0$ and $a_x > 0$ for $x > 0$) nozzles. They applied the geometric singular perturbation theory to provide a rather complete description of stationary waves for both the inviscid and viscous systems, and found classes of new types of transonic waves. In particular, transonic waves from subsonic to supersonic are constructed for contracting-expanding nozzle. In [8], the maximal sub-to-super transonic wave is shown to be linearly stable.

In this paper, we will conduct a case study and examine the stability of stationary non-transonic waves—simplest steady states [6, 7]. Our main result is that supersonic waves with sufficiently low density are spectrally *unstable* as long as $a_x(x)$ changes sign; more precisely, we will establish the existence of positive eigenvalues for the linearization along such waves (see Theorem 5.1). Our result is directly relevant to the stability result in [13] where Liu constructed global solutions for quasilinear hyperbolic systems and studied their asymptotic behaviors. In particular, under some conditions, he established the stability of supersonic and subsonic waves. There seems to be a contradiction between T.P. Liu's stability result with our instability results. Our explanation lies in the following two reasons: first of all, the conditions under which the stability result of T.P. Liu do not hold for supersonic waves with sufficiently low density, secondly, T.P. Liu considered hyperbolic systems and we have the viscosity terms. An interesting observation is that, a certain form of viscosity might cause stable waves for inviscid flows to be unstable.

Our instability result relies on a center manifold reduction of the eigenvalue problem. The reduced eigenvalue problem turns out to be a quadratic eigenvalue problem and it is then studied via the Sturm–Liouville theory. The method of center manifold reduction in stability study has been applied by others (see, e.g., [15]).

The rest of the paper is organized as follows. We recall, in Sect. 2, the relevant existence results on stationary waves from [6,7]. In Sect. 3, we set up the eigenvalue problem and make a center manifold reduction of the eigenvalue problem. Section 4 focuses on the $\varepsilon = 0$ limiting reduced eigenvalue problem and provides a symmetric structure of eigenvalues for symmetric nozzles. We then show that supersonic waves with sufficiently low density are spectrally unstable in Sect. 5 to complete the paper.

2 Steady-State Problem

We recall the relevant existence result on stationary non-transonic waves from [6,7] with a slight extension. To distinguish the variables from those of the linearization, we use $\bar\rho$, etc. for the stationary solutions. We introduce new variables

$$\bar w = \varepsilon a(\bar\rho\bar u)_x - a(\bar\rho\bar u^2 + P(\bar\rho)), \quad \bar v = \varepsilon a\bar\rho_x - a\bar\rho\bar u. \tag{5}$$

The steady-state system of (1) becomes,

$$\begin{cases} \varepsilon\bar\rho_x = \bar v a^{-1} + \bar\rho\bar u, \\ \bar v_x = 0, \\ \varepsilon\bar u_x = a^{-1}\bar\rho^{-1}(\bar w - \bar v\bar u + aP(\bar\rho)), \\ \bar w_x = -a_x P(\bar\rho). \end{cases} \tag{6}$$

Note that $\bar v = \bar v(\varepsilon)$ is constant. We also introduce a variable $\eta \in (-1,1)$ via $\eta_x = 1 - \eta^2$. It is obvious that $\eta(x)$ is increasing in x and $\eta(\pm\infty) = \pm 1$. This well-known trick allows one to replace the x-variable in $a(x)$ with $x(\eta)$ so that system (6) becomes an autonomous system

$$\begin{cases} \varepsilon\bar\rho_x = \bar v a^{-1} + \bar\rho\bar u, \\ \varepsilon\bar u_x = a^{-1}\bar\rho^{-1}(\bar w - \bar v\bar u + aP(\bar\rho)), \\ \bar w_x = -a_x P(\bar\rho), \\ \eta_x = 1 - \eta^2. \end{cases} \tag{7}$$

System (7) is the so-called *slow system*. In terms of the fast time $\xi = x/\varepsilon$, the corresponding *fast system* is

$$\begin{cases} \bar\rho_\xi = \bar v a^{-1} + \bar\rho\bar u, \\ \bar u_\xi = a^{-1}\bar\rho^{-1}(\bar w - \bar v\bar u + aP(\bar\rho)), \\ \bar w_\xi = -\varepsilon a_x P(\bar\rho), \\ \eta_\xi = \varepsilon(1 - \eta^2). \end{cases} \tag{8}$$

The limiting slow and fast systems are, respectively,

$$\begin{cases} 0 = \bar v a^{-1} + \bar\rho\bar u, \\ 0 = a^{-1}\bar\rho^{-1}(\bar w - \bar v\bar u + aP(\bar\rho)), \\ \bar w_x = -a_x P(\bar\rho), \\ \eta_x = 1 - \eta^2, \end{cases} \tag{9}$$

and

$$\begin{cases} \bar{\rho}_\xi = \bar{v}a^{-1} + \bar{\rho}\bar{u}, \\ \bar{u}_\xi = a^{-1}\bar{\rho}^{-1}(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})), \\ \bar{w}_\xi = 0, \\ \eta_\xi = 0. \end{cases} \tag{10}$$

The slow manifold $\mathcal{Z}_0$ is given by

$$\mathcal{Z}_0 = \{a\bar{\rho}\bar{u} + \bar{v} = 0 \text{ and } \bar{w} - \bar{v}\bar{u} + aP(\bar{\rho}) = 0\}.$$

For the linearization of system (10) along $\mathcal{Z}_0$, the eigenvalues r_1 and r_2 in the directions transversal to $\mathcal{Z}_0$ are those of

$$\begin{pmatrix} \bar{u} & \bar{\rho} \\ \bar{\rho}^{-1}P'(\bar{\rho}) & \bar{u} \end{pmatrix},$$

that is,

$$r_1(\bar{\rho}, \bar{u}, \bar{w}, \eta) = \bar{u} - \sqrt{P'(\bar{\rho})} \quad \text{and} \quad r_2(\bar{\rho}, \bar{u}, \bar{w}, \eta) = \bar{u} + \sqrt{P'(\bar{\rho})}.$$

Recall that $\sqrt{P'(\bar{\rho})}$ is the sound speed. Thus, $(\bar{\rho}, \bar{u})$ is called a supersonic (resp. sonic or subsonic) state at x if $\bar{u}(x) > \sqrt{P'(\bar{\rho}(x))}$ (resp. $\bar{u}(x) = \sqrt{P'(\bar{\rho}(x))}$ or $\bar{u}(x) < \sqrt{P'(\bar{\rho}(x))}$). Set

$$\begin{aligned} \mathcal{Z}_0^u &= \{(\bar{\rho}, \bar{u}, \bar{w}, \eta) \in \mathcal{Z}_0 : \bar{u} > \sqrt{P'(\bar{\rho})}, \ \bar{\rho} > 0, \ \eta \in [-1, 1]\}, \\ \mathcal{Z}_0^s &= \{(\bar{\rho}, \bar{u}, \bar{w}, \eta) \in \mathcal{Z}_0 : \bar{u} < \sqrt{P'(\bar{\rho})}, \ \bar{\rho} > 0, \ \eta \in [-1, 1]\}, \\ T &= \{(\bar{\rho}, \bar{u}, \bar{w}, \eta) \in \mathcal{Z}_0 : \bar{u} = \sqrt{P'(\bar{\rho})}, \ \bar{\rho} > 0, \ \eta \in [-1, 1]\}. \end{aligned}$$

Then $\mathcal{Z}_0 = \mathcal{Z}_0^s \cup T \cup \mathcal{Z}_0^u$. The portion $\mathcal{Z}_0^s$ is (normally) saddle and consists of subsonic states, $\mathcal{Z}_0^u$ is (normally) repelling and consists of supersonic states, and T is the set of turning points and consists of sonic states.

For the dynamics of the limiting slow flow on $\mathcal{Z}_0$, we differentiate

$$\bar{u} = -\bar{v}a^{-1}\bar{\rho}^{-1} \quad \text{and} \quad \bar{w} = -\bar{v}^2a^{-1}\bar{\rho}^{-1} - aP(\bar{\rho})$$

with respect to x and use system (9) to get

$$-aP'(\bar{\rho})\bar{\rho}_x + \bar{v}^2a^{-2}a_x\bar{\rho}^{-1} + \bar{v}^2a^{-1}\bar{\rho}^{-2}\bar{\rho}_x = 0.$$

Thus, the limiting slow dynamics on $\mathcal{Z}_0$ can be represented by the system

$$\bar{\rho}_x = \frac{\bar{v}^2a^{-3}a_x\bar{\rho}^{-1}}{P'(\bar{\rho}) - \bar{v}^2a^{-2}\bar{\rho}^{-2}}, \quad \eta_x = 1 - \eta^2. \tag{11}$$

It can be checked directly (see [6,7]) that system (11) has an integral

$$I(\bar\rho,\eta)=E(\bar\rho)+\frac{\bar v^2}{2a^2(x(\eta))\bar\rho^2}\ \text{ where }\ E(\bar\rho)=\int_{\bar\rho_0}^{\bar\rho}\frac{P'(s)}{s}\,\mathrm{d}s. \tag{12}$$

By an inviscid non-transonic wave, we mean, for each fixed $\bar v<0$, a solution $(\bar\rho(x),\bar u(x),\bar w(x),\eta(x))$ of system (9) so that $r_j\neq 0$ for $j=1,2$. By a viscous profile of $(\bar\rho(x),\bar u(x),\bar w(x),\eta(x))$, we mean, for $\bar v(\varepsilon)\to\bar v$ as $\varepsilon\to 0$, a solution $(\bar\rho(x;\varepsilon),\bar u(x;\varepsilon),\bar w(x;\varepsilon),\eta(x))$ of system (7) so that

$$(\bar\rho(x;\varepsilon),\bar u(x;\varepsilon),\bar w(x;\varepsilon),\eta(x))\to(\bar\rho(x),\bar u(x),\bar w(x),\eta(x))$$

as $\varepsilon\to 0$ in L^1_{loc}.

The following result can be readily obtained. For more complete results and proofs, we refer the readers to the papers [6,7].

Theorem 2.1. *Fix $\bar v<0$. For any $\bar\rho_-$ and $\bar\rho_+$, there is an inviscid non-transonic wave $(\bar\rho(x),\bar u(x))$ for system (9) or (11) with $a(x)\bar\rho(x)\bar u(x)+\bar v=0$ and $\bar\rho(x)\to\bar\rho_\pm$ as $x\to\pm\infty$ if and only if*

$$\int_{\bar\rho_0}^{\bar\rho_-}\frac{P'(s)}{s}\,\mathrm{d}s+\frac{\bar v^2}{2a_-^2\bar\rho_-^2}=\int_{\bar\rho_0}^{\bar\rho_+}\frac{P'(s)}{s}\,\mathrm{d}s+\frac{\bar v^2}{2a_+^2\bar\rho_+^2}$$

and the level set L lies entirely in either $\mathcal{Z}_0^s$ or $\mathcal{Z}_0^u$ where

$$L=\{(\bar\rho,\eta):I(\bar\rho,\eta)=I(\bar\rho_-,-1)=I(\bar\rho_+,1)\}.$$

Any such a non-transonic wave admits viscous profiles.

In the rest, we consider the stability of viscous profiles for system (1).

3 The Eigenvalue Problem and a Center Manifold Reduction

With the same new variables introduced in (5), we rewrite system (1) as

$$\begin{cases}\varepsilon\rho_x=a^{-1}v+\rho u,\\ v_x=(a\rho)_t,\\ \varepsilon u_x=a^{-1}\rho^{-1}(w-vu+aP(\rho)),\\ w_x=(a\rho u)_t-a_xP(\rho).\end{cases} \tag{13}$$

Let $(\bar\rho(x;\varepsilon),\bar v(x;\varepsilon),\bar u(x;\varepsilon),\bar w(x;\varepsilon))$ be a stationary wave for $\varepsilon\geq 0$. In the following, we will drop the argument $(x;\varepsilon)$. It should be clear from the context when $\varepsilon=0$ and when $\varepsilon>0$.

The eigenvalue problem for the stationary wave is

$$\begin{cases} \varepsilon\rho_x = a^{-1}v + \bar{\rho}u + \bar{u}\rho, \\ v_x = \lambda a\rho, \\ \varepsilon u_x = -a^{-1}\bar{\rho}^{-2}\left(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})\right)\rho \\ \qquad\quad + a^{-1}\bar{\rho}^{-1}\left(w - \bar{v}u - \bar{u}v + aP'(\bar{\rho})\rho\right), \\ w_x = \lambda a\bar{\rho}u + \lambda a\bar{u}\rho - a_x P'(\bar{\rho})\rho. \end{cases} \tag{14}$$

We will treat this problem as a singularly perturbed problem. In terms of the fast scale $\xi = x/\varepsilon$, it becomes

$$\begin{cases} \rho_\xi = a^{-1}v + \bar{\rho}u + \bar{u}\rho, \\ v_\xi = \varepsilon\lambda a\rho, \\ u_\xi = -a^{-1}\bar{\rho}^{-2}\left(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})\right)\rho \\ \qquad\quad + a^{-1}\bar{\rho}^{-1}\left(w - \bar{v}u - \bar{u}v + aP'(\bar{\rho})\rho\right), \\ w_\xi = \varepsilon\left(\lambda a\bar{\rho}u + \lambda a\bar{u}\rho - a_x P'(\bar{\rho})\rho\right). \end{cases} \tag{15}$$

Next, we augment the eigenvalue problem (15) with the steady-state system (7) to obtain an autonomous problem as the following:

$$\begin{cases} \rho_\xi = a^{-1}v + \bar{\rho}u + \bar{u}\rho, \\ v_\xi = \varepsilon\lambda a\rho, \\ u_\xi = -a^{-1}\bar{\rho}^{-2}\left(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})\right)\rho \\ \qquad\quad + a^{-1}\bar{\rho}^{-1}\left(w - \bar{v}u - \bar{u}v + aP'(\bar{\rho})\rho\right), \\ w_\xi = \varepsilon\left(\lambda a\bar{\rho}u + \lambda a\bar{u}\rho - a_x P'(\bar{\rho})\rho\right), \\ \bar{\rho}_\xi = \bar{v}a^{-1} + \bar{\rho}\bar{u}, \\ \bar{v}_\xi = 0, \\ \bar{u}_\xi = a^{-1}\bar{\rho}^{-1}(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})), \\ \bar{w}_\xi = -\varepsilon a_x P(\bar{\rho}), \\ \eta_\xi = \varepsilon(1 - \eta^2). \end{cases} \tag{16}$$

The phase space of this system is $\mathbb{R}^9$ with the variable $(\rho, v, u, w, \bar{\rho}, \bar{v}, \bar{u}, \bar{w}, \eta)$.

Viewing system (16) as a singularly perturbed autonomous system, the slow manifold $\mathcal{S}_0$ is given by

$$\mathcal{S}_0 = \Big\{\bar{\rho}\bar{u} + \bar{v}a^{-1} = 0, \quad \bar{w} - \bar{v}\bar{u} + aP(\bar{\rho}) = 0, \\ a^{-1}v + \bar{\rho}u + \bar{u}\rho = 0, w - \bar{v}u - \bar{u}v + aP'(\bar{\rho})\rho = 0\Big\}$$

$$= \Big\{ \bar{u} = -\bar{v}a^{-1}\bar{\rho}^{-1}, \bar{w} = -\bar{v}^2 a^{-1}\bar{\rho}^{-1} - aP(\bar{\rho}),$$

$$u = -a^{-1}\bar{\rho}^{-1}v + \bar{v}a^{-1}\bar{\rho}^{-2}\rho, w = \frac{\bar{v}^2}{a\bar{\rho}^2}\rho - \frac{2\bar{v}}{a\bar{\rho}}v - aP'(\bar{\rho})\rho \Big\}. \tag{17}$$

Note that $\mathcal{S}_0$ contains equilibria of the limiting fast system (16) with $\varepsilon = 0$ and $\dim \mathcal{S}_0 = 5$.

The linearized matrix of system (16) with $\varepsilon = 0$ at each point on $\mathcal{S}_0$ has the form

$$\begin{pmatrix} R & * & * \\ 0 & R & * \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$R = \begin{pmatrix} \bar{u} & a^{-1} & \bar{\rho} & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\rho}^{-1}P'(\bar{\rho}) & -a^{-1}\bar{u}\bar{\rho}^{-1} & -a^{-1}\bar{v}\bar{\rho}^{-1} & a^{-1}\bar{\rho}^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

The sets of eigenvalues are $\{0, r_1, r_2\}$ where the zero eigenvalue with multiplicity five corresponds to the dimension of the slow manifold $\mathcal{S}_0$ and where

$$r_1 = \bar{u} - \sqrt{P'(\bar{\rho})}, \quad r_2 = \bar{u} + \sqrt{P'(\bar{\rho})} \tag{19}$$

each with multiplicity two.

We will consider only non-transonic stationary wave $(\bar{\rho}, \bar{v}, \bar{u}, \bar{w})$, that is, for any x, $\bar{u}(x) \neq \sqrt{P'(\bar{\rho}(x))}$. In this case, both r_1 and r_2 are non-zero, and hence, $\mathcal{S}_0$ is normally hyperbolic. The normally hyperbolic invariant manifold theory then implies that, for $\varepsilon > 0$ small, $\mathcal{S}_0$ persists and the perturbed slow manifold has the form

$$\mathcal{S}_\varepsilon = \Big\{ \bar{u} = -\bar{v}a^{-1}\bar{\rho}^{-1} + \varepsilon G, \quad \bar{w} = -\frac{\bar{v}^2}{a\bar{\rho}} - aP(\bar{\rho}) + \varepsilon F,$$

$$u = -a^{-1}\bar{\rho}^{-1}v + \bar{v}a^{-1}\bar{\rho}^{-2}\rho + \varepsilon(h_1 v + h_2 \rho),$$

$$w = \frac{\bar{v}^2}{a\bar{\rho}^2}\rho - \frac{2\bar{v}}{a\bar{\rho}}v - aP'(\bar{\rho})\rho + \varepsilon(F_1 v + F_2 \rho) \Big\}, \tag{20}$$

where the argument for the functions F, G, F_j's and h_j's is $(\bar{\rho}, \bar{v}, \eta, \varepsilon)$.

Remark 3.1. The normally hyperbolic invariant manifold theory ([4, 9]) requires the invariant manifold to be bounded or compact. The slow manifold $\mathcal{S}_0$ is not. We can certainly restrict the $(\bar{\rho}, \bar{v}, \bar{u}, \bar{w}, \eta)$-component to a bounded neighborhood of the steady-states but the eigenvalue problem needs the whole linear (ρ, v, u, w)-component. To get around this non-boundeness, one can view the first four linear equations in system (16) as defined in the projective space of $\mathbb{R}^4$ so that homogeneous (ρ, v, u, w)-component lies in the compact projective space. The normally hyperbolic invariant manifold theory can be applied and, afterwards, one can return back to the original setting. This also explains why the extra terms in u and w for $\mathcal{S}_\varepsilon$ take the special form.

The idea is then to reduce the eigenvalue problem (16) onto $\mathcal{S}_\varepsilon$. To do so, we first substitute (20) into (7) and (14), and after some tedious algebra, we find, up to $O(\varepsilon)$, that

$$\begin{aligned} G &= \frac{\bar{v}^2 a^{-2} a_x \bar{\rho}^{-1}}{a\bar{\rho}\,(P'(\bar{\rho}) - \bar{v}^2 a^{-2} \bar{\rho}^{-2})}, \quad h_1 = \frac{\lambda + 2\bar{v} a^{-2} a_x \bar{\rho}^{-1} + 2\bar{v} a^{-1} \bar{\rho}^{-1} G}{a\bar{\rho}\,(P'(\bar{\rho}) - \bar{v}^2 a^{-2} \bar{\rho}^{-2})}, \\ h_2 &= -\frac{2\lambda \bar{v} \bar{\rho}^{-1} + \bar{v}^2 a^{-2} a_x \bar{\rho}^{-2} + \bar{v}^2 a^{-1} \bar{\rho}^{-2} G + a\bar{\rho} P''(\bar{\rho}) G + a P'(\bar{\rho}) G}{a\bar{\rho}\,(P'(\bar{\rho}) - \bar{v}^2 a^{-2} \bar{\rho}^{-2})}. \end{aligned} \tag{21}$$

On the center manifold $\mathcal{S}_\varepsilon$, up to $O(\varepsilon^2)$, the first four equations in system (16) are reduced to a system of two equations,

$$\begin{cases} \rho_\xi = v a^{-1} + \bar{\rho}(-v a^{-1} \bar{\rho}^{-1} + \bar{v} a^{-1} \bar{\rho}^{-2} \rho + \varepsilon(h_1 v + h_2 \rho)) \\ \qquad + (-\bar{v} a^{-1} \bar{\rho}^{-1} + \varepsilon G)\rho \\ \quad = \varepsilon\, \bar{\rho} h_1 v + \varepsilon(\bar{\rho} h_2 + G)\rho, \\ v_\xi = \varepsilon \lambda a \rho. \end{cases} \tag{22}$$

If we return to the x-variable, the latter system becomes

$$\rho_x = f(\bar{\rho}, \bar{v}; \varepsilon) v + g(\bar{\rho}, \bar{v}; \varepsilon)\rho, \quad v_x = \lambda a \rho \tag{23}$$

where $f = \bar{\rho} h_1 + O(\varepsilon)$ and $g = G + \bar{\rho} h_2 + O(\varepsilon)$. System (23) is referred to as *the reduced eigenvalue problem* via the center manifold reduction.

4 The Limiting Eigenvalue Problem with $\varepsilon = 0$

In this section, we will consider the limiting eigenvalue problem of (23) with $\varepsilon = 0$:

$$\rho_x = f(\bar{\rho}(x), \bar{v}; 0) v + g(\bar{\rho}(x), \bar{v}; 0)\rho, \quad v_x = \lambda a \rho \tag{24}$$

where $\bar{\rho} = \bar{\rho}(x)$ is the ρ-component of the inviscid stationary wave.

System (24) can be cast as

$$v_{xx} - \left(\frac{a_x}{a} + g\right) v_x = \lambda a f v. \tag{25}$$

Setting $y = v e^{-\frac{1}{2}\int(\frac{a_x}{a}+g)}$, we have

$$y_{xx} - \left(\frac{1}{4}\left(\frac{a_x}{a}+g\right)^2 - \frac{1}{2}\left(\frac{a_x}{a}+g\right)_x\right) y = \lambda a f y. \tag{26}$$

If we separate f and g into terms without λ and with λ as $f = f_1 + \lambda f_2$ and $g = g_1 + \lambda g_2$, then (26) becomes

$$y_{xx} - \left(\frac{1}{4}q^2 - \frac{1}{2}q_x\right) y = \lambda w_1 y + \lambda^2 w_2 y, \tag{27}$$

where

$$q = \frac{a_x}{a} + g_1,\ w_1 = a f_1 + \frac{1}{2}\left(\frac{a_x}{a}+g_1\right) g_2 - \frac{1}{2} g_{2x},\ w_2 = a f_2 + \frac{1}{4} g_2^2.$$

It follows from (21) that

$$q = \frac{a_x}{a}\left(1 - \frac{\bar{v}^2 a^{-2}\bar{\rho}^5(P' + \bar{\rho}P'') + \bar{v}^4 a^{-4}\bar{\rho}^3}{(\bar{\rho}^2 P' - \bar{v}^2 a^{-2})^3}\right), \tag{28}$$

$$w_1 = -\frac{\bar{v}^2 a^{-4} a_x (a\bar{u}\bar{\rho} + a^{-2})\bar{\rho}^{-1} P'(\bar{\rho})}{\bar{\rho}^2 P'(\bar{\rho}) - \bar{v}^2 a^{-2}}, \tag{29}$$

and

$$w_2 = \frac{\bar{\rho}^4 P'(\bar{\rho})}{(\bar{\rho}^2 P'(\bar{\rho}) - \bar{v}^2 a^{-2})^2} > 0. \tag{30}$$

For later use, we collect some estimates

Lemma 4.1. *As $\bar{\rho} \to 0^+$, $\bar{\rho}^{-1}P'(\bar{\rho}) \to \infty$, and*

$$q^2 + |q_x| + |q_{xx}| + w_2 + |w_{2x}| = o(\bar{\rho}^{-1}P'(\bar{\rho})),$$
$$w_1 = a_x \bar{\rho}^{-1} P'(\bar{\rho}) O(1), \quad w_{1x} = O(\bar{\rho}^{-1}P'(\bar{\rho})).$$

Proof. It follows from the displays (28), (29) and (30), and (11) that

$$\begin{aligned} q =& \frac{a_x}{a} + O(P' + \bar{\rho}P'')\bar{\rho}^5 + O(\bar{\rho}^3), \\ q_x =& \left(\frac{a_x}{a}\right)_x + O(P' + \bar{\rho}P'' + \bar{\rho}^2 P''')\bar{\rho}^5 \\ & + O(P' + \bar{\rho}P'')\bar{\rho}^8 P'' + O(\bar{\rho}^3), \end{aligned}$$

$$q_{xx} = \left(\frac{a_x}{a}\right)_{xx} + O(P' + \bar{\rho}P'' + \bar{\rho}^2 P''' + \bar{\rho}^3 P^{(4)})\bar{\rho}^5$$
$$+ O(P' + \bar{\rho}P'' + \bar{\rho}^2 P''')\bar{\rho}^8 P'' + O(\bar{\rho}^3),$$
$$w_1 = a_x \bar{\rho}^{-1} P' O(1), \quad w_{1x} = O(\bar{\rho}^{-1}P' + P''),$$
$$w_2 = O(\bar{\rho}^4 P'), \quad w_{2x} = O(P' + \bar{\rho}P'')\bar{\rho}^4.$$

The conclusion is then a direct consequence of the assumption (2). □

We end this section with two simple results.

Lemma 4.2. *If $a_x \le 0$ and $a_x \neq 0$, then, for every subsonic wave $(\bar{\rho}, \bar{u})$, all eigenvalues have negative real parts. If $a_x \ge 0$ and $a_x \neq 0$, then, for every supersonic wave $(\bar{\rho}, \bar{u})$, all eigenvalues have negative real parts.*

Proof. Let λ be an eigenvalue and let $y(x)$ be an eigenfunction associated to λ. We multiply the conjugate $\bar{y}$ of y on (27) and integrate over $(-\infty, \infty)$ to get

$$-\int |y_x|^2 - \frac{1}{4}\int q^2 |y|^2 + \frac{1}{2}\int q_x y\bar{y} = \lambda \int w_1 |y|^2 + \lambda^2 \int w_2 |y|^2.$$

An application of integration by parts for the third term on the left gives

$$-\int \left(y_x + \frac{1}{2}qy\right)\left(\bar{y}_x + \frac{1}{2}q\bar{y}\right) = \lambda \int w_1 |y|^2 + \lambda^2 \int w_2 |y|^2.$$

Note that $w_2 > 0$ from (30). In view of (29), we have $w_1 \ge 0$ and $w_1 \neq 0$ if either $a_x \le 0$, $a_x \neq 0$ and $(\bar{\rho}, \bar{u})$ is subsonic or $a_x \ge 0$, $a_x \neq 0$ and $(\bar{\rho}, \bar{u})$ is supersonic. The conclusion then follows directly. □

Lemma 4.3. *Assume $a(x)$ is symmetric with respect to $x = 0$ and $(\bar{\rho}, \bar{u})$ is a symmetric non-transonic steady-state. Then, eigenvalues occur in pairs $\lambda, -\lambda$.*

Proof. It is easy to check that $\frac{1}{4}q^2 - \frac{1}{2}q_x$ is even, w_2 is even, but w_1 is odd. Suppose $y(x)$ is an eigenfunction associated to λ. Let $\hat{y}(x) = y(-x)$. Then $\hat{y}_{xx}(x) = y_{xx}(-x)$ and

$$\begin{aligned}\hat{y}_{xx}(x) - \left(\frac{q^2(x)}{4} - \frac{q_x(x)}{2}\right)\hat{y} =& y_{xx}(-x) - \left(\frac{q^2(-x)}{4} - \frac{q_x(-x)}{2}\right)y(-x)\\ =& \lambda w_1(-x)y(-x) + \lambda^2 w_2(-x)y(-x)\\ =& (-\lambda)w_1(x)\hat{y}(x) + (-\lambda)^2 w_2(x)\hat{y}(x).\end{aligned}$$

Thus, $-\lambda$ is also an eigenvalue. □

5 Instability of Low Density Steady-States

We will consider now the case where $a_x(x)$ changes sign or simply the set $\{x : a_x(x) < 0\}$ is not empty and show that supersonic waves with sufficiently low density are unstable. Our main result is

Theorem 5.1. *Assume the set $\{x : a_x(x) < 0\}$ is not empty. Then there is a constant $\kappa > 0$ so that, if $(\bar\rho, \bar u)$ is a stationary supersonic wave and $|\bar\rho| \le \kappa$, then $(\bar\rho, \bar u)$ is spectrally unstable.*

Theorem 5.1 is a direct consequence of Lemma 5.3 to be proved later. We first recall a basic result from the Sturm–Liouville Theory.

Proposition 5.2 ((Theorem 5.2 in [5])). *Let $r(t) > 0$ be continuous and of bounded variation on $[x_0, x_1]$. If $y(x) \neq 0$ is a real-valued solution of*

$$y_{xx} + r(x)y = 0$$

and N is the number of its zeros on $(x_0, x_1]$, then

$$\left| N - \frac{1}{\pi}\int_{x_0}^{x_1} \sqrt{r(x)}\,\mathrm{d}x \right| \le 1 + \frac{1}{4\pi}\int_{x_0}^{x_1} \frac{|r_x(x)|}{r(x)}\,\mathrm{d}x.$$

Now, for any real λ, let

$$r(x;\lambda) = -\frac{1}{4}q^2(x) + \frac{1}{2}q_x(x) - \lambda w_1(x) - \lambda^2 w_2(x).$$

Then (27) becomes

$$y_{xx} + r(x;\lambda)y = 0. \tag{31}$$

It can be expressed as

$$\begin{pmatrix} y \\ z \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ -r(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \tag{32}$$

In view of the assumption in (3) and the displays (28)–(30), we have

$$\lim_{x\to\pm\infty} q(x) = \lim_{x\to\pm\infty} w_1(x) = 0,$$

and hence,

$$r_\pm(\lambda) =: \lim_{x\to\pm\infty} r(x;\lambda) = -\lambda^2 \lim_{x\to\pm\infty} w_2(x) = \frac{-\lambda^2 \bar\rho_\pm^4 P'(\bar\rho_\pm)}{\left(\bar\rho_\pm^2 P'(\bar\rho_\pm) - \bar v^2 a_\pm^{-2}\right)^2}.$$

Thus, if λ is real and $\lambda \neq 0$, then $r_\pm(\lambda) < 0$, and hence,

$$\begin{pmatrix} 0 & 1 \\ -r_\pm(\lambda) & 0 \end{pmatrix}$$

has two real non-zero eigenvalues $\pm\sqrt{|r_\pm(\lambda)|}$ with associated eigenvectors

$$v^s_\pm(\lambda) = \frac{1}{1+|r(\lambda)|}\left(1, -\sqrt{|r_\pm(\lambda)|}\right)^T, \; v^u_\pm(\lambda) = \frac{1}{1+|r(\lambda)|}\left(1, \sqrt{|r_\pm(\lambda)|}\right)^T.$$

The unit vector $v^s_-(\lambda)$ (resp. $v^u_-(\lambda)$) is the stable (resp. unstable) eigenvector of system (32) at $x = -\infty$. The unit vector $v^s_+(\lambda)$ (resp. $v^u_+(\lambda)$) is the stable (resp. unstable) eigenvector of system (32) at $x = \infty$.

Therefore, for any real $\lambda \neq 0$, there exists a unique solution $(y_\lambda(x), z_\lambda(x))^T$ of (32) such that

$$|(y_\lambda(0), z_\lambda(0))^T| = 1 \quad \text{and} \quad \frac{(y(x;\lambda), z(x;\lambda))^T}{|(y(x;\lambda), z(x;\lambda))^T|} \to v^u_-(\lambda) \text{ as } x \to -\infty.$$

In particular, $y_\lambda(x) \neq 0$ is a solution of (31) and $y_\lambda(x) \to 0$ as $x \to -\infty$.

Let $N_\lambda = N_\lambda(y_\lambda(x))$ be the number of zeros of $y_\lambda(x)$ on $(-\infty, \infty)$. It follows from the asymptotic hyperbolicity of (32) that N_λ is finite. Note that N_λ is essentially twice the number of full clockwise rotations of the solution $(y_\lambda(x), z_\lambda(x))^T$ of system (32) for $x \in (-\infty, \infty)$.

Lemma 5.3. *Assume the set* $\{x : a_x(x) < 0\}$ *is nonempty. For any* $\Lambda > 0$ *and any positive integer n, there exists* $\kappa > 0$ *such that, if* $\bar\rho \leq \kappa$*, then there are at least n eigenvalues in* $(0, \Lambda)$.

Proof. From the assumption, there is $\delta > 0$ so that $U(\delta) = \{x : a_x(x) < -\delta\}$ is open and non-empty. For fixed $\Lambda > 0$, let

$$I(\delta) = \{x \in U(\delta) : r(x;\Lambda) > 0\}.$$

Due to (29) and Lemma 4.1, for $x \in U(\delta)$, $\lim_{\bar\rho(x)\to 0} r(x;\Lambda) = \infty$ and there exists $K > 0$ such that $\dfrac{|r_x(x;\Lambda)|}{r(x;\Lambda)} \leq K$. Hence, for any positive integer n, there exists $\kappa > 0$ such that, if $\bar\rho \leq \kappa$, then, $I(\delta) = U(\delta)$ and

$$\frac{1}{\pi}\int_{I(\delta)} \sqrt{r(x;\Lambda)}\mathrm{d}x - 1 - \frac{1}{4\pi}\int_{I(\delta)} \frac{|r_x(x;\Lambda)|}{r(x;\Lambda)}\mathrm{d}x > n.$$

Proposition 5.2 implies that

$$N_\Lambda|_{I(\delta)} \geq \frac{1}{\pi}\int_{I(\delta)} \sqrt{r(x;\Lambda)}\mathrm{d}x - 1 - \frac{1}{4\pi}\int_{I(\delta)} \frac{|r_x(x;\Lambda)|}{r(x;\Lambda)}\mathrm{d}x > n.$$

Therefore, for $\bar\rho \leq \kappa$, we have $N_\Lambda \geq N_\Lambda|_{I(\delta)} > n$.

Now, fix any $\bar\rho \leq \kappa$. If $\lambda = 0$, one verifies that

$$y(x) = \exp\left\{-\frac{1}{2}\int_0^x q(s)\mathrm{d}s\right\}$$

is a solutions of (27) so that $N(y(x)) = 0$. By continuous dependence of solutions on parameter λ and the asymptotic hyperbolicity of system (32), it follows that, for any integer $1 \leq j \leq n$, there exists $\lambda_j \in (0,\Lambda)$ and a solution $y_j(x) \neq 0$ of (31) with $\lambda = \lambda_j$ such that $y_j(x) \to 0$ as $x \to \pm\infty$ and $N(y_j(x)) = j$. Each λ_j is then an eigenvalue with an eigenfunction $y_j(x)$. Since, for $i \neq j$, $N(y_i(x)) \neq N(y_j(x))$, we have $y_i(x)$ and $y_j(x)$ are linearly independent, and hence, $\lambda_i \neq \lambda_j$. □

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Authors' Note: After the paper was accepted, C.-H. Hsu and T.-S. Yang pointed out to us that the expression for w_1 in display (29) is wrong and it should be $w_1 = 0$. We then found an error in our calculation and, indeed, $w_1 = 0$. This error aspects the claimed main result (Theorem 5.1) since the proof of Lemma 5.3 in the paper relies on the wrong expression (29) for the term w_1. At this moment, we could not prove Theorem 5.1 with $w_1 = 0$ and we do not know if the statement in Theorem 5.1 is correct or not.

Also, the following corrections should be made due to this error:

1. Relative parts on w1 in Lemma 4.1 should be changed with the statement $w_1 = 0$.
2. Lemma 4.2 should be changed to:

   Lemma 4.2. *For every non-transonic wave $(\bar\rho, \bar u)$, all eigenvalues are pure imaginary.*

3. In the proofs of Lemmas 4.2 and 4.3, one uses $w_1 = 0$ at relevant places and the rest remains the same.

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A Simple Proof of the Stability of Solitary Waves in the Fermi-Pasta-Ulam Model Near the KdV Limit

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Abstract By combining results of Mizumachi and Pego on the stability of solutions for the Toda lattice with a simple rescaling and a careful control of the KdV limit we give a simple proof that small amplitude, long-wavelength solitary waves in the Fermi-Pasta-Ulam (FPU) model are linearly stable and hence by the results of Friesecke and Pego that they are also nonlinearly, asymptotically stable.

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1 Introduction

In a series of four recent papers Friesecke and Pego [1–4] made a detailed study of the existence and stability of solitary wave solutions of the Fermi-Pasta-Ulam (FPU) system:

$$\ddot{q}_j = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}) \qquad j \in \mathbb{Z} \tag{1}$$

which models an infinite chain of anharmonic oscillators with nearest-neighbor interaction potential V. If we make the change of variables $r_j = q_{j+1} - q_j$ and $p_j = \dot{q}_j$, the state variable $u = (r, p)$ satisfies a system of first order Hamiltonian ODEs,

$$u_t = JH'(u) \tag{2}$$

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where the Hamiltonian H is given by

$$H(r,p) = \sum_{k\in\mathbb{Z}} \frac{1}{2}p_k^2 + V(r_k) \ . \tag{3}$$

The symplectic operator J is given by $J = \begin{pmatrix} 0 & S-1 \\ 1-S^{-1} & 0 \end{pmatrix}$ where S is the left shift on bi-infinite sequences, i.e. $(Sx)_n := x_{n+1}$. The problem is well posed in each ℓ^p space, but for concreteness and simplicity we work in ℓ^2, a natural choice since for small solutions the conservation of the Hamiltonian gives a global in time bound on the ℓ^2 norm. Throughout the paper we shall assume that the interaction potential V satisfies the following

$$V \in C^4; \qquad V(0) = V'(0) = 0; \qquad V''(0) > 0; \qquad V'''(0) \neq 0 \ . \tag{4}$$

In the first paper in the series, [1], Friesecke and Pego prove that the system (2) has a family of solitary wave solutions which in the small amplitude, long-wavelength limit have a profile close to that of the KdV soliton. More precisely they show:

Theorem 1.1 (Theorem 1.1 (b) from [1], restated). *Assume that V satisfies (4) and that $c > \sqrt{V''(0)}$ is sufficiently close to $\sqrt{V''(0)}$. Then there exists a solution to the wave profile equation for FPU:*

$$cr_c''(x) = (S + S^{-1} - 2I)V'(r_c) \qquad r_c(\pm\infty) = 0$$

which in addition satisfies

$$\left\| \frac{1}{\varepsilon^2} r_c\left(\frac{\cdot}{\varepsilon}\right) - \phi_1 \right\|_{H^2} \leq C\varepsilon^2 \ , \tag{5}$$

where $\phi_1(x) := \frac{V''(0)}{V'''(0)}\left(\frac{1}{2}\operatorname{sech}(\frac{1}{2}x)\right)^2$ is the KdV soliton and $\varepsilon := 24\sqrt{\frac{c}{V''(0)} - 1}$.

Remark 1.2. In fact, in [1] it is only proven that the traveling wave profile is close to the (rescaled) KdV soliton in the H^1 norm. The strengthening of the estimate to hold in the H^2 norm was done in [5].

Remark 1.3. The first general results about the existence of traveling waves in these general FPU type systems were obtained by Friesecke and Wattis [6] via variational methods.

In the second paper in this series, [2], the authors use the method of modulation equations to prove that if the solitary waves are linearly stable, they are nonlinearly stable as well. More precisely, suppose that the following linear stability condition is satisfied:

Define

$$\omega((r,p),(\rho,\pi)) := \sum_{j\in\mathbb{Z}} \left(\sum_{k=-\infty}^{0} p_{k+j}\rho_j + \sum_{k=-\infty}^{-1} r_{k+j}\pi_j \right) \tag{6}$$

and define

$$\|x\|_a^2 := \sum_{k\in\mathbb{Z}} e^{2aj} x_j^2. \tag{7}$$

Hypothesis L: *There are positive constants K and β' and $c_0 > \sqrt{V''(0)}$ such that whenever $\sqrt{V''(0)} < c_* < c_0$, and w is a solution of the linear equation*

$$\partial_t w = JH''(u_{c_*}(\cdot - c_* t))w \tag{8}$$

with $\|w(t_0)\|_a < \infty$ and such that

$$\omega(\partial_z u_{c_*}(z)|_{z=\cdot - c_* t}, w(t_0)) = \omega(\partial_c u_c(z)|_{c=c_*, z=\cdot - c_* t}, w(t_0)) = 0 \tag{9}$$

holds, then the estimate

$$e^{-c_* t}\|w(t)\|_a \le K e^{-\beta'(t-s)} e^{-ac_* s}\|w(s)\|_a \tag{10}$$

holds for all $t \ge s \ge t_0$.

Theorem 1.4 (Theorem 1.1 from [2], restated). *Assume that V satisfies (4) and hypothesis L holds. Then the solitary wave u_c is stable in the nonlinear system (2) in the following sense: Let $\beta \in (0,\beta')$. Then there are positive constants C_0 and δ_0 such that if for some $\delta \le \delta_0$ and $\gamma_* \in \mathbb{R}$ the initial data satisfy*

$$\|u_0 - u_{c_*}(\cdot - \gamma_*)\| \le \sqrt{\delta} \qquad \text{and} \qquad \|e^{a(\cdot - \gamma_*)}(u_0 - u_{c_*}(\cdot - \gamma_*))\| < \delta$$

then there is an unique asymptotic wave speed c_∞ and phase γ_∞ such that

$$|c_\infty - c_*| + |\gamma_\infty - \gamma_*| \le C_0\delta$$

and

$$\|u(t,\cdot) - u_{c_\infty}(\cdot - c_\infty t - \gamma_\infty)\| \le C_0\sqrt{\delta} \qquad t > 0\,,$$

and in addition

$$\|e^{a(\cdot - c_\infty t - \gamma_\infty)}(u(t,\cdot) - u_{c_\infty}(\cdot - c_\infty t - \gamma_\infty))\| \le C_0\delta e^{-\beta t}.$$

The last two papers in the series, [3, 4], are devoted to verifying that the linear estimate (10) holds for the solitary waves constructed in [1].

In [3], Friesecke and Pego construct a type of Floquet theory to prove estimates like (10). The reason that one needs such an approach is that because of the discreteness introduced by the lattice, the linearized equation (8) is not autonomous in a frame of reference moving with the traveling waves, but only periodic (with a spatial translation.) Finally, in paper [4], Friesecke and Pego verify that the solitary waves constructed in [1] satisfy the hypotheses of their Floquet theory and hence, by the results of [2] are asymptotically stable. This last step involves, among other things, the fact that these solitary waves are well approximated by the KdV soliton and the fact that the linearization of the KdV equation about its solitary wave is well understood.

In this note, we give a simple alternative proof of the estimate (10) which avoids the use of the Floquet theory and spectral analysis of [3, 4].

Our proof combines three observations:

1. The linear stability of the soliton for the Toda lattice, established by Mizumachi and Pego by the construction of an explicit Bäcklund transformation in [7].
2. A transformation of the original FPU equation (2) into a form in which we can prove that its solitary wave solution is close to that of the Toda lattice.
3. A careful control of the way in which various quantities depend on the small parameter ε.

We note that the last two points were originally developed in our study of counter-propagating 2-soliton solutions of the FPU model [5].

We now explain these three points in more detail. Recall first that the Toda lattice is the special case of the FPU model with potential function

$$\tilde{V}(x) = (e^{-x} + x - 1). \tag{11}$$

(Throughout this paper, quantities with tildes will refer to the Toda model.) Note that $\tilde{V}$ satisfies the hypotheses of [1] so the results of that paper imply that the Toda model has a family of solitary waves close to those of the KdV equation. Of course in the case of the Toda model these solutions were explicitly constructed by Toda in the 1960s and indeed the Toda model is one of the classic examples of a completely integrable, infinite-dimensional Hamiltonian system. The stability of the Toda soliton can also be analyzed in a very direct fashion. By constructing a Bäcklund transformation which conjugates the linearization of the Toda model about its soliton to the linearization of the Toda lattice about zero, Mizumachi and Pego proved that the linearized Toda equation satisfies hypothesis L and hence, by the results of [2], that the Toda soliton is asymptotically stable.

Remark 1.5. In [7], Mizumachi and Pego prove that the constant β' in hypothesis L is $\mathcal{O}(a^3)$, for small values of a. In the KdV regime $a = \mathcal{O}(\varepsilon)$, and thus, we can choose $\beta' = \beta\varepsilon^3$ for some constant β independent of ε. In [5] we extended this result by showing that the constant K in (10) can be chosen uniformly in ε in the KdV regime.

Remark 1.6. Although we will be most interested in these results in the long-wavelength, small amplitude regime studied by Friesecke and Pego, in fact the results of [7] apply to Toda lattice solutions of arbitrary size.

In order to extend the estimate on the linear decay from the semigroup of the linearized Toda equation to the linearization of the general FPU model satisfying hypothesis **(H1)**, we first make use of the following simple:

Lemma 1.7. *Without loss of generality we may assume that the potential energy function V in* (1)*, or equivalently* (3)*, satisfies*

$$V''(0) = 1, \;\; V'''(0) = -1.$$

Proof. To see this simply note that if $q(j,t)$ is a solution of (1) with a potential energy function V that satisfies hypothesis **(H1)**, then V can be written as $V(x) = \frac{1}{2}ax^2 + \frac{1}{6}bx^3 + \mathcal{O}(x^4)$ for some $a > 0$ and $b \neq 0$. If we now define $\hat{q}(j,t) = \alpha q(j, \beta t)$, then $\hat{q}$ solves the FPU equations with potential function $\hat{V}(x) = \frac{1}{2}a\beta x^2 + \frac{1}{6}b(\beta^2/\alpha)x^3 + \mathcal{O}(x^4)$, so choosing $\alpha = -b/a$ and $\beta = \sqrt{1/a}$ insures that $\partial_x^2 \hat{V}(0) = 1$ and $\partial_x^3 \hat{V}(0) = -1$, as desired. □

From now on we will assume that the potential function V satisfies this normalization. Note that with this normalization the potential V in (1) differs from the Toda potential only by terms of $\mathcal{O}(x^4)$ or higher. With this observation the existence results of [1] imply:

Proposition 1.8. *Let u_c be the profile of the solitary wave of the FPU model* (1) *with speed c, and let $\tilde{u}_c$ be the profile for the special case of the Toda potential. Define $\xi_1 = \partial_z u_c(z)$ and $\xi_2 = \partial_c u_c(z)$ and let $\tilde{\xi}_{1,2}$ be the corresponding quantities for the Toda lattice. Then there exists $\varepsilon_0, C > 0$ such that for $0 < \varepsilon < \varepsilon_0$, one has the estimates*

$$\|u_c - \tilde{u}_c\|_{\ell^2_a} \leq C\varepsilon^{7/2} \quad \|u_c - \tilde{u}_c\|_{\ell^\infty} \leq C\varepsilon^4, \tag{12}$$

$$\|\xi_1 - \tilde{\xi}_1\|_{\ell^2_a} \leq C\varepsilon^{9/2} \quad \|\xi_1 - \tilde{\xi}_1\|_{\ell^\infty} \leq C\varepsilon^5, \tag{13}$$

$$\|\xi_2 - \tilde{\xi}_2\|_{\ell^2_a} \leq C\varepsilon^{3/2} \quad \|\xi_2 - \tilde{\xi}_2\|_{\ell^\infty} \leq C\varepsilon^2. \tag{14}$$

Remark 1.9. The half-powers of ε that occur in the estimates of the ℓ^2_a norms are a consequence of the scaling of the functions u_c.

Proof. The inequality $\|u_c - \tilde{u}_c\|_{\ell^\infty} \leq C\varepsilon^4$ follows immediately from (5) because to leading (ε^2) order both u_c and $\tilde{u}_c$ agree with the KdV soliton. The estimate on $\|u_c - \tilde{u}_c\|_{\ell^2_a}$ then follows from this estimate because of the prior remark about the scaling of the ℓ^2_a norms. The estimates on $\xi_1 - \tilde{\xi}_1$ then follow from these two since theorem 1.1 shows that a derivative of the solitary wave profile with respect to the spatial variable gains exactly one power of ε. The estimates for $\xi_2 - \tilde{\xi}_2$ follow in a similar fashion. For more details see [5]. □

With these estimates in hand we consider the evolution semigroup generated by

$$\partial_t v = JH''(u_c)v = J\tilde{H}''(\tilde{u}_c)v + J\left(H''(u_c) - \tilde{H}''(\tilde{u}_c)\right)v\,. \tag{15}$$

The idea is now to treat the term $J\left(H''(u_c)-\tilde{H}''(\tilde{u}_c)\right)v$ as a perturbation of the Toda semigroup. Recalling that H and $\tilde{H}$ differ only at quartic order and that u_c and $\tilde{u}_c$ are both of order $\mathcal{O}(\varepsilon^2)$ and differ only by terms of $\mathcal{O}(\varepsilon^4)$ we have

$$\begin{aligned}\|J\left(H''(u_c)-\tilde{H}''(\tilde{u}_c)\right)v\|_{\ell^2_a} &\le C\left(\|u_c-\tilde{u}_c\|_{\ell^\infty}+(\|u_c\|_{\ell^\infty}+\|\tilde{u}_c\|_{\ell^\infty})^2\right)\|v\|_{\ell^2_a}\\ &\le C\varepsilon^4\|v\|_{\ell^2_a}\,. \end{aligned} \tag{16}$$

The other fact we must deal with is that $v\in E^s=\{v\mid \omega(\xi_1,v)=\omega(\xi_2,v)=0\}$, while our decay estimates on the Toda semigroup hold only if the semigroup acts on vectors $\tilde{v}\in\tilde{E}^s=\{\tilde{v}\mid \omega(\tilde{\xi}_1,\tilde{v})=\omega(\tilde{\xi}_2,\tilde{v})=0\}$. To cope with this difference we define the projection operator

$$Qv=v-\left(\frac{\omega(\tilde{\xi}_2,\tilde{\xi}_1)\omega(\tilde{\xi}_2,v)+\omega(\tilde{\xi}_2,\tilde{\xi}_2)\omega(\tilde{\xi}_1,v)}{\omega(\tilde{\xi}_2,\tilde{\xi}_1)^2}\right)\tilde{\xi}_1-\frac{\omega(\tilde{\xi}_1,v)}{\omega(\tilde{\xi}_2,\tilde{\xi}_1)}\tilde{\xi}_2\,, \tag{17}$$

which maps ℓ^2_a to $\tilde{E}_s$. Using the estimates in proposition 1.8 , we have:

Proposition 1.10 (Lemma 4.4 in [5], simplified). *There exists* $\varepsilon_0, C>0$ *such that if* $0<\varepsilon<\varepsilon_0$, *then*

$$\|Qv\|_{\ell^2_a}\le C\|v\|_{\ell^2_a}\,.$$

Furthermore, if $v\in E^s$, *then*

$$\|(I-Q)v\|_{\ell^2_a}\le C\varepsilon^{3/2}\|v\|_{\ell^2_a}\,.$$

Proof. The key factor in the proof of this proposition is that the estimates of theorem 1.1 make it possible to evaluate the leading order in ε behavior of the symplectic products $\omega(\xi_j,\xi_k)$ and $\omega(\tilde{\xi}_j,\tilde{\xi}_k)$. This was first used in [1] and was utilized repeatedly in [5]. For instance, one has $\omega(\xi_1,\xi_1)=0$, while $\omega(\xi_2,\xi_2)=c_{22}\varepsilon^{-2}+\mathcal{O}(\varepsilon^{-1})$ and the cross term $\omega(\xi_1,\xi_2)=c_{12}\varepsilon+\mathcal{O}(\varepsilon^2)$ with the constants c_{12} and c_{22} both non-zero. Similarly, the leading order behavior in ε of the norms of ξ_1 and ξ_2 can be computed by relating them to the derivatives of the profile of the KdV soliton using the estimates of theorem 1.1. The same estimates also hold for the symplectic inner products of $\tilde{\xi}_{1,2}$ and with these estimates the first bound in the proposition follows immediately.

The second estimate follows by rewriting the projection operator as

$$\begin{aligned}&\omega(\tilde{\xi}_2,\tilde{\xi}_1)(I-Q)v\\ &=\left(\frac{\omega(\tilde{\xi}_2,\tilde{\xi}_1)\omega(\tilde{\xi}_2,v)+\omega(\tilde{\xi}_2,\tilde{\xi}_2)\omega(\tilde{\xi}_1,v)}{\omega(\tilde{\xi}_2,\tilde{\xi}_1)}\right)\tilde{\xi}_1-\omega(\tilde{\xi}_1,v)\tilde{\xi}_2\end{aligned}$$

$$= \left(\frac{\omega(\tilde{\xi}_2, \tilde{\xi}_1)\omega(\tilde{\xi}_2 - \xi_2, v) + \omega(\tilde{\xi}_2, \tilde{\xi}_2)\omega(\tilde{\xi}_1, v)}{\omega(\tilde{\xi}_2, \tilde{\xi}_1)} \right) \tilde{\xi}_1 - \omega(\tilde{\xi}_1 - \xi_1, v)\tilde{\xi}_2 \;, \quad (18)$$

where the last step used the fact that since $v \in E^s$, $\omega(\xi_1, v) = \omega(\xi_2, v) = 0$. But now note that each term on the right hand side contains a factor of either $\tilde{\xi}_1 - \xi_1$ or $\tilde{\xi}_2 - \xi_2$ and these are small due to the estimates in proposition 1.8 . With the aid of these estimates the second estimate in the proposition follows in a straightforward fashion. For more details see the proof of lemma 4.4 in [5]. □

Corollary 1.11. *There exists* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$ *and* $v \in E^s$*, then*

$$\|v\|_{\ell^2_a} \leq 2\|Qv\|_{\ell^2_a} \,.$$

Now write $v(t)$, the solution of (15) as

$$v(t) = \tilde{S}(t,0)v(0) + \int_0^t \tilde{S}(t,s)J\left(H''(u_c) - \tilde{H}''(\tilde{u}_c)\right)v(s)\mathrm{d}s \;, \quad (19)$$

where $\tilde{S}$ is the evolution semigroup generated by the linearized Toda system. Then

$$Qv(t) = \tilde{S}(t,0)Qv(0) + \int_0^t \tilde{S}(t,s)Q\left(J\left(H''(u_c) - \tilde{H}''(\tilde{u}_c)\right)\right)v(s)\mathrm{d}s \;, \quad (20)$$

Taking the norm of both sides and using the estimates on the Toda semigroup (10) obtained in [7] we find:

$$\begin{aligned} \|Qv\|_{\ell^2_a} &\leq K\mathrm{e}^{-b\varepsilon^3 t}\|v(0)\|_{\ell^2_a} \\ &\qquad + K\int_0^t \mathrm{e}^{-b\varepsilon^3(t-s)} \left\|Q\left(J\left(H''(u_c) - \tilde{H}''(\tilde{u}_c)\right)\right)v(s)\right\|_{\ell^2_a} \mathrm{d}s \quad (21) \\ &\leq K\mathrm{e}^{-b\varepsilon^3 t}\|v(0)\|_{\ell^2_a} + K_2\varepsilon^4 \int_0^t \mathrm{e}^{-b\varepsilon^3(t-s)}\|v(s)\|_{\ell^2_a}\mathrm{d}s \quad (22) \\ &\leq K\mathrm{e}^{-b\varepsilon^3 t}\|v(0)\|_{\ell^2_a} + K_3\varepsilon^4 \int_0^t \mathrm{e}^{-b\varepsilon^3(t-s)}\|Qv(s)\|_{\ell^2_a}\mathrm{d}s \,. \quad (23) \end{aligned}$$

Here we have used (16) and proposition 1.10. Note that the constants K, K_2, and K_3 are all independent of ε. Now setting $\phi(t) = \sup_{0\leq\tau\leq t} \mathrm{e}^{b'\varepsilon^3\tau}\|Qv(\tau)\|_{\ell^2_a}$ for some $0 < b' < b$ and taking the supremum in the above equation we have

$$\begin{aligned} \phi(t) &\leq K\|v(0)\|_{\,ell^2_a} + K_3\varepsilon^4\mathrm{e}^{-(b-b')\varepsilon^3 t}\int_0^t \mathrm{e}^{-(b-b')\varepsilon^3 s}\phi(s)\mathrm{d}s \quad (24) \\ &\leq K\|v(0)\|_{\ell^2_a} + \frac{K_3}{b-b'}\varepsilon\phi(t) \,. \quad (25) \end{aligned}$$

Note that $b - b'$ can be chosen uniformly in ε. Thus, if ε is sufficiently small, we conclude that $\phi(t)$ is uniformly bounded for all t and hence:

Proposition 1.12. *There exists $K', \beta' > 0$, independent of ε, and $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $v(t) \in E^s$ is a solution of (15) then*

$$\|v\|_{\ell^2_a} \leq K' e^{-b'\varepsilon^3 t} \|v(0)\|_{\ell^2_a} .$$

This verifies that the linearized FPU semigroup satisfies hypothesis L and hence by the results of [2] that the FPU solitary wave is asymptotically stable in the small amplitude, long-wavelength regime.

Remark 1.13. We note that in the proof of proposition 1.12 it is important to carefully control the dependence of the semigroup on the parameter ε. It is not surprising that a perturbation argument permits one to extend the results of Mizumachi and Pego to solitary waves in small perturbations of the Toda model. Indeed, this was already noted in [7] . What we do find noteworthy is that this simple argument can cover all FPU solitary waves in the KdV regime, and this requires the detailed study of the small ε asymptotics contained in propositions 1.8 and 1.10.

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Littlewood Problem for a Singular Subquadratic Potential

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Dedicated to Professor George R. Sell on the occasion of his 70th birthday.

Abstract We consider a periodically forced singular oscillator in which the potential has subquadratic growth at infinity and admits a singularity. Using Moser's twist theorem of invariant curves, we show the existence of quasiperiodic solutions. This solves the Littlewood problem on the boundedness of all solutions for such a system.

Mathematics Subject Classification (2010): Primary 34C15; Secondary 58F27

1 Introduction

In this chapter, we will consider the Littlewood problem for the forced nonlinear oscillator

$$\ddot{x} + V'(x) = e(t), \tag{1}$$

where $e(t)$ is a 1-periodic continuous function.

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In the early 1960s, Littlewood [6] suggested to study the boundedness of all solutions of systems like (1) for the following two cases:

(1) Superlinear case: $\frac{V'(x)}{x} \to +\infty$ as $x \to \pm\infty$

(2) Sublinear case: $\operatorname{sign}(x)V'(x) \to +\infty$ and $\dfrac{V'(x)}{x} \to 0$ as $x \to \pm\infty$

The first result in the superlinear case is due to Morris [11], who proved the boundedness of all solutions of (1) for $V(x) = \frac{1}{2}x^4$. Later, Dieckerhoff and Zehnder [2] considered the polynomial case

$$\ddot{x} + x^{2n+1} + \sum_{i=0}^{2n} p_i(t)x^i = 0,$$

where p_i, $i = 0, 1, \ldots, 2n$, are 1-periodic C^∞ functions, and showed the boundedness of all solutions. Subsequently, this result was extended to more general cases by several authors (see [4, 5, 7, 16, 17], and references therein).

Recently, the boundedness of all solutions was shown by Küpper and You [3] for the sublinear equation

$$\ddot{x} + |x|^{\alpha-1}x = e(t), \tag{2}$$

where $0 < \alpha < 1$ and e is smooth. The general sublinear case was later considered by Liu [8] under certain reasonable conditions.

The Littlewood problem for singular potentials is known to be challenging, and there are only a very few results. A case of such oscillators with semilinearly growing potential was recently considered by Capietto–Dambrosio–Liu [1] for which the boundedness of all solutions was shown under some nonresonance conditions (see also Liu [9] for an extension of the result to the isochronous case).

This chapter is devoted to the study of Littlewood problem for forced oscillators with singular sublinearly growing potential. While our result and technique hold for general potentials of the like, we will consider a model problem of (1) with

$$V(x) = x_+^{\alpha+1} + \frac{1}{1-x_-^2} - 1 = \begin{cases} x^{\alpha+1}, & \text{if } x \geq 0; \\ \frac{1}{1-x^2} - 1, & \text{if } -1 < x < 0, \end{cases} \tag{3}$$

where $\frac{1}{3} < \alpha < 1$, $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$.

The main result in this chapter is the following.

Theorem 1.1. *Consider the forced oscillator* (1) *with the singular potential* (3) *and* $e(t) \in C^4(\mathbb{R}/\mathbb{Z})$. *Then any solution* $x(t)$ *of the oscillator with the initial value* $x(0) > -1$ *satisfies*

$$\inf_{t\in\mathbb{R}} x(t) > -1, \qquad \sup_{t\in\mathbb{R}}(|x(t)| + |\dot{x}(t)|) < +\infty.$$

The power range $\frac{1}{3} < \alpha < 1$ and the smoothness requirement of $e(t)$ in the theorem are due to the technique we employ. We do not know whether they are essential to the validity of the theorem.

The proof of theorem concerns the formal reduction to normal form and estimates. The formal reduction consists of the following sequence of transformations:

$$\left(x,\dot{x},t;\mathcal{H}=\frac{\dot{x}^2}{2}+V(x)+yE(t)\right)\xrightarrow{A}(\theta,I,t;H=H_0(I)+H_1(\theta,I,t))$$
$$\xrightarrow{B}(t,H,\theta;I=I_0(H)+I_1(t,H,\theta)),$$

where $E(t)=\int_0^t e(s)\mathrm{d}s$, in which A is the standard action-angle reduction of $(x,\dot{x})$ into (θ,I), and B is the change of time and energy into the new position and momentum with the angle θ playing the role of new time. For each step of transformations, detailed estimates will be given. In particular, due to the existence of singularity in the potential, special cares are needed on estimates with respect to $H_0(I)$, $x(\theta,I)$, and $y(\theta,I)$, for which we will employ some techniques developed in [1, 4, 8]. An application of Moser's twist theorem will then yield the theorem.

The rest of the chapter is organized as follows. In Sects. 2 and 3, we introduce transformations A, B, respectively, along with some technical estimates. The theorem will be proved in Sect. 4. We will also give another result on the existence of quasiperiodic solutions, Aubry–Mather sets, and unlinked periodic solutions in this section. The Appendix is devoted to the estimates on $I_0(H)$, $x(\theta,I)$, and $y(\theta,I)$.

For simplicity, throughout this chapter, we use symbols c,C to denote appropriate universal constants which are in $(0,1)$, $(1,\infty)$, respectively.

2 Action-Angle Variables

Without loss of generality, we assume that the average value of $e(t)$ vanishes, i.e., the function

$$E(t)=\int_0^t e(s)\,\mathrm{d}s$$

is 1-periodic and of class C^5. Indeed, if $\bar{e}=\int_0^1 e(t)\neq 0$, then we can use

$$\tilde{V}(x)=\begin{cases} x^{\alpha+1}-\bar{e}x, & \text{if } x\geq 0;\\ \frac{1}{1-x^2}-1, & \text{if } -1<x<0,\end{cases}$$

and $\tilde{e}(t)=e(t)-\bar{e}$ in place of $V(x)$ and $e(t)$, respectively in (1).

It is clear that the oscillator (1) is a Hamiltonian system of one-and-a-half degree of freedom:

$$\dot{x}=\partial_y\mathcal{H},\qquad \dot{y}=-\partial_x\mathcal{H}, \tag{4}$$

where

$$\mathcal{H}(x,y,t) = \frac{1}{2}y^2 + V(x) + yE(t).$$

For each $H > 0$, we denote by $I_0(H)$ the area enclosed by the closed curve

$$\Gamma_H : \ \frac{1}{2}y^2 + V(x) = H.$$

Let $-1 < -\alpha_H < 0 < \beta_H$ be such that

$$V(-\alpha_H) = V(\beta_H) = H,$$

i.e.,

$$\alpha_H = \sqrt{\frac{H}{H+1}}, \qquad \beta_H = H^{\frac{1}{1+\alpha}}.$$

It is easy to see that

$$I_0(H) = 2\int_{-\alpha_H}^{\beta_H} \sqrt{2(H - V(\xi))}\,\mathrm{d}\xi$$

and

$$T_0(H) := I_0'(H) = 2\int_{-\alpha_H}^{\beta_H} \frac{\mathrm{d}\xi}{\sqrt{2(H - V(\xi))}}.$$

Denote

$$I_0^+(H) = 2\int_0^{\beta_H} \sqrt{2(H - V(\xi))}\,\mathrm{d}\xi, \qquad I_0^-(H) = 2\int_{-\alpha_H}^0 \sqrt{2(H - V(\xi))}\,\mathrm{d}\xi,$$

$$T_0^+(H) = 2\int_0^{\beta_H} \frac{\mathrm{d}\xi}{\sqrt{2(H - V(\xi))}}, \qquad T_0^-(H) = 2\int_{-\alpha_H}^0 \frac{\mathrm{d}\xi}{\sqrt{2(H - V(\xi))}}.$$

Then

$$I_0(H) = I_0^+(H) + I_0^-(H), \qquad T_0(H) = T_0^+(H) + T_0^-(H).$$

Now, the action-angle variable (I,θ), $I > 0$, $\theta \in \mathbb{R} \pmod 1$, can be introduced as usual:

$$I = I_0(H(x,y)) \tag{5}$$

and

$$\theta = \theta(x,y) = \begin{cases} \frac{1}{T_0(H(x,y))}\left(\int_{-\alpha_H}^{x} \frac{\mathrm{d}\xi}{\sqrt{2(H(x,y) - V(\xi))}}\right), & \text{if } y > 0; \\ 1 - \frac{1}{T_0(H(x,y))}\left(\int_{-\alpha_H}^{x} \frac{\mathrm{d}\xi}{\sqrt{2(H(x,y) - V(\xi))}}\right), & \text{if } y < 0 \end{cases} \tag{6}$$

for $(x,y) \in (-1,+\infty) \times \mathbb{R} \setminus \{(0,0)\}$, where

$$H(x,y) = \frac{1}{2}y^2 + V(x).$$

Also, we let $x(\theta,I)$, $y(\theta,I)$ be inverse functions of (5) and (6).

With the action-angle variable (I,θ), system (4) becomes

$$\dot{I} = -\partial_\theta H, \qquad \dot{\theta} = \partial_I H, \tag{7}$$

where

$$H = H(\theta,I,t) = H_0(I) + H_1(\theta,I,t), \tag{8}$$

in which $H_0(I)$ is the inverse function of $I_0(H)$ and

$$H_1(\theta,I,t) = y(\theta,I)E(t).$$

We now give some estimates on the functions $H_0(I)$, $H_1(\theta,I,t)$, $x(\theta,I)$, and $y(\theta,I)$.

Lemma 2.1. *The following hold:*

(1) $\sqrt{2}H^{\frac{1}{2}+\frac{1}{1+\alpha}} \le I_0^+(H) \le 2\sqrt{2}H^{\frac{1}{2}+\frac{1}{1+\alpha}}$

(2) $\sqrt{2}H^{\frac{1}{2}}\left(\frac{H}{H+1}\right)^{\frac{1}{2}} \le I_0^-(H) \le 2\sqrt{2}H^{\frac{1}{2}}\left(\frac{H}{H+1}\right)^{\frac{1}{2}}$

(3) $T_0^+(H) = \frac{\mathrm{d}I_0^+(H)}{\mathrm{d}H} = \left(\frac{1}{2}+\frac{1}{1+\alpha}\right)H^{-1}I_0^+(H)$

(4) $\frac{\mathrm{d}T_0^+(H)}{\mathrm{d}H} = \frac{\mathrm{d}^2 I_0^+(H)}{\mathrm{d}H^2} = \left(\frac{1}{(1+\alpha)^2} - \frac{1}{4}\right)H^{-2}I_0^+(H)$

Proof. (1) (Resp. (2)) can be easily proved by comparing the area bounded by Γ_H in the right half plane (resp. the left half plane) with the area of the triangle or rectangle with sides $\sqrt{2H}$ and β_H (resp. α_H).

Similar to the proof of [4, A3.2], we have

$$\begin{aligned}
T_0^+(H) &= \frac{\mathrm{d}I_0^+(H)}{\mathrm{d}H} = \frac{2}{H}\int_0^{\beta_H}\left(\frac{1}{2}+W'(\xi)\right)\sqrt{2(H-V(\xi))}\,\mathrm{d}\xi \\
&= \frac{2}{H}\left(\frac{1}{2}+\frac{1}{1+\alpha}\right)\int_0^{\beta_H}\sqrt{2(H-V(\xi))}\,\mathrm{d}\xi = \left(\frac{1}{2}+\frac{1}{1+\alpha}\right)\frac{I_0^+(H)}{H}
\end{aligned}$$

with $W(x) = \frac{V(x)}{V'(x)}$. This proves (3).

(4) easily follows from (3). □

Lemma 2.2. *For all* $n = 0,1,\ldots,$

$$\frac{\mathrm{d}^n T_0^-(H)}{\mathrm{d}H^n} = (-1)^n\frac{(2n-1)!!}{2^n}\frac{\sqrt{2}}{H^{(2n+1)/2}} + o\left(\frac{1}{H^{(2n+1)/2}}\right)$$

as $H \to +\infty$. Consequently,

$$\left|\frac{\mathrm{d}^n I_0^-(H)}{\mathrm{d}H^n}\right| \le CH^{-n} I_0^-(H)$$

for all $n \ge 0$ as $H \gg 1$.

Proof. This lemma was proved in [1]. For the readers' convenience, we include the proof in the Appendix. □

Remark 2.3. (1). It follows from the Lemmas 2.1 and 2.2 that

$$\lim_{H\to+\infty} T_0^+(H) = +\infty, \quad \lim_{H\to+\infty} T_0^-(H) = 0.$$

Thus the time period $T_0(H)$ of the integral curve Γ_H is dominated by $T_0^+(H)$ when H is sufficiently large.

(2). It also follows from Lemmas 2.1 and 2.2 that

$$cH^{-n} I_0(H) \le \left|\frac{\mathrm{d}^n I_0(H)}{\mathrm{d}H^n}\right| \le CH^{-n} I_0(H) \tag{9}$$

for all $n = 0, 1, \ldots$ as H is sufficiently large. In particular,

$$cH^{\frac{1}{2}+\frac{1}{1+\alpha}} \le I_0(H) \le CH^{\frac{1}{2}+\frac{1}{1+\alpha}} \tag{10}$$

as H is sufficiently large.

(3). Note that $H_0(I)$ is the inverse function of $I_0(H)$. We have by (9) and (10) that

$$\left|\frac{\mathrm{d}^n H_0(I)}{\mathrm{d}I^n}\right| \le CI^{-n} H_0(I),$$

$$cI^{\frac{3+\alpha}{2+2\alpha}} \le H_0(I) \le CI^{\frac{3+\alpha}{2+2\alpha}}$$

for all $n = 0, 1, \ldots$ as I is sufficiently large.

Lemma 2.4. *As $x > -1$ and I sufficiently large,*

$$|\partial_I^n x(\theta, I)| \le CI^{-n}|x(\theta, I) + 1|, \quad |\partial_I^n y(\theta, I)| \le CI^{-n}|y(\theta, I)|$$

for all $n = 0, 1, \ldots$.

Proof. See Appendix. □

Lemma 2.5. *As I is sufficiently large,*

$$\left|I^k \partial_I^k \partial_t^l H_1(\theta, I, t)\right| \le C\sqrt{H_0(I)}, \qquad 0 \le k + l \le 5.$$

Proof. It follows from the definition of $H_1(\theta,I,t)$, Lemma 2.1 (2), and the fact that $|y(\theta,I)| \leq \sqrt{2H_0(I)}$. □

3 New Action-Angle Variables

Now we consider the forced Hamiltonian $H(\theta,I,t)$ in (8). The identity

$$I\mathrm{d}\theta - H(\theta,I,t)\mathrm{d}t = -(H\mathrm{d}t - I(t,H,\theta)\mathrm{d}\theta)$$

implies that if one can solve $I = I(t,H,\theta)$ in H from (8) with θ and t as parameters, then the Hamiltonian system (7) becomes

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \partial_H I(t,H,\theta), \quad \frac{\mathrm{d}H}{\mathrm{d}\theta} = -\partial_t I(t,H,\theta) \tag{11}$$

with Hamiltonian $I(t,H,\theta)$ and new action-angle variables H,t, and time variable θ. This point of view has already been adopted in [4].

As I is sufficient, because $\partial_I H(\theta,I,t) \neq 0$, one can indeed solve $I(t,H,\theta)$ as the inverse function of $H(\theta,I,t)$ with t,θ playing the role of parameters. Hence (11) is well defined when H is sufficiently large. We define $I_1(t,H,\theta)$ as such that

$$I(t,H,\theta) = I_0(H) + I_1(t,H,\theta).$$

Lemma 3.1. *As H is sufficiently large,*

$$\left|\partial_H^k \partial_t^l I_1(t,H,\theta)\right| \leq CH^{-k-\frac{1}{2}} I_0(H), \quad 0 \leq k+l \leq 5.$$

Proof. There are three cases to consider. Below, for the sake of brevity, we suspend the θ,t-dependence in most terms.

Case 1. $l = 0, k = 0$. From the definition of $I(t,H,\theta)$, we have

$$H_0(I(H)) + H_1(I(H)) = H, \tag{12}$$

or equivalently

$$I(H) = H_0^{-1}(H - H_1(I(H))) = I_0(H - H_1(I(H))). \tag{13}$$

Denote $H_1 = H_1(I(H))$. Then

$$I_1(H) = I(H) - I_0(H) = I_0(H - H_1(I(H))) - I_0(H) \tag{14}$$

$$= -I_0'(H)H_1 + \int_0^{H_1} s I_0''(H - H_1 + s)\,\mathrm{d}s. \tag{15}$$

We will estimate $I_1(H)$ through that of $H_1(I(H))$. By Lemmas 2.1 and 2.4 and Remark 2.3, we have that $I(H) \to \infty$ as $H \to \infty$ and $|H_1(I)| < \frac{1}{2}H_0(I)$ for all θ, t as $I \gg 1$. It follows that

$$|H_1(I(H))| < \frac{1}{2}H_0(I(H)) \tag{16}$$

for all t, θ as $H \gg 1$. Using (13), (16) and the monotonicity of I_0 in H, we have

$$I_0\left(\frac{1}{2}H\right) < I(H) < I_0\left(\frac{3}{2}H\right).$$

It follows from Remark 2.3 that

$$I_0\left(\frac{1}{2}H\right) > cI_0(H), \quad I_0\left(\frac{3}{2}H\right) < CI_0(H),$$

which implies that

$$cI_0(H) < I(H) < CI_0(H).$$

Using Remark 2.3 and Lemma 2.4 again, we obtain

$$|H_1(I(H))| \le C\sqrt{H_0(I(H))} \le C\sqrt{H_0(CI_0(H))} \le C\sqrt{H}.$$

This gives the desired estimate for the first term of (15) as

$$|I_0'(H)H_1(I(H))| \le CH^{-\frac{1}{2}}I_0(H).$$

The second term of (15) is bounded by

$$H_1^2 \sup_{H-H_1\le\widetilde{H}\le H} I_0''(\widetilde{H}) \le CH_1^2H^{-2}I_0(H) \le CH^{-1}I_0(H).$$

This completes Case 1.

Case 2. $l = 0, k \ge 1$. Differentiating (15) k times with respect to H yields

$$\partial_H^k I_1(H) = -\sum_{i=0}^{k} C_{ki}\partial_H^{i+1}I_0(H)\partial_H^{k-i}H_1 + \partial_H^k \int_0^{H_1} sI_0''(H-H_1+s)\,\mathrm{d}s,$$

where $H_1 = H_1(I(H))$ and, for each i, C_{ki} is an integer which only depends on k and i. Since

$$\partial_H^k H_1(I(H)) = \sum_{k_1+\cdots+k_j=k} C_{kj}\partial_I^j H_1(I(H))\partial_H^{k_1}I(H)\cdots\partial_H^{k_j}I(H)$$

and

$$\partial_H^k \int_0^{H_1} s I_0''(H - H_1 + s)\,\mathrm{d}s$$

$$= \sum_{i,j=0}^{k} C_{kji} \partial_H^j H_1 \partial_H^i H_1 \partial_H^{k-i-j+2} I_0(H)$$

$$+ \sum_{l_1+\cdots+l_j=k} C_{kjl} \partial_H^{l_1} H_0 \cdots \partial_H^{l_j} H_0 \int_0^{H_1} s\, \partial_H^{j+2} I_0(H - H_1 + s)\,\mathrm{d}s,$$

the proof of Case 2 is reduced to that of

$$\left|\partial_H^k I(H)\right| \leq C H^{-k} I(H), \quad H \gg 1, 1 \leq k \leq 5. \tag{17}$$

Differentiating (12) with respect to H yields

$$I'(H) = I_0'(H - H_1)[1 - H_1' I'(H)] = \frac{I_0'(H - H_1)}{1 + I_0'(H - H_1) H_1'}. \tag{18}$$

As H is sufficiently large, the denominator of the above is close to one and we thus have

$$I'(H) < 2 I_0'(H - H_1) < C(H - H_1)^{-1} I_0(H - H_1) < C H^{-1} I(H),$$

i.e., (17) holds when $k = 1$.

Using induction, we now assume that for some $n < 5$, (17) holds for all $1 \leq k \leq n$. Differentiating (18) n times with respect to H yields

$$\partial_H^{n+1} I(H) = \sum_{i=0}^{n} C_{ni} \partial_H^i I_0'(H - H_1) \partial_H^{n-i} \Delta,$$

where $\Delta = [1 + I_0'(H - H_1) H_1']^{-1}$. Hence the proof of (17) for $k = n+1$ is reduced to the that of

$$\left|\partial_H^m I_0'(H - H_1)\right| \leq C H^{-m} I_0'(H - H_1), \tag{19}$$

$$|\partial_H^m \Delta| \leq C H^{-m} \tag{20}$$

for all $m \leq n$ and $H \gg 1$.

Since

$$\partial_H^m I_0'(H - H_1) = \sum_{m_1+\cdots+m_j=m} C_{mj} I_0^{(j+1)}(H - H_1) \partial_H^{m_1}(H - H_1) \cdots \partial_H^{m_j}(H - H_1),$$

$$\partial_H^l (H - H_1) = \partial_H^l H_0(I(H)) = \sum_{l_1+\cdots+l_i=l} C_{li} \partial_I^i H_0(I(H)) \partial_H^{l_1} I(H) \cdots \partial_H^{l_i} I(H),$$

the proof of (19) easily follows from the induction hypothesis and Remark 2.3.

Note that

$$\partial_H^m \Delta = \sum C_{mj}[1+I_0'(H-H_1)H_1']^{-1-j}$$
$$\partial_H^{m_1}[1+I_0'(H-H_1)H_1'] \cdots \partial_H^{m_j}[1+I_0'(H-H_1)H_1'],$$

where $0 \le j \le m, m_1 + \cdots + m_j = m$. The proof of (20) follows from (19) and Lemma 2.5.

Case 3. $k \ge 0, l \ge 1$. Similar to the above, the proof of Case 3 is reduced to that of

$$\left|\partial_H^k \partial_t^l I(t,H,\theta)\right| \le CH^{-k} I(t,H,\theta)$$

for all $k \ge 0, l \ge 1$ and $H \gg 1$. In fact, according to (18), Remark 2.3, and Lemma 2.4, we conclude that the differentiation of $\partial_H^k I(t,H,\theta)$ l times with respect to t does not increase the order of the upper bound. □

4 Proof of the Main Result

We rewrite (11) more explicitly as

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = I_0'(H) + \partial_H I_1(t,H,\theta), \qquad \frac{\mathrm{d}H}{\mathrm{d}\theta} = -\partial_t I_1(t,H,\theta), \tag{21}$$

where, as shown in the previous two sections, the functions $I_0(H)$ and $I_1(t,H,\theta)$ for sufficiently large H satisfy the following estimates:

$$cH^{\frac{1}{2}+\frac{1}{1+\alpha}} \le I_0(H) \le CH^{\frac{1}{2}+\frac{1}{1+\alpha}}, \tag{22}$$

$$\frac{\mathrm{d}^k I_0(H)}{\mathrm{d}H^k} \ge cH^{-k} I_0(H), \qquad k = 1,2, \tag{23}$$

$$\left|\frac{\mathrm{d}^k I_0(H)}{\mathrm{d}H^k}\right| \le CH^{-k} I_0(H), \qquad 0 \le k \le 5, \tag{24}$$

$$|\partial_H^k \partial_t^l I_1(t,H,\theta)| \le CH^{-k-\frac{1}{2}} I_0(H), \qquad 0 \le k+l \le 5. \tag{25}$$

For $H_0 > 0$, we consider the domain

$$A_{H_0} = \{(t,H) | H \ge H_0, t \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}\}$$

and the diffeomorphism Ψ: $A_{H_0} \times \mathbb{S}^1 \to A_{H_0} \times \mathbb{S}^1$:

$$\lambda = I_0'(H), \;\; t = t, \;\; \theta = \theta.$$

Then system (21) under the transformation Ψ becomes

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} = f_1(\lambda,t,\theta), \qquad \frac{\mathrm{d}t}{\mathrm{d}\theta} = \lambda + f_2(\lambda,t,\theta), \tag{26}$$

where

$$f_1(\lambda,t,\theta) = -I_0''(H)\partial_t I_1(t,H,\theta),\ \ f_2(\lambda,t,\theta) = \partial_H I_1(t,H,\theta),$$

with $H = H(\lambda)$ defined through the transformation Ψ.

To estimate f_1 and f_2, we note by (22)–(24) that

$$cH^{\frac{1}{1+\alpha}-\frac{1}{2}} \le I_0'(H) \le CH^{\frac{1}{1+\alpha}-\frac{1}{2}}.$$

It follows that

$$c\lambda^{\frac{2(1+\alpha)}{1-\alpha}} \le H(\lambda) \le C\lambda^{\frac{2(1+\alpha)}{1-\alpha}}.$$

Thus $H \gg 1$ if and only if $\lambda \gg 1$. Moreover, by (23), (24), as $\lambda \gg 1$,

$$\left|\frac{\mathrm{d}^k H(\lambda)}{\mathrm{d}\lambda^k}\right| \le C\lambda^{-k}H(\lambda),$$

and consequently

$$|\lambda^k\partial_\lambda^k\partial_t^l f_1(\lambda,t,\theta)| \le CH^{-2-\frac{1}{2}}I_0(H)^2 \le CH^{-\frac{3\alpha-1}{2+2\alpha}} \le C\lambda^{-\frac{3\alpha-1}{1-\alpha}} \le C\lambda^{-\sigma},$$
$$|\lambda^k\partial_\lambda^k\partial_t^l f_2(\lambda,t,\theta)| \le CH^{-1-\frac{1}{2}}I_0(H) \le CH^{-\frac{\alpha}{1+\alpha}} \le C\lambda^{-\frac{2\alpha}{1-\alpha}} \le C\lambda^{-\sigma},$$

for all $0 \le k+l \le 4$, where $\sigma = \min\{\frac{3\alpha-1}{1-\alpha}, \frac{2\alpha}{1-\alpha}\} = \frac{3\alpha-1}{1-\alpha} > 0$.

Since f_1 and f_2 are sufficiently small as $\lambda \gg 1$, all solutions of (26) exist for $0 \le \theta \le 1$ when the initial values $\lambda(0) = \lambda$ are sufficiently large. Hence the Poincaré map Φ associated to (26) is well defined on A_{λ_0} as $\lambda_0 \gg 1$. In fact, by integrating (26) from $\theta = 0$ to $\theta = 1$, we see that Φ has the form

$$\Phi: \quad t_1 = t_0 + \lambda_0 + \Xi_1(t_0,\lambda_0),\ \ \lambda_1 = \lambda_0 + \Xi_2(t_0,\lambda_0),$$

where Ξ_1 and Ξ_2 satisfy the same estimates as those of f_1 and f_2, i.e.,

$$\left|\partial_{\lambda_0}^k\partial_{t_0}^l\Xi_i\right| \le C\lambda_0^{-\sigma}, \quad i = 1,2,\ 0 \le k+l \le 4.$$

According to [2], the Poincaré map $\mathcal{P}$ associated to (21) admits the intersection property on A_{H_0}, i.e., if Γ is an embedded circle in A_{H_0} homotopic to a circle H = constant, then $\mathcal{P}(\Gamma)\cap\Gamma \neq \emptyset$. It follows that Φ also admits the intersection property on A_{λ_0}. Hence Φ satisfies all the assumptions of Moser's twist theorem [12, 14], from which we conclude that for any $\omega \gg 1$ satisfying

$$\left|\omega - \frac{p}{q}\right| \geq c_0|q|^{-\frac{5}{2}}, \qquad \frac{p}{q} \in \mathbb{Q}, \tag{27}$$

there is an invariant curve Γ_ω of Φ which is conjugated to a pure rotation of the circle with rotation number ω. Tracing back to the system (4), Γ_ω gives rise to an invariant closed curve of the Poincaré map of (4) with rotation number $\frac{1}{\omega}$ which surrounds and is arbitrarily far away from the origin. Hence all solutions of (1) are bounded. This completes the proof of the theorem. □

In fact, also applying Aubry–Mather theory [10, 13] to the Poincaré map Φ, one can obtain more precise dynamics of (1) as follows.

Proposition *Under conditions of the theorem, there is an* $\varepsilon_0 > 0$ *such that the following hold:*

(a) *For any rational number* $\frac{p}{q} \in (0, \varepsilon_0)$*,* (1) *admits an unlinked periodic solution (of Birkhoff type) with period* q*.*
(b) *For any irrational number* $\omega \in (0, \varepsilon_0)$*,* (1) *admits weak quasiperiodic solutions with frequency* $(1, \omega)$ *corresponding to an Aubry–Mather set.*
(c) *For any irrational number* $\omega \in (0, \varepsilon_0)$ *with* $\frac{1}{\omega}$ *satisfying the Diophantine condition* (27)*,* (1) *admits a quasiperiodic solution with frequency* $(1, \omega)$*.*

We finally remark that the weak quasiperiodic solutions asserted by Aubry–Mather theory form a special class of almost automorphic functions generating the almost periodic ones [15].

Appendix

A1. Some Properties on the Potential $V(x)$

Let

$$W(x) = \frac{V(x)}{V'(x)} = \begin{cases} \frac{x}{\alpha+1}, & \text{if } x \geq 0; \\ \frac{1}{2}x(1-x)(1+x), & \text{if } -1 < x < 0. \end{cases}$$

Then

$$W'(x) = \begin{cases} \frac{1}{\alpha+1}, & \text{if } x > 0; \\ \frac{1}{2}(1-3x^2), & \text{if } -1 < x < 0, \end{cases}$$

$$W''(x) = \begin{cases} 0, & \text{if } x > 0; \\ -3x, & \text{if } -1 < x < 0, \end{cases}$$

$$W^{(3)}(x) = \begin{cases} 0, & \text{if } x > 0; \\ -3, & \text{if } -1 < x < 0, \end{cases}$$

and

$$W^{(4)}(x) = 0, \qquad x \in \mathbb{R} \setminus \{0\}.$$

Also, if $n \geq 1$, then

$$V^{(n)}(x) = \begin{cases} (\alpha+1)\alpha \cdots [(\alpha+1)-(n-1)]x^{(\alpha+1)-n}, & \text{if } x > 0; \\ g_n(x)(1+x)^{-n-1}, & \text{if } -1 < x < 0, \end{cases}$$

where $g_n(x)$ is defined recursively by the following formulas:

$$\begin{cases} g_1(x) &= 2x(1-x)^{-2}, \\ g_{n+1}(x) &= (1+x)g_n'(x) - (n+1)g_n(x), \qquad n \geq 1. \end{cases}$$

It is clear that there exists a constant $C = C_n$ such that for all $x \in (-1, 0)$,

$$|V^{(n)}(x)| \leq C(1+x)^{-n-1}.$$

A2. Some Derivative Formulas

Here we list some derivative formulas which are very similar to those in [4]. Let $K(x, H)$ be a smooth function and define

$$\mathscr{I}(H) = \int_{-\alpha_H}^{0} K(\xi, H)\sqrt{H - V(\xi)}\,\mathrm{d}\xi, \tag{A2.1}$$

$$\mathscr{X}(I, \theta) = \int_{-\alpha_H}^{x(\theta, I)} K(\xi, I)\frac{\mathrm{d}\xi}{\sqrt{H(I) - V(\xi)}}. \tag{A2.2}$$

Then

$$\frac{\mathrm{d}\mathscr{I}(H)}{\mathrm{d}H} = \int_{-\alpha_H}^{0} \mathscr{H}(K)(\xi, H)\sqrt{H - V(\xi)}\mathrm{d}\xi,$$

$$\begin{aligned} \partial_I \mathscr{X}(I, \theta) = K(x, I)\int_{-\alpha_H}^{x(\theta, I)} L(\xi, I)\frac{\mathrm{d}\xi}{\sqrt{H(I) - V(\xi)}} \\ + \int_{-\alpha_H}^{x(\theta, I)} \mathscr{L}(K)(\xi, I)\frac{\mathrm{d}\xi}{\sqrt{H(I) - V(\xi)}}, \end{aligned}$$

where

$$\mathscr{H}(K)(x,H) = \frac{1}{H}\left[H\partial_H K(x,H) + \frac{1}{2}K(x,H) + \partial_x\left(K(x,H)W(x)\right)\right],$$
$$L(x,I) = -\frac{H''}{H'} + \frac{H'}{H}\left(W' - \frac{1}{2}\right),$$
$$\mathscr{L}(K)(x,I) = \frac{H'}{H}\left(\partial_x\left(K(I,x)W(x)\right) - \frac{1}{2}K(x,I)\right) + \partial_I K(x,I).$$

When applying (A2.1) to $I_0(H)$, we will obtain its derivatives of any order.

When applying (A2.2) to $x(I,\theta), y(I,\theta)$, we particularly have

$$\partial_I x(\theta,I) = \sqrt{H-V(x)}\int_{-\alpha_H}^{x(\theta,I)} L(\xi,I)\frac{\mathrm{d}\xi}{\sqrt{H-V(\xi)}} + \frac{H'}{H}W(x) \tag{A2.3}$$

and

$$\partial_I y(\theta,I) = \frac{H'}{H}\sqrt{2(H-V(x))} - V'(x)\int_{-\alpha_H}^{x(\theta,I)} L(\xi,I)\frac{\mathrm{d}\xi}{\sqrt{2(H-V(\xi))}}, \tag{A2.4}$$

where $H = H_0(I)$.

A3. Proof of Lemma 2.2

We recall that

$$I_0^-(H) = 2\int_{-\alpha_H}^{0} \sqrt{2(H-V(\xi))}\,\mathrm{d}\xi.$$

By (A2.2), differentiating the equality n times with respect to H yields

$$\frac{\mathrm{d}^n I_0^-(H)}{\mathrm{d}H^n} = 2\sqrt{2}\int_{-\alpha_H}^{0} \mathscr{H}^n(1)(\xi,H)\sqrt{H-V(\xi)}\,\mathrm{d}\xi.$$

Using induction (see also [4, Lemma 6.2.1] and [1, Proposition 5.1]), we have that for every $n \geq 1$

$$\mathscr{H}^n(1)(x,H) = \frac{1}{H^n}P_n(x),$$

where $P_n(x)$ is recursively defined by

$$\begin{cases} P_1(x) &= \frac{1}{2} + W'(x), \\ P_{n+1}(x) &= \left(\frac{1}{2} - n\right)P_n(x) + (WP_n)'(x), \qquad n \geq 1. \end{cases}$$

Then

$$\frac{\mathrm{d}^n I_0^-(H)}{\mathrm{d}H^n} = \frac{2\sqrt{2}}{H^n}\int_{-\alpha_H}^{0} P_n(\xi)\sqrt{H-V(\xi)}\,\mathrm{d}\xi,$$

or

$$H^{n-\frac{1}{2}}\frac{\mathrm{d}^n I_0^-(H)}{\mathrm{d}H^n} = 2\sqrt{2}\int_{-\alpha_H}^{0} P_n(\xi)\sqrt{1-\frac{V(\xi)}{H}}\,\mathrm{d}\xi.$$

Since $\alpha_H \to 1$ as $H \to +\infty$, we have

$$\lim_{H\to+\infty} H^{n-\frac{1}{2}}\frac{\mathrm{d}^n I_0^-(H)}{\mathrm{d}H^n} = 2\sqrt{2}\int_{-1}^{0} P_n(\xi)\mathrm{d}\xi.$$

Therefore the proof of Lemma 2.2 is reduced to the computation of the integral $\int_{-1}^{0} P_n(x)\mathrm{d}x$, which had be done in [1]. Since $W(0) = W(-1) = 0$ and $\int_{-1}^{0} P_1(x)\mathrm{d}x = \frac{1}{2}$, we have that

$$\begin{aligned}
\int_{-1}^{0} P_n(x)\mathrm{d}x &= \left(\frac{3}{2}-n\right)\int_{-1}^{0} P_{n-1}(x)\,\mathrm{d}x \\
&= \cdots = \left(\frac{3}{2}-n\right)\left(\frac{3}{2}-(n-1)\right)\cdots\left(\frac{3}{2}-2)\right)\int_{-1}^{0} P_1(x)\mathrm{d}x \\
&= \cdots = \left(\frac{3}{2}-n\right)\left(\frac{3}{2}-(n-1)\right)\cdots\left(\frac{3}{2}-2)\right)\frac{1}{2} \\
&= (-1)^{n-1}\frac{(2n-3)!!}{2^n}.
\end{aligned}$$

This proves Lemma 2.2.

A4. Proof of Lemma 2.4

We first consider the case $n = 1$.

When $x \geq 0$, we rewrite (A2.3) as

$$\begin{aligned}
\partial_I x(\theta,I) = \sqrt{H-V(x)}&\int_{-\alpha_H}^{0} L(\xi,I)\frac{\mathrm{d}\xi}{\sqrt{H-V(\xi)}} \\
&+\sqrt{H-V(x)}\int_{0}^{x} L(\xi,I)\frac{\mathrm{d}\xi}{\sqrt{H-V(\xi)}} + \frac{H'}{H}W(x).
\end{aligned} \tag{A4.1}$$

Since, for $-1 < x < 0$, $-1 < W'(x) < \frac{1}{2}$ and $|\frac{H''}{H'}|, |\frac{H'}{H}|$ are bounded by CI^{-1} for sufficiently large I (Remark 2.1), then $|L| \leq CI^{-1}$. It follows from Lemma 2.2 that, for sufficiently large I, the absolute value of the first term on the right-hand side of (A4.1) is bounded by $CI^{-1}\sqrt{H}T_0^-(H) \le CI^{-1}$. Since, for $0 \le \xi \le x$, $0 \le \dfrac{\sqrt{H-V(x)}}{\sqrt{H-V(\xi)}} \le 1$, the absolute value of the second term on the right-hand side of (A4.1) is bounded by $CI^{-1}x$. Using Remark 2.1 and the fact that $W(x) = \frac{x}{\alpha+1}$ for $x \ge 0$, the absolute value of the third term on the right-hand side of (A4.1) is also bounded above by $CI^{-1}x$. Thus the estimate of $x(\theta, I)$ in Lemma 2.4 holds when $n = 1$ and $x \ge 0$. When $-1 < x < 0$, we note that $-1 < \dfrac{W(x)}{1+x} = \frac{1}{2}x(1-x) < 0$ and $|\frac{H'}{H}|$ is bounded by CI^{-1} for sufficiently large I (Remark 2.1). Hence the absolute value of the second term on the right-hand side of (A2.3) is bounded by $CI^{-1}(1+x)$. To estimate the first term on the right-hand side of (A2.3), we let

$$A(x) = \int_{-\alpha_H}^{x} L(\xi, I)\frac{\mathrm{d}\xi}{\sqrt{H-V(\xi)}}, \qquad B(x) = \frac{I^{-1}(\alpha_H + x)}{\sqrt{H-V(x)}}.$$

We note that $A(-\alpha_H) = 0$ and

$$B(-\alpha_H) = \lim_{x\to -\alpha_H} \frac{I^{-1}\cdot(\alpha_H + x)}{\sqrt{H-V(x)}} = -2I^{-1}\lim_{x\to -\alpha_H}\frac{\sqrt{H-V(x)}}{V'(x)} = 0.$$

Since $V'(x) < 0$ is monotonely increasing in $(-1,0)$, the mean value Theorem implies that there exists a $\eta \in [-\alpha_H, x]$ such that

$$V(x) - H = V(x) - V(-\alpha_H) = V'(\eta)(x + \alpha_H).$$

Hence

$$\left|\frac{V'(x)(\alpha_H + x)}{H - V(x)}\right| \le 1,$$

from which it easily follows that for some sufficiently large constant C

$$-CI^{-1}\left[1 + \frac{1}{2}\frac{V'(x)(\alpha_H + x)}{H-V(x)}\right] \le L(x,I) \le CI^{-1}\left[1 + \frac{1}{2}\frac{V'(x)(\alpha_H + x)}{H-V(x)}\right],$$

i.e.,

$$-CB'(x) \le A'(x) \le CB'(x)$$

for $-\alpha_H \le x \le 0$. Integrating the above inequalities from $-\alpha_H$ to x yields that

$$\left|\sqrt{H-V(x)}\int_{-\alpha_H}^{x} L(\xi, I)\frac{\mathrm{d}\xi}{\sqrt{H-V(\xi)}}\right| \le CI^{-1}(\alpha_H + x).$$

Thus the estimate of $x(\theta, I)$ in Lemma 2.4 holds when $n = 1$ and $-1 < x < 0$.

The estimate of $x(\theta,I)$ in Lemma 2.4 for general n follows from similar arguments and induction.

Similarly, we obtain the desired estimate on $y(\theta,I)$.

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Semiflows for Neutral Equations with State-Dependent Delays

Hans-Otto Walther

Abstract We show that under mild hypotheses neutral functional differential equations where delays may be state-dependent generate continuous semiflows, a larger one on a thin set in a Banach space of C^1-functions and a smaller one, with better smoothness properties, on a closed subset in a Banach manifold of C^2-functions. The hypotheses are satisfied for a prototype equation of the form

$$x'(t) = ax'(t + d(x(t))) + f(x(t))$$

with $-h < d(x(t)) < 0$, which for certain d and f models the interaction between following a trend and negative feedback with respect to some equilibrium state.

Mathematics Subject Classification (2010): Primary 34K40, 37L05; Secondary 34K05, 58B99

1 Introduction

This paper deals with retarded functional differential equations of neutral type where delays may be state-dependent. Let us begin with a simple example, namely

$$x'(t) = ax'(t + d(x(t))) + f(x(t)) \tag{1}$$

for given functions $d : \mathbb{R} \to (-h, 0)$ and $f : \mathbb{R} \to \mathbb{R}$, and $a > 0$. In case d is increasing for $\xi < 0$ and decreasing for $\xi > 0$, $f(0) = 0$ and $\xi f(\xi) < 0$ for $\xi \neq 0$, Eq. (1) has

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the following interpretation: The state tries to keep a fraction of its rate of change, under the influence of immediate negative feedback with respect to the equilibrium at $\xi = 0$. The larger the deviation from equilibrium, the closer to the present the (fraction of the) rate of change which is to be continued. In short this may be called a strategy of *cautiously following a trend*.

Equation (1) is a modification of a (non-neutral) differential equation with constant delay which was introduced as a model for a currency exchange rate by Brunovský and studied in [2, 24, 25]. A further version of Brunovský's model, non-neutral and with state-dependent delay, is investigated by Stumpf [20].

Notice that in Eq. (1) the delay is never zero. Therefore a solution which is known up to some $t = t_0$ can be determined for $t > t_0$ not too large from a simpler nonautonomous (ordinary, for Eq. (1)) differential equation. All results in this paper are built on this observation. Related ideas were used in [19, 21].

Other simple-looking neutral equations with state-dependent delay arise, for example, as modifications of the business cycle model studied by Preisenberger [18] and from the soft landing model in [26].

We consider the equation

$$x'(t) = g(\partial x_t, x_t) \tag{2}$$

for a functional

$$g : C \times C^1 \supset W \to \mathbb{R}^n,\ W \text{ open}.$$

Here C denotes the Banach space of continuous functions $[-h,0] \to \mathbb{R}^n$, for some $h > 0$ and $n \in \mathbb{N}$, with the norm given by $|\phi| = \max_{-h \le t \le 0} |\phi(t)|$. C^1 is the Banach space of continuously differentiable functions $[-h,0] \to \mathbb{R}^n$, with the norm given by $|\phi|_1 = |\phi| + |\partial \phi|$, and $\partial : C^1 \to C$ is the continuous linear operator of differentiation. The segment $x_t \in C$ of a continuous function $x : \mathbb{R} \supset I \to \mathbb{R}^n$ with $[t-h,t] \subset I$ is defined by $x_t(s) = x(t+s)$.

Equation (1) is the special case of Eq. (2) where $n = 1$, $W = C \times C^1$, and

$$g(\phi, \psi) = a\phi(d(\psi(0))) + f(\psi(0)).$$

A solution of Eq. (2) is a continuously differentiable function $x : [t_0 - h, t_e) \to \mathbb{R}^n$, with $t_0 < t_e \le \infty$ and $(\partial x_t, x_t) \in W$ for $t_0 \le t < t_e$, which satisfies Eq. (2) for $t_0 < t < t_e$. Solutions on compact intervals and solutions of certain nonautonomous equations are defined analogously.

For any solution $x : [t_0 - h, t_e) \to \mathbb{R}^n$ all segments belong to the open subset

$$U_1 = \{\psi \in C^1 : (\partial \psi, \psi) \in W\}$$

of C^1, and Eq. (2) shows that for $t_0 < t < t_e$ the segment x_t satisfies

$$\psi'(0) = g(\partial \psi, \psi). \tag{3}$$

It will be convenient to define

$$X_1 = X_{g,1} = \{\psi \in U_1 : (3) \text{ holds}\}.$$

We are interested in existence, uniqueness and smoothness properties of solutions $x : [-h, t_e) \to \mathbb{R}^n$ to the initial value problem (IVP)

$$x'(t) = g(\partial x_t, x_t),\ x_0 = \psi,$$

under mild conditions on g which are suitable for applications to equations with state-dependent delay.

Results for non-neutral equations of the form

$$x'(t) = g_{nn}(x_t) \tag{4}$$

which cover state-dependent delays suggest that existence and uniqueness of solutions to the IVP of Eq. (2) require a Lipschitz continuous argument $(\partial x_t, x_t)$ of g; in particular initial data $x_0 = \psi$ should have a Lipschitz continuous derivative—see for example [12, 16, 17, 21–23]. Furthermore, a continuous semiflow might be found on the set $X_{g,1}$ or on subsets of $X_{g,1}$, but not on the open set $U_1 \subset C^1$. This is supported by a result of Krisztin and Wu [15] who obtained a semiflow for neutral equations with constant delay on an analogue of $X_{1,g}$. Differentiability of solutions with respect to initial data can be expected if the argument $(\partial x_t, x_t)$ of g is continuously differentiable, so for this initial data $x_0 = \psi$ ought to be taken from the space C^2 of two times continuously differentiable functions $[-h, 0] \to \mathbb{R}^n$, with the norm given by $|\psi|_2 = |\psi| + |\partial\psi|_1$.

The results which we obtain are more complicated than in the non-neutral case (4), and differentiability properties of solutions are weaker. Let us turn to the hypotheses on $g : W \to \mathbb{R}^n$, $W \subset C \times C^1$ open, which are used in the sequel. In Sect. 3 below it is shown that all of them are satisfied for Eq. (1) if d and f are differentiable with derivatives locally Lipschitz continuous.

We assume that

(g0) *g is continuous.*

The next hypothesis is an abstract version of the property that in the neutral term in Eq. (1) the delay is never zero.

(g1) *For every $\psi \in U_1 \subset C^1$ there exist $\Delta \in (0, h)$ and a neighbourhood $N \subset W$ of $(\partial\psi, \psi)$ in $C \times C^1$ such that for all $(\phi_1, \chi), (\phi_2, \chi)$ in N with*

$$\phi_1(t) = \phi_2(t) \text{ for all } t \in [-h, -\Delta]$$

we have

$$g(\phi_1, \chi) = g(\phi_2, \chi).$$

Recall that the Lipschitz constant of a map $a : B \supset M \to B_1$, B and B_1 Banach spaces, is given by

$$Lip(a) = \sup_{x \neq y} \frac{|a(x) - a(y)|}{|x - y|} \leq \infty.$$

For the basic result on existence and uniqueness in Sect. 4 below we need, in addition to (g0) and (g1), the following property.

(g2) *For every $\psi \in U_1 \subset C^1$ there exist $L \geq 0$ and a neighbourhood $N \subset W$ of $(\partial\psi, \psi)$ in $C \times C^1$ such that for all $(\phi_1, \psi_1), (\phi_2, \psi_2)$ in N we have*

$$|g(\phi_2, \psi_2) - g(\phi_1, \psi_1)| \leq L(|\phi_2 - \phi_1| + (Lip(\phi_2) + 1)|\psi_2 - \psi_1|).$$

In the non-neutral case where g does not depend on its first argument, i. e. for Eq. (4), property (g2) reduces to local estimates which were essential for the construction of solutions in [22]; see also the property of being *locally almost Lipschitzian* in [16] and Section 3.2 in [12]. Notice that on the right hand side of these estimates the smaller norm $|\cdot|$ appears, and not $|\cdot|_1$—they are stronger than local Lipschitz continuity.

In Sect. 4 below we use the hypotheses (g0)–(g2) in order to construct a semiflow $(t, x_0) \mapsto x_t$ of solutions to Eq. (2) on the set

$$X_{1+} = \{\psi \in X_1 : Lip(\partial\,\psi) < \infty\}.$$

This semiflow is continuous with respect to the topologies given by $|\cdot|_1$. The construction in Sect. 4 is done without recourse to earlier results from [22, 23], in order to keep the present paper self-contained.

One can show that in case of Eq. (1) the set X_{1+} is a global graph

$$Y \to L, \text{ where } C^{1+} = \{\psi \in C^1 : Lip(\partial\,\psi) < \infty\} = Y \oplus L, \ \dim L = 1.$$

In order to obtain a semiflow with better smoothness properties we assume the following.

(g3) *The restriction g_1 of g to the open subset $W_1 = W \cap (C^1 \times C^1)$ of the space $C^1 \times C^1$ is continuously differentiable, every derivative $Dg_1(\phi, \psi) : C^1 \times C^1 \to \mathbb{R}^n$, $(\phi, \psi) \in W_1$, has a linear extension*

$$D_e g_1(\psi, \phi) : C \times C \to \mathbb{R}^n,$$

and the map

$$W_1 \times C \times C \ni (\phi, \psi, \chi, \rho) \mapsto D_e g_1(\phi, \psi)(\chi, \rho) \in \mathbb{R}^n$$

is continuous.

Hypothesis (g3) generalizes a smoothness property which was crucial in [22, 23] and in [14], and which occurred earlier in [16] under the name of *almost Frechet differentiability*. The existence and continuity of the extended derivatives $D_e g_1(\phi,\psi)$ in hypothesis (g3) yields in Sect. 5 that the set

$$X_2 = X_1 \cap C^2,$$

if non-empty, is a continuously differentiable submanifold in C^2, with codimension n. Its tangent spaces $T_\psi X_2 \subset C^2$, $\psi \in X_2$, are given by the equation

$$\chi'(0) = Dg_1(\partial\,\psi,\psi)(\partial\,\chi,\chi). \tag{5}$$

We define extended tangent spaces $T_{e,\psi}X_2 \subset C^1$ by

$$\chi'(0) = D_e g_1(\partial\,\psi,\psi)(\partial\,\chi,\chi). \tag{6}$$

In Sect. 6 it is shown that under the hypotheses (g0)–(g3) the solutions of Eq. (2) define a semiflow on the closed subset

$$\begin{aligned} X_{2*} &= \{\psi \in X_2 : \partial\,\psi \in T_{e,\psi}X_2\} \\ &= \{\psi \in X_2 : \partial\,\psi \text{ satisfies } (6)\} \end{aligned}$$

of the manifold X_2. This semiflow $G_2 : \Omega_2 \to X_{2*}$, $\Omega_2 \subset [0,\infty) \times X_{2*}$, is continuous with respect to the topologies given by $|\cdot|_2$.

The hypotheses (g0)–(g3) are also used in the first part of Sect. 7 where a linear variational equation with right hand side given by $D_e g$ is discussed, in order to prepare the proof of a differentiability property of the solution operators $G_2(t,\cdot) : X_{2*} \supset \Omega_{2,t} \to X_{2*}$ in Sect. 8. Existence and uniqueness of solutions to this variational equation are not immediate from the sections of the monographs [3, 9] about linear (homogeneous and inhomogeneous) nonautonomous retarded functional differential equations. An appendix, Sect. 11, provides the little we need.

For the proof in Sect. 8 we require a further hypothesis.

(g4) *The restriction g_1 of g to the open subset $W_1 = W \cap (C^1 \times C^1)$ of the space $C^1 \times C^1$ is continuously differentiable, and for every $(\phi_0,\psi_0) \in W_1$ there exist $c \geq 0$ and a neighbourhood $N \subset W_1$ of (ϕ_0,ψ_0) in $C^1 \times C^1$ such that for all $(\phi,\psi),(\phi_1,\psi_1)$ in N and for all $\chi \in C^1$ we have*

$$|(Dg_1(\phi,\psi) - Dg_1(\phi_1,\psi_1))(\chi,0)| \leq c|\partial\chi||\psi - \psi_1|.$$

Under the hypotheses (g0)–(g4) we show in Sect. 8 that compositions $j \circ G_2(t,\cdot) \circ p$ of $G_2(t,\cdot)$ with the inclusion map $j : C^2 \to C^1$ and continuously differentiable maps $p : Q \to C^2$, Q an open subset of some Banach space B and $p(Q) \subset \Omega_{2,t}$, are differentiable.

The question arises whether the derivatives obtained in Sect. 8 are continuous. In Sects. 9 (about the variational equation) and 10 we establish continuity at points $q \in Q$ with $Lip(\partial\partial p(q)) < \infty$, provided (g0)–(g4) hold and the following final hypothesis is satisfied.

(g5) *The restriction g_1 of g to the open subset $W_1 = W \cap (C^1 \times C^1)$ of the space $C^1 \times C^1$ is continuously differentiable, every derivative $Dg_1(\phi,\psi) : C^1 \times C^1 \to \mathbb{R}^n$, $(\phi,\psi) \in W_1$, has a linear extension*

$$D_e g_1(\psi,\phi) : C \times C \to \mathbb{R}^n,$$

and for every $(\phi_0,\psi_0) \in W_1$ there exist $c \geq 0$ and a neighbourhood $N \subset W_1$ of (ϕ_0,ψ_0) in $C^1 \times C^1$ such that for all $(\phi,\psi),(\phi_1,\psi_1)$ in N and for all $(\chi,\rho) \in C \times C$ with $|(\chi,\rho)|_{C\times C} = 1$ we have

$$|(D_e g_1(\phi,\psi) - D_e g_1(\phi_1,\psi_1))(\chi,\rho)| \leq |(\phi,\psi) - (\phi_1,\psi_1)|_{C^1\times C^1}\, c \cdot (Lip(\chi) + Lip(\partial\phi) + 1).$$

For earlier results on IVPs for classes of neutral differential equations with state-dependent delays see e.g. [1, 8, 10, 11, 13], and Driver's work about special cases of the two-body problem of electrodynamics and related IVPs [4–7]. The general equations of motion for two charged particles in space can be viewed as an implicit, neutral system with unbounded state-dependent deviating arguments. This system is beyond the scope of the present results.

2 Preliminaries

We begin with some notation. For a finite Cartesian product of sets the projection onto the kth component is denoted by pr_k. The closure and boundary of a subset M of a topological space are denoted by $cl\,M$ and $bd\,M$, respectively. Norms on Cartesian products $X \times Y$ of normed vectorspaces X, Y are always given by addition, $|(x,y)| = |x| + |y|$. For given Banach spaces E, F the Banach space of continuous linear maps $E \to F$ is denoted by $L_c(E,F)$.
For derivatives of functions $x : I \to E$, $I \subset \mathbb{R}$, we have $x'(t) = Dx(t)1$.

The proofs of the first four propositions are omitted.

Proposition 2.1. *For reals $a < b < c$ and functions $\alpha : [a,b] \to \mathbb{R}$, $\beta : [b,c] \to \mathbb{R}$ with $\alpha(b) = \beta(b)$ the concatenation $\alpha\beta : [a,c] \to \mathbb{R}$ (given by α on $[a,b]$ and by β on $[b,c]$) satisfies*

$$Lip(\alpha\beta) \leq Lip(\alpha) + Lip(\beta).$$

Let $h > 0, n \in \mathbb{N}$. Recall the spaces C, C^1, C^2 from the introduction. The evaluation map

$$ev : [-h,0] \times C \ni (s,\psi) \mapsto \psi(s) \in \mathbb{R}^n$$

is continuous, but not locally Lipschitz continuous. The linear maps $ev(s,\cdot) : C \to \mathbb{R}^n$, $-h \leq s \leq 0$, are continuous, as well as their restrictions $ev_1(s,\cdot)$ to C^1 (with respect to $|\cdot|_1$). The map

$$[-h,0] \ni t \mapsto ev_1(t,\cdot) \in L_c(C^1,\mathbb{R}^n)$$

is continuous, due to the estimate

$$|ev(s,\phi) - ev(t,\phi)| = |\phi(s) - \phi(t)| \leq |\partial\phi||s-t|$$

for $-h \leq s \leq t \leq 0$ and $\phi \in C^1$. The restriction ev_1 of ev to $(0,h) \times C^1$ is continuously differentiable (with respect to the norm on $\mathbb{R} \times C^1$) , with

$$D\,ev_1(s,\phi)(t,\chi) = t\,\partial\phi(s) + \chi(s).$$

Proposition 2.2. *Let $x : [t_0 - h, t_e) \to \mathbb{R}^n$, $t_0 < t_e \leq \infty$, be continuously differentiable. Then the maps $[t_0,t_e) \ni t \mapsto x_t \in C^1$ and $[t_0,t_e) \ni t \mapsto \partial x_t \in C$ are continuous, and*

$$\tilde{x} : [t_0,t_e) \ni t \mapsto x_t \in C$$

is continuously differentiable with $\tilde{x}'(t) = \partial x_t = (x')_t$ for $t_0 \leq t < t_e$.

Proposition 2.3. *Let $x : [t-h,s] \to \mathbb{R}^n$, $t < s$, be continuously differentiable, with $x_t \in C^2$, and let $r > 0$. Then there is a twice continuously differentiable function $y : [t-h,s] \to \mathbb{R}^n$ with $y_t = x_t$ and*

$$|y'(u) - x'(u)| < r \text{ for all } u \in [t,s].$$

We need three prolongations of functions $\phi : [-h,0] \to \mathbb{R}^n$ to functions on $[-h,\infty)$: $\phi^c : [-h,\infty) \to \mathbb{R}^n$ is given by $\phi^c(t) = \phi(0)$ for $t \geq 0$. In case $\phi \in C^1$ we define $\phi^d : [-h,\infty) \to \mathbb{R}^n$ by

$$\phi^d(t) = \phi(0) + t\phi'(0) \text{ for } t > 0,$$

and for $\phi \in C^2$ we define $\phi^{dd} : [-h,\infty) \to \mathbb{R}^n$ by

$$\phi^{dd}(t) = \phi(0) + t\phi'(0) + t^2\frac{\phi''(0)}{2} \text{ for } t > 0.$$

Proposition 2.4. *(i) For $\phi \in C$ and $t \geq 0$, $|(\phi^c)_t| \leq |\phi|$.*
(ii) The map $[0,\infty) \times C^1 \ni (t,\phi) \mapsto (\phi^d)_t \in C^1$ is continuous. For each $\phi \in C^1$,

$$(\phi^d)' = (\partial\phi)^c \text{ and } Lip((\phi^d)') \leq Lip(\partial\phi),$$

and for all $t \geq 0$,

$$|(\phi^d)_t|_1 \leq (1+t)|\phi|_1.$$

(iii) *The map* $[0,\infty) \times C^2 \ni (t,\phi) \mapsto (\phi^{dd})_t \in C^2$ *is continuous. For each* $\phi \in C^2$,

$$(\phi^{dd})' = (\partial\phi)^d \text{ and } Lip((\phi^{dd})'') \leq Lip(\partial\partial\phi),$$

and for all $t \geq 0$,

$$|(\phi^{dd})_t|_2 \leq \left(1+t+\frac{t^2}{2}\right)|\phi|_2.$$

Next we draw conclusions from the properties (g0)–(g5) stated in Sect. 1.

Proposition 2.5. *Assume* $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ *open, satisfies (g0). Then all segments* $\psi = x_t$ *of any solution* $x : [t_0 - h, t_e) \to \mathbb{R}^n$ *of Eq. (2) belong to* X_1.

Proof. For $t_0 < t < t_e$ this was observed in Sect. 1. The definition of solutions yields $x_{t_0} \in U_1$. By continuity,

$$x'(t_0) = \lim_{t_0<t\to t_0} x'(t) = \lim_{t_0<t\to t_0} g(\partial x_t, x_t) = g(\partial x_{t_0}, x_{t_0}). \qquad \square$$

Proposition 2.6. *Assume* $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ *open, satisfies (g3). Then the map* $D_e g_1 : W_1 \to L_c(C \times C, \mathbb{R}^n)$ *is locally bounded.*

Proof. Let $(\phi_0, \psi_0) \in W_1$ be given. By continuity at $(\phi_0, \psi_0, 0, 0)$, there are a neighbourhood $N \subset W_1$ of (ϕ_0, ψ_0) in $C^1 \times C^1$ and $r > 0$ such that $|D_e g_1(\phi,\psi)(\chi,\rho)| \leq 1$ for $(\phi,\psi) \in N_1$ and $|(\chi,\rho)|_{C\times C} \leq r$. Hence

$$|D_e g_1(\phi,\psi)|_{L_c(C\times C,\mathbb{R}^n)} \leq \frac{1}{r} \text{ for } (\phi,\psi) \in N. \qquad \square$$

We need versions of $(g1), (g2), (g4), (g5)$ on compact sets.

Proposition 2.7. *(i) Assume* $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ *open, satisfies (g1) and* $K \subset W$ *is compact. Then there exist* $\Delta \in (0,h)$, *a neighbourhood* $N \subset W$ *of* K *in* $C \times C^1$, *and* $r > 0$ *so that for all* $(\phi,\psi), (\phi_1,\psi)$ *in* N *with*

$$|(\phi,\psi) - (\phi_1,\psi)|_{C\times C^1} = |\phi - \phi_1| < r \text{ and } \phi(t) = \phi_1(t) \text{ in } [-h,-\Delta]$$

we have $g(\phi,\psi) = g(\phi_1,\psi)$.

(ii) Assume $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ *open, satisfies (g4), and let* $K \subset W_1$ *be a compact subset of* $C^1 \times C^1$. *Then there exist a neighbourhood* $N_1 \subset W_1$ *of* K *in* $C^1 \times C^1$, $c \geq 0$ *and* $r > 0$ *so that for all* $(\phi,\psi), (\phi_1,\psi_1)$ *in* N_1 *with*

$$|(\phi,\psi) - (\phi_1,\psi_1)|_{C^1\times C^1} < r$$

and for all $\chi \in C^1$ *we have*

$$|(Dg_1(\phi,\psi) - Dg_1(\phi_1,\psi_1))(\chi,0)| \le c|\partial\chi||\psi - \psi_1|.$$

(iii) Assume $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ *open, satisfies (g5), and let* $K \subset W_1$ *be a compact subset of* $C^1 \times C^1$. *Then there exist a neighbourhood* $N_1 \subset W_1$ *of* K *in* $C^1 \times C^1$, $c \ge 0$ *and* $r > 0$ *so that for all* $(\phi,\psi),(\phi_1,\psi_1)$ *in* N_1 *with*

$$|(\phi,\psi) - (\phi_1,\psi_1)|_{C^1\times C^1} < r$$

and for all $(\chi,\rho) \in C \times C$ *with* $|(\chi,\rho)|_{C\times C} = 1$ *we have*

$$|(D_e g_1(\phi,\psi) - D_e g_1(\phi_1,\psi_1))(\chi,\rho)| \le |(\phi,\psi) - (\phi_1,\psi_1)|_{C^1\times C^1}\, c\cdot(Lip(\chi) + Lip(\partial\phi) + 1).$$

Proof. 1. Proof of (i). For every $(\phi,\psi) \in K$ choose $\varepsilon = \varepsilon(\phi,\psi) > 0$ and $\Delta(\phi,\psi) \in (0,h)$ so that with

$$N = N(\phi,\psi) = \{(\overline{\phi},\overline{\psi}) \in C \times C^1 : |(\overline{\phi},\overline{\psi}) - (\phi,\psi)|_{C\times C^1} < \varepsilon\}$$

the statement in (g1) holds. A finite number of smaller balls

$$N'(\phi,\psi) = \left\{(\overline{\phi},\overline{\psi}) \in C \times C^1 : |(\overline{\phi},\overline{\psi}) - (\phi,\psi)|_{C\times C^1} < \frac{\varepsilon}{2}\right\}$$

covers K, centered at, say, $(\phi_1,\psi_1),\ldots,(\phi_m,\psi_m)$. Set

$$\Delta = \min_{i=1,\ldots,m} \Delta(\phi_i,\psi_i),\ N = \cup_1^m N'(\phi_i,\psi_i),\ r = \min_{i=1,\ldots,m} \frac{\varepsilon(\phi_i,\psi_i)}{2}.$$

2. *Proof of (ii).* For every $(\phi,\psi) \in K$ choose $\varepsilon(\phi,\psi) > 0$ and $c(\phi,\psi) \ge 0$ so that the open ball

$$N(\phi,\psi) = \{(\overline{\phi},\overline{\psi}) \in C^1 \times C^1 : |(\overline{\phi},\overline{\psi}) - (\phi,\psi)|_{C^1\times C^1} < \varepsilon(\phi,\psi)\}$$

is contained in W^1, and for all (ϕ_2,ψ_2) and (ϕ_1,ψ_1) in $N(\phi,\psi)$ and all $\chi \in C^1$ we have the estimate

$$|(Dg_1(\phi_2,\psi_2) - Dg_1(\phi_1,\psi_1))(\chi,0)| \le c(\phi,\psi)|\partial\chi||\psi_2 - \psi_1|.$$

A finite number of smaller balls

$$N'(\phi,\psi) = \left\{(\overline{\phi},\overline{\psi}) \in C^1 \times C^1 : |(\overline{\phi},\overline{\psi}) - (\phi,\psi)|_{C^1\times C^1} < \frac{\varepsilon(\phi,\psi)}{2}\right\}$$

covers K, centered at, say, $(\phi_1,\psi_1),\dots,(\phi_m,\psi_m)$. Set

$$N_1=\cup_{k=1}^m N'(\phi_k,\psi_k),\ c=\max_{k=1,\dots,m} c(\phi_k,\psi_k),\ \text{and}\ r=\min_{k=1,\dots,m}\frac{\varepsilon(\phi_k,\psi_k)}{2}.$$

For any $(\phi,\psi),(\overline{\phi},\overline{\psi})$ in N_1 with

$$|(\phi,\psi)-(\overline{\phi},\overline{\psi})|_{C^1\times C^1}<r$$

there exists $k\in\{1,\dots,m\}$ with $(\phi,\psi)\in N'(\phi_k,\psi_k)$. The choice of r yields $(\overline{\phi},\overline{\psi})\in N(\phi_k,\psi_k)$. For every $\chi\in C^1$ we get the estimate

$$\begin{aligned}|(Dg_1(\phi,\psi)-Dg_1(\phi_1,\psi_1))(\chi,0)| &\le c(\phi_k,\psi_k)|\partial\chi||\psi-\psi_1|\\ &\le c|\partial\chi||\psi-\psi_1|.\end{aligned}$$

3. The proof of (iii) is analogous to the previous one. □

The next result is a consequence of Proposition 2.7 (i).

Corollary 2.8. *Assume $g: C\times C^1\supset W\to\mathbb{R}^n$, $W\subset C\times C^1$ open, satisfies (g1) and (g3), and let $K\subset W_1$ be a compact subset of $C^1\times C^1$. Then there are $\Delta\in(0,h)$, a neighbourhood $N_1\subset W_1$ of K in $C^1\times C^1$, and $r>0$ so that for all $(\phi,\psi),(\phi_1,\psi)$ in N_1 with*

$$|(\phi,\psi)-(\phi_1,\psi)|_{C^1\times C^1}=|\phi-\phi_1|_1<r\ \textit{and}\ \phi(t)=\phi_1(t)\ \textit{in}\ [-h,-\Delta]$$

and for all $(\chi,\rho),(\chi_1,\rho)$ in $C\times C$ with

$$\chi(t)=\chi_1(t)\ \textit{in}\ [-h,-\Delta]$$

we have

$$D_e g_1(\phi,\psi)(\chi,\rho)=D_e g_1(\phi_1,\psi)(\chi_1,\rho).$$

Proof. 1. K is also a compact subset of $C\times C^1$. Choose Δ and a neighbourhood N and $r>0$ according to Proposition 2.7 (i). Then $N_1=N\cap(C^1\times C^1)\subset W_1$ is a neighbourhood of K in $C^1\times C^1$. Let $(\phi,\psi),(\phi_1,\psi)$ in N_1 be given with

$$|(\phi,\psi)-(\phi_1,\psi)|_{C^1\times C^1}=|\phi-\phi_1|_1<r$$

and $\phi(t)=\phi_1(t)$ for $-h\le t\le-\Delta$.

2. Consider χ,χ_1 in C^1 with $\chi(t)=\chi_1(t)$ for $-h\le t\le-\Delta$. For $s\in\mathbb{R}$ sufficiently small $(\phi+s\chi,\psi)$ and $(\phi_1+s\chi_1,\psi)$ belong to $N_1\subset N$, and Proposition 2.7 (i) yields

$$g_1(\phi+s\chi,\psi)-g_1(\phi_1,\psi)=g_1(\phi_1+s\chi_1,\psi)-g_1(\phi_1,\psi).$$

It follows that

$$D_1 g_1(\phi,\psi)\chi = D_1 g_1(\phi_1,\psi)\chi_1.$$

Now consider $\rho \in C^1$. For $s \in \mathbb{R}$ sufficiently small we infer from Proposition 2.7 (i) that

$$g_1(\phi,\psi+s\rho) - g_1(\phi,\psi) = g_1(\phi_1,\psi+s\rho) - g_1(\phi_1,\psi).$$

It follows that

$$D_2 g_1(\phi,\psi)\rho = D_2 g_1(\phi_1,\psi)\rho.$$

Altogether,

$$Dg_1(\phi,\psi)(\chi,\rho) = Dg_1(\phi_1,\psi)(\chi_1,\rho).$$

3. Finally, let χ,χ_1 in C be given with $\chi(t) = \chi_1(t)$ for $-h \le t \le -\Delta$. Choose sequences $(\chi^m)_0^\infty$ and $(\chi_1^m)_0^\infty$ in C^1 which converge in C to χ and χ_1, respectively, and satisfy

$$\chi^m(t) = \chi_1^m(t) \text{ for all } m \in \mathbb{N}_0 \text{ and } t \in [-h,-\Delta].$$

Let $\rho \in C$ be given. Choose a sequence $(\rho_m)_0^\infty$ in C^1 which converges in C to ρ. Using (g3) and the result of part 2 we infer

$$\begin{aligned} D_e g_1(\phi,\psi)(\chi,\rho) &= \lim_{m\to\infty} D_e g_1(\phi,\psi)(\chi^m,\rho_m) \\ &= \lim_{m\to\infty} Dg_1(\phi,\psi)(\chi^m,\rho_m) \\ &= \lim_{m\to\infty} Dg_1(\phi_1,\psi)(\chi_1^m,\rho_m) \\ &= \lim_{m\to\infty} D_e g_1(\phi_1,\psi)(\chi_1^m,\rho_m) \\ &= D_e g_1(\phi_1,\psi)(\chi_1,\rho). \end{aligned}$$

□

Proposition 2.9. *Assume $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ open, satisfies (g3). Let $t_0 < t_e \le \infty$. Suppose $q : [t_0-h,t_e) \to \mathbb{R}^n$ and $c : [t_0-h,t_e) \to \mathbb{R}^n$ are continuously differentiable, and $(q_t,c_t) \in W_1$ for all $t \in [t_0,t_e)$. Then the function*

$$g_{q,c} : [t_0,t_e) \to \mathbb{R}^n \text{ given by } g_{q,c}(t) = g_1(q_t,c_t)$$

is continuously differentiable with

$$g'_{q,c}(t) = D_e g_1(q_t,c_t)(\partial q_t, \partial c_t) \text{ for all } t \in [t_0,t_e).$$

Remark 2.10. As in general the maps $[t_0,t_e) \ni t \mapsto q_t \in C^1$ and $[t_0,t_e) \ni t \mapsto c_t \in C^1$ are not differentiable, one can not simply apply the chain rule in order to obtain the previous result.

Proof of Proposition 2.9. (1) Let $t \in [t_0, t_e)$. Choose a convex neighbourhood $N \subset W_1$ of (q_t, c_t) in $C^1 \times C^1$. By the continuity of $[t_0, t_e) \ni s \mapsto (q_s, c_s) \in C^1 \times C^1$, there exists $\varepsilon > 0$ with

$$(q_{t+s}, c_{t+s}) \in N \text{ for all } s \in (-\varepsilon, \varepsilon) \text{ with } t_0 \le t+s < t_e.$$

For such $s \neq 0$ we have

$$\begin{aligned}
&g_{q,c}(t+s) - g_{q,c}(t) - sD_e g_1(q_t, c_t)(\partial q_t, \partial c_t)\\
&\quad = \int_0^1 Dg_1((q_t, c_t) + \theta(q_{t+s} - q_t, c_{t+s} - c_t))[(q_{t+s} - q_t, c_{t+s} - c_t)]d\theta\\
&\qquad - sD_e g_1(q_t, c_t)(\partial q_t, \partial c_t)\\
&\quad = s \cdot \int_0^1 [D_e g_1((q_t, c_t) + \theta(q_{t+s} - q_t, c_{t+s} - c_t))\left(\frac{1}{s}(q_{t+s} - q_t), \frac{1}{s}(c_{t+s} - c_t)\right)\\
&\qquad - D_e g_1(q_t, c_t)(\partial q_t, \partial c_t)]d\theta.
\end{aligned}$$

The continuity of the map $[0, t_e) \ni s \mapsto (q_s, c_s) \in C^1 \times C^1$ yields that for $0 \neq s \to 0$ the argument $(q_t, c_t) + \theta(q_{t+s} - q_t, c_{t+s} - c_t)$ of $D_e g_1$ tends to (q_t, c_t) in $C^1 \times C^1$ uniformly with respect to $\theta \in [0, 1]$. Using uniform continuity of q' and of c' on compact sets one easily obtains

$$\frac{1}{s}(q_{t+s} - q_t) \to \partial q_t \text{ in } C \text{ as } 0 \neq s \to 0 \text{ and } \frac{1}{s}(c_{t+s} - c_t) \to \partial c_t \text{ in } C \text{ as } 0 \neq s \to 0.$$

Using the continuity of the map

$$W_1 \times C \times C \ni (\phi, \psi, \chi, \rho) \mapsto D_e g_1(\phi, \psi)(\chi, \rho) \in \mathbb{R}^n$$

at $(q_t, c_t, \partial q_t, \partial c_t)$, which is guaranteed by (g3), we infer that the last integral tends to $0 \in \mathbb{R}^n$ as $0 \neq s \to 0$. This shows that $g_{q,c}$ is differentiable at t, with

$$g'_{q,c}(t) = D_e g_1(q_t, c_t)(\partial q_t, \partial c_t).$$

(2) The continuity of $g'_{q,c}$ follows by means of (g3) from the preceding formula since the maps $[t_0, t_e) \ni s \mapsto (q_s, c_s) \in C^1 \times C^1$ and $[t_0, t_e) \ni s \mapsto (\partial q_s, \partial c_s) \in C \times C$ are continuous. □

Corollary 2.11. *Assume $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ open, satisfies (g0) and (g3). Then all segments $\psi = x_t$ of any twice continuously differentiable solution $x : [t_0 - h, t_e) \to \mathbb{R}^n$ of Eq. (2) belong to X_{2*}.*

Proof. All segments x_t belong to $X_2 = X_1 \cap C^2$, due to Proposition 2.5. It remains to show that for $t_0 \le t < t_e$ the function $\partial x_t \in C^1$ satisfies (6). As $x : [t_0 - h, t_e) \to \mathbb{R}^n$ is twice continuously differentiable we obtain from Proposition 2.9 that the function given by the right hand side of Eq. (2), namely

$$g_* : [t_0, t_e) \ni t \mapsto g((x')_t, x_t) \in \mathbb{R}^n,$$

is continuously differentiable with

$$g_*'(t) = D_e g_1((x')_t, x_t)(\partial (x')_t, \partial x_t) \text{ for } 0 \le t < t_e.$$

It follows that

$$x''(t) = D_e g_1((x')_t, x_t)(\partial (x')_t, \partial x_t) = D_e g_1(\partial x_t, x_t)(\partial \partial x_t, \partial x_t)$$

for such t. Finally, use $(\partial x_t)'(0) = x''(t)$. □

3 The Example

In this section we consider the spaces C and C^1 with $h > 0$ and $n = 1$. Let continuous functions $d : \mathbb{R} \to (-h, 0)$ and $f : \mathbb{R} \to \mathbb{R}$ be given, and let $a > 0$. Define $g : C \times C^1 \to \mathbb{R}$ by

$$g(\phi, \psi) = a\,\phi(d(\psi(0))) + f(\psi(0)) = a\,ev(d(\psi(0)), \phi) + f(ev(0, \psi)),$$

or

$$g = a\,ev \circ ((d \circ ev_1(0, \cdot) \circ pr_2) \times pr_1) + f \circ ev_1(0, \cdot) \circ pr_2. \tag{7}$$

Proposition 3.1. *g has the properties (g0) and (g1), and $X_2 \neq \emptyset$. If d and f are locally Lipschitz continuous then (g2) holds. If d and f are continuously differentiable then (g3) and (g4) hold. If in addition d' and f' are locally Lipschitz continuous then also (g5) holds.*

Proof. 1. From (7) we infer that g is continuous, which is property (g0).

2. Proof of (g1). Let $\psi \in C^1$ be given. There exists $\Delta \in (0, h)$ with $d(ev(0, \psi)) = d(\psi(0)) < -\Delta$. By continuity there is a neighbourhood N of ψ in C^1 with $d(ev_1(0, \chi)) < -\Delta$ for all $\chi \in N$. For all $\chi \in N$ and for all ϕ, ϕ_1 in C with $\phi(t) = \phi_1(t)$ on $[-h, -\Delta]$ we get

$$g(\phi, \chi) = a\,\phi(d(\chi(0))) + f(\chi(0)) = a\,\phi_1(d(\chi(0))) + f(\chi(0)) = g(\phi_1, \chi).$$

   Now (g1) follows easily.

3. Proof of $X_2 \neq \emptyset$. Choose $\xi \in \mathbb{R}$. Let $\delta = d(\xi) \in (-h, 0)$. There exist $\psi \in C^2$ with $\psi(0) = \xi$ and $\psi'(0) = a\,\psi'(\delta) + f(\xi)$. Such ψ belong to X_2.

4. Assume that d and f are locally Lipschitz continuous. Let $\psi \in C^1$ be given. Choose $r > |\psi|_1$ and $L > Lip(d|[-r,r]) + Lip(f|[-r,r])$. For all ψ_1, ψ_2 in C^1 with $|\psi_1|_1 < r, |\psi_2|_1 < r$ and for all ϕ_1, ϕ_2 in C we obtain

$$\begin{aligned}|g(\phi_2,\psi_2) - g(\phi_1,\psi_1)| &\leq a(|\phi_2(d(\psi_2(0))) - \phi_2(d(\psi_1(0)))| \\ &\quad +|\phi_2(d(\psi_1(0)))-\phi_1(d(\psi_1(0)))|)+|f(\psi_2(0))-f(\psi_1(0))| \\ &\leq a(Lip(\phi_2)L|\psi_2-\psi_1| + |\phi_2-\phi_1|) + L|\psi_2-\psi_1|.\end{aligned}$$

This implies (g2).

5. Let d and f be continuously differentiable.

5.1 Proof of (g3). Using (7) and the chain rule one finds that $g_1 = g|C^1 \times C^1$ is continuously differentiable, with

$$\begin{aligned}Dg_1(\phi,\psi)(\chi,\rho) &= a(ev(d(ev(0,\psi)),\chi) + ev(d(ev(0,\psi)),\partial\phi) \\ &\quad \cdot d'(ev(0,\psi))ev(0,\rho)) + f'(ev(0,\psi))ev(0,\rho)\end{aligned}$$

for all ϕ, ψ, χ, ρ in C^1. For $(\phi,\psi) \in C^1 \times C^1$ we define $D_e g_1(\phi,\psi) : C \times C \to \mathbb{R}^n$ by the right hand side of the preceding equation and observe that the map

$$C^1 \times C^1 \times C \times C \ni (\phi,\psi,\chi,\rho) \mapsto D_e g_1(\phi,\psi)(\chi,\rho) \in \mathbb{R}^n$$

is continuous.

5.2 Proof of (g4). We just saw that $g_1 : C^1 \times C^1 \to \mathbb{R}$ is continuously differentiable. For $(\phi,\psi),(\phi_1,\psi_1)$ in $C^1 \times C^1$ and $\chi \in C^1$ the mean value theorem yields the estimate

$$\begin{aligned}|(Dg_1(\phi,\psi) &- Dg_1(\phi_1,\psi_1))(\chi,0)| \\ &= |a(\chi(d(\psi(0))) - \chi(d(\psi_1(0))))| \\ &\leq a|\partial\chi| \max_{|\xi|\leq|\psi|+|\psi_1|} |d'(\xi)||\psi-\psi_1|.\end{aligned}$$

6. Proof of (g5), for d and f differentiable with locally Lipschitz continuous derivatives. We saw already that $g_1 : C^1 \times C^1 \to \mathbb{R}$ is continuously differentiable and that there are linear extensions $D_e g_1(\phi,\psi) : C \times C \to \mathbb{R}$, for every $(\phi,\psi) \in C^1 \times C^1$. For $r > 0$ and for $(\phi,\psi),(\phi_1,\psi_1)$ in $C^1 \times C^1$ with $|\phi_1|_1 \leq r$, $|\psi|_1 \leq r$, $|\psi_1|_1 \leq r$, and for $(\chi,\rho) \in C \times C$ with $|(\chi,\rho)|_{C\times C} = 1$ we have

$$\begin{aligned}&|(D_e g_1(\phi,\psi) - D_e g_1(\phi_1,\psi_1))(\chi,\rho)| \\ &\quad \leq a|\chi(d(\psi(0))) - \chi(d(\psi_1(0))) \\ &\qquad +\partial\phi(d(\psi(0)))d'(\psi(0))\rho(0) - \partial\phi_1(d(\psi_1(0)))d'(\psi_1(0))\rho(0)| \\ &\qquad +|f'(\psi(0))\rho(0) - f'(\psi_1(0))\rho(0)|\end{aligned}$$

$$\leq a[Lip(\chi)Lip(d|[-r,r])|\psi-\psi_1|$$
$$+|\rho|(|\partial\phi(d(\psi(0)))-\partial\phi(d(\psi_1(0)))||d'(\psi(0))|$$
$$+|\partial\phi(d(\psi_1(0)))-\partial\phi_1(d(\psi_1(0)))||d'(\psi(0))|$$
$$+|\partial\phi_1(d(\psi_1(0)))||d'(\psi(0))-d'(\psi_1(0))|)]+|\rho|Lip(f'|[-r,r])|\psi-\psi_1|$$
$$\leq a[Lip(\chi)Lip(d|[-r,r])|\psi-\psi_1|$$
$$+|\rho|(Lip(\partial\phi)Lip(d|[-r,r])|\psi-\psi_1||d'(\psi(0))|$$
$$+|\partial\phi-\partial\phi_1||d'(\psi(0))|+|\partial\phi_1|Lip(d'|[-r,r])|\psi-\psi_1|)]$$
$$+|\rho|Lip(f'|[-r,r])|\psi-\psi_1|.$$

Using $|\rho|\leq 1$ and $Lip(d|[-r,r])\leq \max_{-r\leq\xi\leq r}|d'(\xi)|=c_r$ we see that the preceding term is majorized by

$$a(Lip(\chi)c_r|\psi-\psi_1|+Lip(\partial\phi)c_r^2|\psi-\psi_1|+c_r|\partial\phi-\partial\phi_1|$$
$$+r\,Lip(d'|[-r,r])|\psi-\psi_1|)+Lip(f'|[-r,r])|\psi-\psi_1|$$
$$\leq(|\phi-\phi_1|_1+|\psi-\psi_1|)$$
$$\cdot(a\,Lip(\chi)c_r+a\,Lip(\partial\phi)c_r^2+c_r+r\,Lip(d'|[-r,r])+Lip(f'|[-r,r]))$$
$$\leq(|\phi-\phi_1|_1+|\psi-\psi_1|)\cdot(ac_r+ac_r^2+c_r+r\,Lip(d'|[-r,r])+Lip(f'|[-r,r]))$$
$$\cdot(Lip(\chi)+Lip(\partial\phi)+1).$$

Now property (g5) is obvious. □

4 The Semiflow on a Subset of C^1

In this section we assume that $g:C\times C^1\supset W\to\mathbb{R}^n$, $W\subset C\times C^1$ open, satisfies(g0)–(g2). Recall that, due to Proposition 2.5, any solution of Eq. (2) has all its segments in the set $X_1\subset C^1$. In the sequel we construct solutions to the IVP

$$x'(t)=g(\partial x_t,x_t),\ x_0=\psi\in X_{1+} \tag{8}$$

and obtain a semiflow on the set $X_{1+}\subset X_1$.

Proposition 4.1 (Local existence). *For every $\psi\in X_{1+}$ there is a solution $x:[-h,t_e)\to\mathbb{R}^n$, $0<t_e\leq\infty$ to the IVP (8).*

The idea of the subsequent proof is based on the fact that, with $\Delta\in(0,h)$ and a neighbourhood N of $(\partial\psi,\psi)$ chosen according to (g1), any solution of the IVP satisfies

$$g(\partial x_t,x_t)=g(\partial\psi_t^d,x_t)$$

as long as $0 \le t \le \Delta$ and $(\partial x_t, x_t)$ and $(\partial \psi_t^d, x_t)$ remain in N. Using this and (8) we find that for small $t > 0$ the function u given by $x(t) = u(t) + \psi^d(t)$ satisfies the equation

$$\begin{aligned} u(t) &= x(t) - \psi^d(t) = x(t) - (\psi(0) + \psi'(0)t) = x(t) - \psi(0) - t\,g(\partial\psi, \psi) \\ &= \int_0^t (x'(s) - g(\partial\psi, \psi))ds = \int_0^t (g(\partial x_s, x_s) - g(\partial\psi, \psi))ds \\ &= \int_0^t (g(\partial\psi_s^d, x_s) - g(\partial\psi, \psi))ds \\ &= \int_0^t (g(\partial\psi_s^d, \psi_s^d + u_s) - g(\partial\psi, \psi))ds, \end{aligned}$$

which can be solved by a contraction.

Proof of Proposition 4.1. 1. For $T \ge 0$ let C_T^1 denote the Banach space of continuously differentiable functions $u : [-h, T] \to \mathbb{R}^n$ with $u_0 = 0$, with the norm given by

$$|u|_{1,T} = \max_{-h \le t \le T} |u(t)| + \max_{-h \le t \le T} |u'(t)|.$$

Let $\psi \in X_{1+}$ be given. Set $\lambda = Lip(\partial\psi)$. Choose a neighbourhood $N \subset W$ of $(\partial\psi, \psi)$ in $C \times C^1$ and $\Delta \in (0, h), L \ge 0$ according to (g1) and (g2). Choose $r > 0$ and $T \in (0, \Delta)$ so small that for all $u \in C_T^1$ with $|u|_{1,T} \le r$ and for all $t \in [0, T]$ we have

$$(\partial\psi_t^d, \psi_t^d + u_t) \in N \text{ and } (\partial(\psi_t^d + u_t), \psi_t^d + u_t) \in N, \tag{9}$$

and

$$2LT(\lambda + 1) < 1,\ 2LT(\lambda + 1)r + (T + 1)\max_{0 \le s \le T} |g(\partial\psi_s^d, \psi_s^d) - g(\partial\psi, \psi)| \le r.$$

2. Observe that for any $u \in C_T^1$ the map

$$[0, T] \ni t \mapsto g(\partial\psi_t^d, \psi_t^d + u_t) \in \mathbb{R}^n$$

is continuous. Define a map $A = A_{\psi,T}$ from C_T^1 into the set F_T of all functions $[-h, T] \to \mathbb{R}^n$ by

$$A(u)(t) = 0 \text{ for } -h \le t \le 0,$$

$$A(u)(t) = \int_0^t (g(\partial\psi_s^d, \psi_s^d + u_s) - g(\partial\psi, \psi))ds \text{ for } 0 < t \le T.$$

For $u \in C_T^1$ the function $A(u)$ is differentiable on $[-h, 0) \cup (0, T]$. At $t = 0$ the left derivative exists and is zero, and the right derivative exists and equals

$$g(\partial\psi_0^d, \psi_0^d + u_0) - g(\partial\psi, \psi) = 0.$$

It follows that $A(u)$ is differentiable, with continuous derivative. Hence $A(u) \in C^1_T$.

3. (Lipschitz estimate) Let u, v in C^1_T be given with $|u|_{1,T} \le r$, $|v|_{1,T} \le r$. Set $w = A(u), z = A(v)$. Let $t \in [0,T]$. Then

$$|w(t) - z(t)| \le t \max_{0 \le s \le t} |g(\partial \psi^d_s, \psi^d_s + u_s) - g(\partial \psi^d_s, \psi^d_s + v_s)|$$
$$\le t L(\lambda + 1) \max_{0 \le s \le t} |u_s - v_s|$$

(with (g2) and $Lip(\partial \psi^d_s) \le Lip((\psi^d)') \le Lip(\partial \psi) = \lambda$)

$$\le t L(\lambda + 1) \max_{-h \le s \le t} |u(s) - v(s)|$$

and similarly

$$|w'(t) - z'(t)| = |g(\partial \psi^d_t, \psi^d_t + u_t) - g(\partial \psi^d_t, \psi^d_t + v_t)|$$
$$\le L(\lambda + 1) \max_{-h \le s \le t} |u(s) - v(s)|.$$

Using the estimate

$$|u(s) - v(s)| = \left| \int_0^s (u'(\sigma) - v'(\sigma)) d\sigma \right| \le t |u - v|_{1,T}$$

for $0 \le s \le t$ we obtain

$$|w'(t) - z'(t)| \le L(\lambda + 1) t |u - v|_{1,T}.$$

It follows that

$$|A(u) - A(v)|_{1,T} \le 2L(\lambda + 1) T |u - v|_{1,T},$$

and the restriction of A to the closed ball $\{u \in C^1_T : |u|_{1,T} \le r\}$ is a strict contraction.

4. (Bound) For $u \in C^1_T$ with $|u|_{1,T} \le r$ we have

$$\begin{aligned} |A(u)|_{1,T} &\le |A(u) - A(0)|_{1,T} + |A(0)|_{1,T} \\ &\le 2L(\lambda + 1) T |u|_{1,T} + \int_0^T |g(\partial \psi^d_s, \psi^d_s) - g(\partial \psi, \psi)| ds \\ &\quad + \max_{0 \le s \le T} |g(\partial \psi^d_s, \psi^d_s) - g(\partial \psi, \psi)| \\ &\le 2L(\lambda + 1) T r + (T + 1) \max_{0 \le s \le T} |g(\partial \psi^d_s, \psi^d_s) - g(\partial \psi, \psi)| \le r. \end{aligned}$$

5. The contraction mapping principle yields a fixed point $u \in C^1_T$ of A with $|u|_{1,T} \le r$. The function $x : [-h, T] \to \mathbb{R}^n$ given by $x(t) = \psi^d(t) + u(t)$ is continuously differentiable with $x_0 = \psi$,

$$(\partial x_t, x_t) = (\partial(\psi_t^d + u_t), \psi_t^d + u_t) \in N \subset W \text{ (due to (9))}$$

and

$$x'(t) = (\psi^d)'(t) + u'(t) = \psi'(0) + g(\partial\,\psi_t^d, \psi_t^d + u_t) - g(\partial\,\psi, \psi) = g(\partial\,\psi_t^d, \psi_t^d + u_t)$$

for $0 \le t \le T$. For $0 \le t \le T$ and for $-h \le s \le -\Delta$ we have $-h \le t+s \le T-\Delta \le 0$, and consequently

$$\partial\,\psi_t^d(s) = (\psi^d)'(t+s) = (\psi^d)'(t+s) + u'(t+s) = \partial\,\psi_t^d(s) + \partial\,u_t(s) = \partial(\psi_t^d + u_t)(s).$$

Recall (9). Property (g1) now yields

$$g(\partial\,\psi_t^d, \psi_t^d + u_t) = g(\partial(\psi_t^d + u_t), \psi_t^d + u_t) = g(\partial x_t, x_t)$$

for $0 \le t \le T$, and we see that x is a solution to Eq. (2). □

Proposition 4.1 is the only result on existence of solutions to Eq. (2) which we need the sequel.

Proposition 4.2 (Smoothness, invariance). *For any solution $x : [t_0 - h, t_e) \to \mathbb{R}^n$, $t_0 < t_e \le \infty$, of Eq. (2) with $x_{t_0} \in X_{1+}$ the derivative $x' : [t_0 - h, t_e) \to \mathbb{R}^n$ is locally Lipschitz continuous.*

Proof. For $t_0 = 0$. We have

$$\emptyset \ne A = \{t \in [0, t_e) : x'|[-h, t] \text{ Lipschitz continuous}\}$$

since $0 \in A$. Assume $s = \sup A < t_e$. Choose a neighbourhood N of $(\partial x_s, x_s)$, Δ and L according to (g1) and (g2). By continuity there exists $\varepsilon \in (0, \frac{\Delta}{2})$ so that $s + \varepsilon < t_e$ and $(\partial x_t, x_t) \in N$ for $t \ge 0$ in $[s-\varepsilon, s+\varepsilon]$. Also, for some $u \ge 0$ in $(s-\varepsilon, s]$ the restriction $x'|[-h, u]$ is Lipschitz continuous. There exists a Lipschitz continuous continuation $y : [-h, s+\varepsilon] \to \mathbb{R}^n$ of $x'|[-h, u]$ with $\max_{u \le t \le s+\varepsilon} |y(t) - x'(t)|$ so small that $(y_t, x_t) \in N$ for $u \le t \le s+\varepsilon$. For such t and $-h \le w \le -\Delta$ we have $-h \le t + w \le u$, hence $y_t(w) = y(t+w) = x'(t+w) = \partial x_t(w)$. Using (g1) we obtain that for t, τ in $[u, s+\varepsilon]$ the following estimate holds:

$$\begin{aligned}
|x'(t) - x'(\tau)| &= |g(\partial x_t, x_t) - g(\partial x_\tau, x_\tau)| = |g(y_t, x_t) - g(y_\tau, x_\tau)| \\
&\le L(|y_t - y_\tau| + (Lip(y_t) + 1)|x_t - x_\tau|) \qquad \text{(by (g2))} \\
&\le L(Lip(y)|t-\tau| + (Lip(y)+1) \max_{-h \le v \le s+\varepsilon} |x'(v)||t-\tau|).
\end{aligned}$$

We infer that $x'|[-h, s+\varepsilon]$ is Lipschitz continuous, in contradiction to $s = \sup A$. □

Corollary 4.3 (Uniqueness). *If $x : [-h, t_e) \to \mathbb{R}^n$, $0 < t_e \le \infty$, and $y : [-h, t_e) \to \mathbb{R}^n$ are solutions of the IVP (8) with $x_0 = y_0 \in X_{1+}$ then $x = y$.*

Proof. Suppose the assertion is false. Then the set

$$A = \{t \in [0,t_e) : x(u) = y(u) \text{ for } -h \le u \le t\} \ne \emptyset$$

is bounded and $s = \sup A < t_e$. By continuity, $x(u) = y(u)$ for $-h \le u \le s$. Choose a neighbourhood N of $(\partial x_s, x_s)$, $\Delta \in (0,h)$ and $L \ge 0$ according to (g1) and (g2). By continuity of $[0,t_e) \ni t \mapsto x_t \in C^1$ and $[0,t_e) \ni t \mapsto y_t \in C^1$ there exists $\sigma \in (s,t_e) \cap (s, s+\Delta)$ so that for $s \le u \le \sigma$ we have

$$(\partial x_u, x_u) \in N, \; (\partial y_u, y_u) \in N, \text{ and } (\partial x_u, y_u) \in N.$$

Observe that for such u and for $-h \le w \le -\Delta$ we have $u + w \le s$, and consequently

$$\partial x_u(w) = x'(u+w) = y'(u+w) = \partial y_u(w).$$

Let $s \le t \le \sigma$. Using (g1) we get

$$\begin{aligned}
|x(t) - y(t)| &\le \int_s^t |x'(u) - y'(u)|du = \int_s^t |g(\partial x_u, x_u) - g(\partial y_u, y_u)|du \\
&= \int_s^t |g(\partial x_u, x_u) - g(\partial x_u, y_u)|du \quad \text{(with (g1))} \\
&\le \int_s^t L(Lip(x'|[-h,\sigma]) + 1)|x_u - y_u|du \quad \text{(with (g2))} \\
&\le L(t-s)(Lip(x'|[-h,\sigma]) + 1) \max_{s \le u \le t} |x(u) - y(u)|.
\end{aligned}$$

Choose $T \in (s, \sigma)$ so that

$$L(T-s)(Lip(x'|[-h,\sigma]) + 1) < 1. \tag{10}$$

Consider $t \in [s,T]$ with $|x(t) - y(t)| = \max_{s \le u \le T} |x(u) - y(u)|$. It follows that $|x(t) - y(t)| = \max_{s \le u \le t} |x(u) - y(u)|$. The preceding estimate and (9) combined yield

$$0 = |x(t) - y(t)| = \max_{s \le u \le t} |x(u) - y(u)| \quad (= \max_{s \le u \le T} |x(u) - y(u)|),$$

hence $x(u) = y(u)$ for all $u \in [-h, T]$, and thereby $T \le s$, in contradiction to $s < T$. □

Now maximal solutions of the IVP (8) are obtained in the usual way: For $\psi \in X_{1+}$ let

$$t_\psi = \sup\{t_e > 0 : \text{There is a solution } x : [-h, t_e) \to \mathbb{R}^n \text{ of the IVP } (4.1)\} \le \infty$$

and define the solution $x^{\psi}:[-h,t_{\psi})\to\mathbb{R}^n$ of the IVP (8) by the solutions on intervals $[-h,t_e)$ with $t_e<t_{\psi}$. It follows that

$$x_t^{\psi}\in X_{1+} \text{ for all } t\in[0,t_{\psi}).$$

Set

$$\Omega_1=\bigcup_{\psi\in X_{1+}}[0,t_{\psi})\times\{\psi\}$$

and define $G_1:\Omega_1\to X_{1+}$ by $G_1(t,\psi)=x_t^{\psi}$. Then $\{0\}\times X_{1+}\subset\Omega_1$ and for all $\psi\in X_{1+}$, $G_1(0,\psi)=\psi$. For $0\le t<t_{\psi}$ and $0\le s<t_{G_1(t,\psi)}$ one easily shows that

$$0\le t+s<t_{\psi} \text{ and } G_1(t+s,\psi)=G_1(s,G_1(t,\psi)).$$

We shall prove that Ω_1 is open in $[0,\infty)\times X_{1+}$ and that G_1 is continuous, both with respect to the topologies given by the norms on $\mathbb{R}\times C^1$ and on C^1, respectively. This requires some preparation.

Incidentally, notice that in the proof of Proposition 4.1 one does *not* immediately get that the fixed point (the local solution) depends continuously on the initial value with respect to the norm $|\cdot|_1$ since the Lipschitz constant for A depends on $Lip(\partial\psi)$; contraction is not locally uniform with respect to the parameter ψ.

Proposition 4.4. *Let $\psi\in X_{1+}$. There exist $T=T(\psi)\in(0,t_{\psi})$ and a neighbourhood $V=V_{\psi}$ of ψ in C^1 with $x_t^{\psi}\in V$ for $0\le t\le T$ such that the following holds:*

(i) For each $\chi\in X_{1+}$, either $T<t_{\chi}$ or $x_t^{\chi}\notin V$ for some $t\in[0,T]\cap[0,t_{\chi})$.

(ii) For each $\rho\in X_{1+}$ with $T<t_{\rho}$ and $x_t^{\rho}\in V$ for all $t\in[0,T]$ there exists $c(\rho)\ge 0$ such that for every $\eta\in X_{1+}$ and for every $t\in[0,T]\cap[0,t_{\eta})$ with $x_s^{\eta}\in V$ for all $s\in[0,t]$ we have

$$|x_s^{\eta}-x_s^{\rho}|_1\le c(\rho)|\eta-\rho|_1 \textit{ for all } s\in[0,t].$$

Proof. 1. Choose a neighbourhood $N\subset W$ of $(\partial\psi,\psi)$ in $C\times C^1$ and $\Delta\in(0,h)$, $L\ge 0$ according to (g1) and (g2). By continuity there are an open bounded neighbourhood V of ψ in C^1 and $T\in(0,\Delta)\cap(0,t_{\psi})$ so that for all χ,ϕ in $cl\,V$ and for all $t\in[0,T]$ we have

$$(\partial\chi_t^d,\phi)\in N,\ (\partial\phi,\phi)\in N, \text{ and } x_t^{\psi}\in V.$$

2. *Proof of (i).* 2.1. Let $\chi\in X_{1+}$.] Set $x=x^{\chi}$. Assume $x_t\in V$ for $t\in[0,T]\cap[0,t_{\chi})$. We show that $x'|[-h,T]\cap[-h,t_{\chi})$ is Lipschitz continuous. For $t\in[0,T]\cap[0,t_{\chi})$ we have $(\partial\chi_t^d,x_t)\in N$ (because of $\chi=x_0\in V$ and $x_t\in V$) and $(\partial x_t,x_t)\in N$ (by $x_t\in V$). For such t and for $-h\le s\le-\Delta$ we also have $-h\le t+s\le T-\Delta\le 0$, hence

$$\partial\chi_t^d(s)=(\chi^d)'(t+s)=x'(t+s)=\partial x_t(s).$$

For every t,u in $[0,T]\cap[0,t_\chi)$ we infer

$$
\begin{aligned}
&|x'(t)-x'(u)|\\
&= |g(\partial x_t,x_t)-g(\partial x_u,x_u)| = |g(\partial\chi_t^d,x_t)-g(\partial\chi_u^d,x_u)| \text{ (by (g1))}\\
&\le L(|\partial\chi_t^d-\partial\chi_u^d| + (Lip(\partial\chi_t^d)+1)|x_t-x_u|) \text{ (by (g2))}\\
&\le L(Lip((\chi^d)')|t-u| + (Lip((\chi^d)')+1)\sup_{s\in[-h,T]\cap[-h,t_\psi)}|x'(s)||t-u|)\\
&\le L(Lip(\partial\chi)|t-u| + (Lip(\partial\chi)+1)\sup_{s\in[0,T]\cap[0,t_\psi)}|x_s|_1|t-u|)\\
&\le L(Lip(\partial\chi)|t-u| + (Lip(\partial\chi)+1)\sup_{\phi\in V}|\phi|_1|t-u|).
\end{aligned}
$$

2.2. Assume in addition $t_\chi \le T$. The result in part 2.1 implies that x' has a Lipschitz continuous extension defined on $[-h,t_\chi]$. It follows that there is continuation of x to a differentiable map $\hat{x}:[-h,t_\chi]\to\mathbb{R}^n$ with Lipschitz continuous derivative. The segment $\phi=\hat{x}_{t_\chi}$ belongs to $cl\,V$, and $\phi\in X_{1+}$. Using the solution x^ϕ and the algebraic semiflow properties we obtain a continuation of x as a solution beyond t_χ, which is in contradiction to the definition of t_χ.

3. *Proof of (ii).* Let $\rho\in X_{1+}$, $T<t_\rho$, $x=x^\rho$, $x_t\in V$ for all $t\in[0,T]$. Let $\eta\in X_{1+}$, $y=x^\eta$, $t\in[0,T]\cap[0,t_\eta)$, and $y_s\in V$ for all $s\in[0,t]$. For $0\le s\le t$ we proceed as in part 2.1 and get

$$
\begin{aligned}
&|x'(s)-y'(s)|\\
&= |g(\partial x_s,x_s)-g(\partial y_s,y_s)|\\
&= |g(\partial\rho_s^d,x_s)-g(\partial\eta_s^d,y_s)| \text{ (by (g1))}\\
&\le L(|\partial\rho_s^d-\partial\eta_s^d| + (Lip(\partial\rho_s^d)+1)|x_s-y_s|) \text{ (by (g2))}\\
&\le L(|\partial\rho-\partial\eta| + (Lip(\partial\rho)+1)(|\rho-\eta| + \int_0^s |x'(u)-y'(u)|du)),
\end{aligned}
$$

because of $|x_s(u)-y_s(u)|\le|\rho-\eta|$ for $s+u\le 0$ and $|x_s(u)-y_s(u)|\le|x(0)-y(0)|+\int_0^{s+u}|x'(v)-y'(v)|dv\le|\rho-\eta|+\int_0^s|x'(v)-y'(v)|dv$ for $0\le s+u$. Using Gronwall's lemma we infer

$$|x'(s)-y'(s)|\le|\rho-\eta|_1 c_1(\rho)$$

with

$$c_1(\rho)=L(Lip(\partial\rho)+1)e^{TL(Lip(\partial\rho)+1))}.$$

Integration yields

$$|x(s)-y(s)| \le |x(0)-y(0)| + T|\rho-\eta|_1 c_1(\rho) \text{ for } 0 \le s \le t.$$

For $-h \le s \le t$ we get

$$|x(s)-y(s)| \le (1+T\,c_1(\rho))|\rho-\eta|_1 \text{ and } |x'(s)-y'(s)| \le (1+c_1(\rho))|\rho-\eta|_1.$$

Finally,

$$|x_s-y_s|_1 \le |\rho-\eta|_1(2+(T+1)c_1(\rho)).$$ □

Corollary 4.5 (Uniform continuous dependence locally). *Let $\psi \in X_{1+}$. Then there exist $T=T(\psi)\in(0,t_\psi)$ and a neighbourhood $V_{0,\psi}$ of ψ in C^1 with the following properties.*

(i) For all $\chi \in X_{1+}\cap V_{0,\psi}$, $T<t_\chi$.
(ii) For each $\rho \in X_{1+}\cap V_{0,\psi}$ there exists $c(\rho)\ge 0$ such that for all $\chi \in X_{1+}\cap V_{0,\psi}$ and for all $t\in[0,T]$ we have

$$|x_t^\chi - x_t^\rho|_1 \le c(\rho)|\chi-\rho|_1.$$

Proof. 1. Let $\psi \in X_{1+}$ be given. Set $x=x^\psi$. Choose $T=T(\psi)$ and an open neighbourhood $V=V_\psi$ according to Proposition 4.4. As $x_t\in V$ for $0\le t\le T$ we also get a constant $c=c(\psi)$ according to part (ii) of Proposition 4.4. As $K=\{x_t : 0\le t\le T\}\subset V$ is compact there exists $\varepsilon>0$ so that all $\phi\in C^1$ with $|\phi-x_t|_1<\varepsilon$ for some $t\in[0,T]$ belong to V. Choose a positive

$$\delta < \frac{\varepsilon}{1+c}$$

and set

$$V_0 = V_{0,\psi} = \{\chi\in C^1 : |\chi-\psi|_1<\delta\}\cap V.$$

2. *Proof of (i).* Let $\chi\in X_{1+}\cap V_0$, $y=x^\chi$.

2.1. Proof of $y_t\in V$ for all $t\in[0,T]\cap[0,t_\chi)$.
Suppose $y_t\notin V$ for some $t\in[0,T]\cap[0,t_\chi)$. As $y_0=\chi\in V_0\subset V$ we obtain $s\in[0,T]\cap[0,t_\chi)$ with $y_t\in V$ for $0\le t<s$ and $y_s\notin V$. Part (ii) of Proposition 4.4 gives

$$|y_t-x_t|_1 \le c|\chi-\psi|_1 \le c\,\delta \text{ for } 0\le t<s.$$

By continuity, $|y_s-x_s|_1\le c\,\delta<\varepsilon$, which implies a contradiction to $y_s\notin V$.
2.2 Part (i) of Proposition 4.4 yields $T<t_\chi$.

3. *Proof of (ii).* As $T<t_\chi$ for every $\chi\in X_{1+}\cap V_0$ the same argument as in part 2.1 now shows that $y_t^\chi\in V$ for all $t\in[0,T]$. Using this and part (ii) of Proposition 4.4 we infer that assertion (ii) of the corollary holds. □

Standard arguments which involve the preceding corollary, the continuity of the curves $[0,t_\psi)\ni t\mapsto x_t^\psi\in C^1$, $\psi\in X_{1+}$, and the algebraic semiflow properties now yield the final result of this section.

Corollary 4.6. Ω_1 *is an open subset of* $[0,\infty)\times X_{1+}$ *with respect to the topology induced by* $\mathbb{R}\times C^1$*, and* $G_1:\Omega_1\to X_{1+}$ *is continuous with respect to the topology on* X_{1+} *induced by* C^1.

Proof. 1. Let $\psi\in X_{1+}$ be given. Corollary 4.5 provides $T\in(0,t_\psi)$ and a neighbourhood V_0 of ψ in C^1 so that $[0,T]\times(X_{1+}\cap V_0)\subset\Omega_1$. We show that the restriction of G_1 to $[0,T]\times(X_{1+}\cap V_0)$ is continuous with respect to the norm on C^1. Let $\rho\in X_{1+}\cap V_0$ and $t\in[0,T]$ be given. Choose $c=c(\rho)$ according to part (ii) of Corollary 4.5. For all $\chi\in X_{1+}\cap V_0$ and all $s\in[0,T]$ we infer

$$
\begin{aligned}
|G_1(s,\chi)-G_1(t,\rho)|_1 &\le |G_1(s,\chi)-G_1(s,\rho)|_1+|G_1(s,\rho)-G_1(t,\rho)|_1\\
&\le c|\chi-\rho|_1+|G_1(s,\rho)-G_1(t,\rho)|_1,
\end{aligned}
$$

and the continuity of G_1 at (t,ρ) becomes obvious as $[0,t_\rho)\ni s\mapsto x_s^\rho=G_1(s,\rho)\in C^1$ is continuous.

2. Part 1 shows that the set $M=M_\psi$ of all $t\in(0,t_\psi)$ for which there is a neighbourhood V_t of ψ in C^1 so that $[0,t]\times(X_{1+}\cap V_t)\subset\Omega_1$ and $G_1|[0,t]\times(X_{1+}\cap V_t)$ is continuous with respect to the norms on $\mathbb{R}\times C^1$ and C^1 is not empty. Proof of

$$M_\psi=(0,t_\psi):$$

2.1 Suppose there exists $t\in(0,t_\psi)\setminus M$. For every $t\in M$, $(0,t]\subset M$. It follows that $0<t_M=\sup M<t_\psi$ and $(0,t_M)\subset M$. Let $\rho=G_1(t_M,\psi)\in X_{1+}$. Part 1 shows that there are $T=T_\rho>0$ and an open neighbourhood $V_{0,\rho}$ of ρ in C^1 so that $[0,T]\times(X_{1+}\cap V_{0,\rho})\subset\Omega_1$ and $G_1|[0,T]\times(X_{1+}\cap V_{0,\rho})$ is continuous with respect to the norms on $\mathbb{R}\times C^1$ and C^1, respectively. By continuity there exist $t>u>0$ in (t_M-T,t_M) with $G_1(t,\psi)\in X_{1+}\cap V_{0,\rho}$ and $G_1(u,\psi)\in X_{1+}\cap V_{0,\rho}$. As $t\in M$ we also have an open neighbourhood V_t of ψ in C^1 so that $[0,t]\times(X_{1+}\cap V_t)\subset\Omega_1$ and $G_1|[0,t]\times(X_{1+}\cap V_t)$ is continuous with respect to the norms on $\mathbb{R}\times C^1$ and C^1. We may assume that V_t is so small that

$$G_1(u,\phi)\in V_{0,\rho}\ \text{for all}\ \phi\in V_t.$$

2.2 Proof of $[u,u+T]\times(X_{1+}\cap V_t)\subset\Omega_1$ and continuity of $G_1|[u,u+T]\times(X_{1+}\cap V_t)$ with respect to the norms on $\mathbb{R}\times C^1$ and C^1: For $(t,\chi)\in[u,u+T]\times(X_{1+}\cap V_t)$ we have $t-u\in[0,T]$, $(u,\chi)\in\Omega_1$, and $G_1(u,\chi)\in X_{1+}\cap V_{0,\rho}$. It follows that $(t-u,G_1(u,\chi))\in[0,T]\times(X_{1+}\cap V_{0,\rho})\subset\Omega_1$, and thereby $(t,\chi)\in\Omega_1$ and $G_1(t,\chi)=G_1(t-u,G_1(u,\chi))$. The last equation shows that the restriction of G_1 to $[u,u+T]\times(X_{1+}\cap V_t)$ is continuous with respect to the norms on $\mathbb{R}\times C^1$ and on C^1.

2.3 Using $u < t$ we infer $[0, u+T] \times (X_{1+} \cap V_t) \subset \Omega_1$ and continuity of $G_1|[0, u+T] \times (X_{1+} \cap V_t)$, which contradicts the fact that $t_M < u+T$ is an upper bound for M.

3. Now let $(t, \psi) \in \Omega_1$. Then $0 \le t < t_\psi$. Choose $s \in (t, t_\psi)$. By part 2, $s \in M_\psi$, and there exists a neighbourhood V_s of ψ in C^1 so that the neighbourhood $[0, s] \times (X_{1+} \cap V_s)$ of (t, ψ) (in the topology given by $\mathbb{R} \times C^1$ on $[0, \infty) \times X_{1+}$) is contained in Ω_1, and the restriction of G_1 to this neighbourhood is continuous with respect to the norms on $\mathbb{R} \times C^1$ and on C^1. This yields the assertions about Ω_1 and G_1.

□

5 A Submanifold in C^2

Proposition 5.1. *Suppose $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ open, has properties (g1) and (g3), and $X_2 \neq \emptyset$. Then X_2 is a continuously differentiable submanifold of codimension n in C^2, with tangent space*

$$T_\psi X_2 = \{\chi \in C^2 : \chi'(0) = Dg_1(\partial\psi, \psi)(\partial\chi, \chi)\}$$

at $\psi \in X_2$.

Proof. 1. We have $X_2 = A^{-1}(0)$, for the continuously differentiable map

$$A : C^2 \supset U_2 \ni \psi \mapsto (ev_1(0, \cdot) \circ \partial)(\psi) - g_1(\partial\psi, \psi) \in \mathbb{R}^n$$

defined on the open subset $U_2 = \{\psi \in C^2 : (\partial\psi, \psi) \in W\}$ of C^2. For X_2 to be a continuously differentiable submanifold of C^2 it is sufficient to show that all derivatives $DA(\psi)$, $\psi \in X_2$, are surjective because then for each $\psi \in X_2$ there is a decomposition

$$C^2 = K \oplus Q \text{ with } K = (DA(\psi))^{-1}(0) \text{ and } \dim Q = n,$$

and $DA(\psi)$ defines an isomorphism $Q \to \mathbb{R}^n$; the Implicit Function Theorem yields a local graph representation of $A^{-1}(0) = X_2$ given by a map from an open subset of K into Q.

2. (Surjectivity of derivatives) Let $\psi \in X_2 = A^{-1}(0)$ be given. Choose $\Delta \in (0, h)$ and a convex neighbourhood N of $(\partial\psi, \psi)$ in $C \times C^1$ according to (g1). Consider $\chi \in C^2$ with $\chi(t) = 0$ for $-h \le t \le -\Delta$. For $s \in \mathbb{R}$ sufficiently small we have $(\partial\psi + s\partial\chi, \psi) \in N$ and

$$g(\partial\psi + s\partial\chi, \psi) = g(\partial\psi, \psi)$$

(by (g1), as $\partial\psi(t) = (\partial\psi + s\partial\chi)(t)$ for $-h \le t \le -\Delta$). It follows that

$$0 = \lim_{0\neq s\to 0} \frac{1}{s}(g_1(\partial\psi + s\partial\chi,\psi) - g_1(\partial\psi,\psi)) = D_1 g_1(\partial\psi,\psi)\partial\chi,$$

and thereby

$$\begin{aligned} DA(\psi)\chi &= \chi'(0) - Dg_1(\partial\psi,\psi)(\partial\chi,\chi) \\ &= \chi'(0) - 0 - D_2 g_1(\partial\psi,\psi)\chi \\ &= \chi'(0) - Dg_1(\partial\psi,\psi)(0,\chi) \\ &= \chi'(0) - D_e g_1(\partial\psi,\psi)(0,\chi). \end{aligned} \tag{11}$$

For $i \in \{1,\ldots,n\}$ consider $e_i \in \mathbb{R}^n$ with $pr_i e_i = 1$ and $pr_k e_i = 0$ for $k \neq i$ in $\{1,\ldots,n\}$. There exists a sequence of functions $\chi_m \in C^2$, $m \in \mathbb{N}$, such that for every $m \in \mathbb{N}$ we have $\chi_m'(0) = e_i$, $\chi_m(t) = 0$ for $-h \le t \le -\Delta$, and $\lim_{m\to\infty} |\chi_m| = 0$. Due to (g3) the map $D_e g_1(\partial\psi,\psi) : C \times C \to \mathbb{R}^n$ is continuous at $(0,0)$. Using (11) we obtain

$$DA(\psi)\chi_m = e_i - D_e g_1(\partial\psi,\psi)(0,\chi_m) \to e_i \text{ as } m \to \infty.$$

We conclude that the image $DA(\psi)C^2$ contains a basis of $\mathbb{R}^n$, and $DA(\psi)$ is surjective.

3. As mentioned above, the result of part 2 yields that $X_2 = A^{-1}(0)$ is a continuously differentiable submanifold with codimension n in C^2. Let $\psi \in X_2$. The definition of tangent vectors, the chain rule, and the fact that A vanishes on X_2 combined yield

$$T_\psi X_2 \subset (DA(\psi))^{-1}(0) = \{\chi \in C^2 : \chi'(0) = Dg_1(\partial\psi,\psi)(\partial\chi,\chi)\}.$$

Both spaces are closed and have codimension n, so they are equal. □

6 A Semiflow on a Subset of the Manifold $X_2 \subset C^2$

In this section we assume that $g : C \times C^1 \supset W \to \mathbb{R}^n$, W open, has the properties (g0)–(g3). We show that Eq. (2) defines a continuous semiflow on the closed subset X_{2*} of the manifold X_2. Recall from Corollary 2.11 that all segments of twice continuously differentiable solutions belong to X_{2*}, and $X_{2*} \subset X_2 \subset X_{1+}$.

Proposition 6.1. *For each $\psi \in X_{2*}$ the solution x^ψ is twice continuously differentiable, and for all $t \in [0,t_\psi)$, $x_t^\psi \in X_{2*}$.*

Proof. Let $\psi \in X_{2*}$ be given. Set $x = x^{\psi}$. It is sufficient to show that the nonempty set

$$M = \{t \in [0,t_\psi) : x'|[-h,t] \text{ is continuously differentiable}\}$$

coincides with $[-h,t_\psi)$. Suppose this is false. Then $0 \le s = \sup M < t_\psi$, and $x'|[-h,s)$ is twice continuously differentiable. Choose a neighbourhood $N \subset W$ of $(\partial x_s, x_s)$ in $C \times C^1$ and $\Delta \in (0,h)$ according to (g1). By continuity there exists $\delta \in (0,\Delta)$ so that $s+\delta < t_\psi$ and $(\partial x_t, x_t) \in N$ for all $t \ge 0$ in $(s-\delta, s+\delta)$. Choose $r > 0$ so that $(\chi,\rho) \in N$ for all $\chi \in C$ with $|\chi - \partial x_s| < r$ and for all $\rho \in C^1$ with $|\rho - x_s|_1 < r$. Choose $\varepsilon \in (0,\delta)$ so that for all $u \ge 0$ in $[s-\varepsilon, s+\varepsilon]$,

$$|x_u - x_s|_1 < \frac{r}{2}.$$

In case $s = 0$ we have $s \in M$. Set $t = 0$ in this case. In case $0 < s$ choose $t \ge 0$ in $(s-\varepsilon, s)$. In both cases, $t \in M$, hence $x_t \in C^2$. Proposition 2.3 provides a twice continuously differentiable function $y : [t-h, s+\varepsilon] \to \mathbb{R}^n$ with $y_t = x_t$ and $|y'(u) - x'(u)| < \frac{r}{2}$ for $t < u \le s+\varepsilon$. Consequently,

$$|\partial y_u - \partial x_s| \le |\partial y_u - \partial x_u| + |x_u - x_s|_1 < \frac{r}{2} + \frac{r}{2} = r \text{ for } t \le u \le s+\varepsilon.$$

It follows that $(\partial y_u, x_u) \in N$ for such u. For $t \le u \le t+\varepsilon$ and for $-h \le v \le -\Delta$ we have $t-h \le u+v \le t+\Delta-\Delta = t$, hence

$$\partial y_u(v) = y'(u+v) = x'(u+v) = \partial x_u(v).$$

Using (g1) we infer

$$x'(u) = g(\partial x_u, x_u) = g(\partial y_u, x_u) = g_1((y')_u, x_u) \text{ for } t \le u \le t+\varepsilon.$$

As y' and x are continuously differentiable Proposition 2.9 and the preceding equation combined yield that $x'|[t,t+\varepsilon)$ is continuously differentiable. In case $0 < s$, $t < s < t+\varepsilon$, we infer that $x'|[-h,t+\varepsilon)$ is continuously differentiable, which implies a contradiction to $s = \sup M$. In case $s = 0$ and $t = 0$ the relation $\psi = x_0 \in X_{2*}$ and Proposition 2.9 combined show that the left and right derivatives of x' at $t = 0$ coincide. It follows that also in this case $x'|[-h,t+\varepsilon)$ is continuously differentiable, and we have a contradiction as before. □

We define

$$\Omega_2 = \{(t,\psi) \in [0,\infty) \times X_{2*} : t < t_\psi\}$$

and consider the map

$$G_2 : \Omega_2 \ni (t,\psi) \mapsto x_t^{\psi} \in X_{2*}.$$

Proposition 6.1 and the fact that G_1 is a semiflow combined yield that G_2 has the algebraic properties of a semiflow. Ω_2 is an open subset of $[0,\infty)\times X_{2,*}$ (in the topology given by $\mathbb{R}\times C^2$) as it is the preimage of the open set Ω_1 under the inclusion map

$$[0,\infty)\times X_{2*}\to[0,\infty)\times X_{1+}$$

which is continuous with respect to the topologies given by $\mathbb{R}\times C^1$ on its range and by $\mathbb{R}\times C^2$ on its domain. It also follows that the map

$$\Omega_2\ni(t,\psi)\mapsto G_2(t,\psi)\in C^1$$

is continuous. In order to obtain continuity of G_2 with respect to the norms on $\mathbb{R}\times C^2$ and of C^2, respectively, we first establish a local result.

Proposition 6.2. *For every $\psi\in X_{2*}$ there exist $T=T(\psi)>0$ and a neighbourhood $V=V(\psi)$ of ψ in X_{2*} (with respect to the topology given by C^2) so that $[0,T]\times V\subset\Omega_2$ and*

$$[0,T]\times V\ni(t,\phi)\mapsto x_t^{\phi}\in C^2$$

is continuous (with respect to the norm of $\mathbb{R}\times C^2$).

Proof. 1. Let $\psi\in X_{2,*}$ be given. Let $x=x^{\psi}$. There exist $c\geq 0$ and an open neighbourhood $N_1\subset W_1$ of $(\partial\psi,\psi)$ in $C^1\times C^1$ so that $|D_eg(\chi,\rho)|_{L_c(C\times C,\mathbb{R}^n)}\leq c$ for all $(\chi,\rho)\in N_1$ (see Proposition 2.6), and such that the assertions of Corollary 2.8 hold, with some $\Delta\in(0,h)$ and for $(\phi,\psi),(\phi_1,\psi_1)$ in N_1 with $\phi(t)=\phi_1(t)$ for $-h\leq t\leq-\Delta$.

2. Recall Proposition 2.4 (iii) and the facts that Ω_1 is open in $[0,\infty)\times X_{1+}$ (with respect to the topology given by $\mathbb{R}\times C^1$) and that G_1 is continuous (with respect to the norms on $\mathbb{R}\times C^1$ and on C^1). An application of Proposition 2.2 to the continuously differentiable map x' yields that $[0,t_{\psi})\ni t\mapsto\partial x_t\in C^1$ is continuous. We conclude that there exist a neighbourhood $V_1=V_1(\psi)\subset X_{2*}$ of ψ (with respect to the topology given by C^2) and $T=T(\psi)\in(0,\Delta)$ with the following properties:

$$[0,T]\times V_1\subset\Omega_1,$$
$$(\partial\phi_t^{dd},x_t^{\phi})\in N_1 \text{ for } 0\leq t\leq T \text{ and } \phi\in V_1,$$
$$(\partial x_t,x_t)\in N_1 \text{ for } 0\leq t\leq T.$$

Observe that we also have $[0,T]\times V_1\subset\Omega_2$. For $\phi\in V_1$, $t\in[0,T]$, and $s\in[-h,-\Delta]$ we get $-h\leq t+s\leq\Delta-\Delta=0$, hence

$$\partial x_t^{\phi}(s)=(x^{\phi})'(t+s)=\phi'(t+s)=\partial\phi_t^{dd}(s)\tag{12}$$

and

$$\partial\partial x_t^{\phi}(s) = (x^{\phi})''(t+s) = \phi''(t+s) = \partial\partial\,\phi_t^{dd}(s). \tag{13}$$

3. (Estimate) Let ϕ, ρ in V_1 be given. Set $y = x^{\phi}, z = x^{\rho}$. For every $t \in [0,T]$ with

$$(\partial y_t, y_t) \in N_1 \text{ and } (\partial z_t, z_t) \in N_1$$

we obtain

$$\begin{aligned}
|y''(t) - z''(t)| &= |D_e g_1(\partial y_t, y_t)(\partial\partial y_t, \partial y_t) - D_e g_1(\partial z_t, z_t)(\partial\partial z_t, \partial z_t)| \\
&\quad \text{(as } y_t \in X_{2*}, z_t \in X_{2*}) \\
&\leq |[D_e g_1(\partial y_t, y_t) - D_e g_1(\partial z_t, z_t)](\partial\partial y_t, \partial y_t)| \\
&\quad + |D_e g_1(\partial z_t, z_t)[(\partial\partial y_t, \partial y_t) - (\partial\partial z_t, \partial z_t)]| \\
&= |[D_e g_1(\partial\,\phi_t^{dd}, y_t) - D_e g_1(\partial\,\rho_t^{dd}, z_t)](\partial\partial\,\phi_t^{dd}, \partial y_t)| \\
&\quad + |D_e g_1(\partial\,\rho_t^{dd}, z_t)[(\partial\partial\,\phi_t^{dd}, \partial y_t) - (\partial\partial\,\rho_t^{dd}, \partial z_t)]| \\
&\quad \text{(recall Corollary 2.8, the choice of } N_1 \text{ and } \Delta, \\
&\quad \text{and (12), (13))} \\
&\leq |[D_e g_1(\partial\,\phi_t^{dd}, y_t) - D_e g_1(\partial\,\rho_t^{dd}, z_t)](\partial\partial\,\phi_t^{dd}, \partial y_t)| \\
&\quad + c(|\phi_t^{dd} - \rho_t^{dd}|_2 + |y_t - z_t|_1)
\end{aligned}$$

4. We show that there is a neighbourhood $V_3 = V_3(\psi) \subset V_1$ of ψ in X_{2*} (with respect of the topology given by C^2) so that for all $t \in [0,T]$ and for all $\phi \in V_3$,

$$(\partial x_t^{\phi}, x_t^{\phi}) \in N_1.$$

4.1 Due to continuity the set $K = \{x_t \in C^2 : 0 \leq t \leq T\}$ is a compact subset of C^2. Its image under $\partial \times id$ belongs to the open subset N_1 of $C^1 \times C^1$. It follows that there exists $\varepsilon > 0$ with $(\partial\,\phi, \phi) \in N_1$ for all $\phi \in C^2$ with $\text{dist}(\phi, K) \leq \varepsilon$.

4.2 The continuity of G_1 and Proposition 2.4 combined imply that there exists an open neighbourhood $V_2 \subset V_1$ of ψ in X_{2*} (with respect of the topology given by C^2) so that for all $t \in [0,T]$ and for all $\phi \in V_2$ we have

$$|x_t^{\phi} - x_t|_1 < \frac{\varepsilon}{4(c+1)} \text{ and } |\phi_t^{dd} - \psi_t^{dd}|_2 < \frac{\varepsilon}{4(c+1)}. \tag{14}$$

4.3 The map

$$[0,T] \times V_2 \ni (t,\phi) \mapsto [D_e g_1(\partial\,\phi_t^{dd}, x_t^{\phi}) - D_e g_1(\partial\,\psi_t^{dd}, x_t)](\partial\partial\,\phi_t^{dd}, \partial x_t^{\phi}) \in \mathbb{R}^n$$

is continuous and vanishes on the compact set $[0,T]\times\{\psi\}$ (in $\mathbb{R}\times C^2$). It follows that there exists an open neighbourhood $V_3\subset V_2$ of ψ in X_{2*} (with respect to the topology given by C^2) so that for all $t\in[0,T]$ and all $\phi\in V_3$ we have

$$|\phi-\psi|_2<\frac{3\varepsilon}{4}$$

and

$$|[D_e g_1(\partial\phi_t^{dd},x_t^\phi)-D_e g_1(\partial\psi_t^{dd},x_t)](\partial\partial\phi_t^{dd},\partial x_t^\phi)|<\frac{\varepsilon}{4}.$$

4.4 Let $\phi\in V_3$ be given and assume $(\partial x_t^\phi,x_t^\phi)\notin N_1$ for some $t\in[0,T]$. As $(\partial\phi,\phi)\in N_1$ and N_1 is open and $[0,T]\ni u\mapsto x_u^\phi\in C^2$ is continuous there exists $s\in(0,T]$ with $(\partial x_t^\phi,x_t^\phi)\in N_1$ for $0\le t<s$ and $(\partial x_s^\phi,x_s^\phi)\notin N_1$. For $0\le u<s$ the estimates of part 3 combined with (14) yield

$$|(x^\phi)''(u)-x''(u)|<\frac{\varepsilon}{4}+c\left(\frac{\varepsilon}{4(c+1)}+\frac{\varepsilon}{4(c+1)}\right)<\frac{3\varepsilon}{4}.$$

Using this and the first estimate in (14) we find

$$|x_t^\phi-x_t|_2<\varepsilon \text{ for } 0\le u<s.$$

It follows that $|x_s^\phi-x_s|_2\le\varepsilon$, hence $\text{dist}(x_s^\phi,K)\le\varepsilon$ which implies a contradiction to $(\partial x_s^\phi,x_s^\phi)\notin N_1$.

5. Let $\phi\in V_3$ and $\varepsilon>0$. We show that there is a neighbourhood $V_\phi\subset V_3$ of ϕ (with respect to the topology given by C^2) so that for $0\le t\le T$ and for $\rho\in V_\phi$ we have $|x_t^\rho-x_t^\phi|_2<\varepsilon$. Let $y=x^\phi$, and for $\rho\in V_3$, $z=x^\rho$. The results of parts 3 and 4 combined yield the estimate

$$\begin{aligned}|z''(t)-y''(t)|\le{}&|[D_e g_1(\partial\rho_t^{dd},z_t)-D_e g_1(\partial\phi_t^{dd},y_t)](\partial\partial\phi_t^{dd},\partial y_t)|\\&+c(|\rho_t^{dd}-\phi_t^{dd}|_2+|z_t-y_t|_1)\end{aligned}$$

for all $t\in[0,T]$. The assertion follows from this estimate by arguments as in part 4.
6. (Continuity of the map $[0,T]\times V_3\ni(t,\rho)\mapsto x_t^\rho\in C^2$). Let $t\in[0,T]$ and $\phi\in V_3$ be given. For all $s\in[0,T]$ and all $\rho\in V_3$ we have

$$\begin{aligned}|x_s^\rho-x_t^\phi|_2&\le|x_s^\rho-x_s^\phi|_2+|x_s^\phi-x_t^\phi|_2\\&\le\sup_{0\le u\le T}|x_u^\rho-x_u^\phi|_2+|x_s^\phi-x_t^\phi|_2\end{aligned}$$

Use part 5 and the continuity of $[0,T] \ni s \mapsto x_s^\phi \in C^2$ at $s=t$ in order to complete the proof. □

Corollary 6.3. *The map $G_2 : \Omega_2 \to X_{2*}$ is continuous (with respect to the norms on $\mathbb{R} \times C^2$ and C^2).*

Proof. Let $(t,\psi) \in \Omega_2$ be given. Choose $t_0 \in (t,t_\psi)$. Define M to be the set of all $s \in (0,t_0]$ for which there exists a neighbourhood $V \subset X_{2*}$ of ψ (with respect to the topology given by C^2) such that $[0,s] \times V \subset \Omega_2$ and $G_2|[0,s] \times V$ is continuous (with respect to the norms on $\mathbb{R} \times C^2$ and C^2). Proposition 6.2 yields $M \neq \emptyset$. Consider $t_1 = \sup M \in (0,t_0]$. In case $t_1 = t_0$ it follows that G_2 is continuous at (t,ψ) (with respect to the norms on $\mathbb{R} \times C^2$ and C^2). We exclude the case $t_1 < t_0$: In this case we argue as before and obtain $T > 0$ and a neighbourhood $V_1 \subset X_{2*}$ of $\rho = x_{t_1}^\psi$ (with respect to the topology given by C^2) so that $[0,T] \times V_1 \subset \Omega_2$ and $G_2|[0,T] \times V_1$ is continuous (with respect to the norms on $\mathbb{R} \times C^2$ and C^2). There exist $s > 0$ in $(t_1 - T, t_1]$ with $x_s^\psi \in V_1$, and $u \in (s,t_1] \cap M$. We obtain a neighbourhood $V_u \subset X_{2*}$ of ψ (with respect to the topology given by C^2) so that $[0,u] \times V_u \subset \Omega_2$ and $G_2|[0,u] \times V_u$ is continuous (with respect to the norms on $\mathbb{R} \times C^2$ and C^2). In addition we may assume that $G_2(\{s\} \times V_u) \subset V_1$. For $s \le w \le s+T$ and $\chi \in V_u$ we have $0 \le w-s \le T$ and $G_2(s,\chi) \in V_1$, hence $w = (w-s)+s < t_\chi$, $(w,\chi) \in \Omega_2$ and

$$G_2(w,\chi) = G_2(w-s, G_2(s,\chi)).$$

This decomposition shows that the restriction $G_2|[s,s+T] \times V_u$ is continuous (with respect to the norms on $\mathbb{R} \times C^2$ and C^2). As $s < u$ it follows that $[0,s+T] \times V_u \subset \Omega_2$ and that the restriction $G_2|[0,s+T] \times V_u$ is continuous (with respect to the norms on $\mathbb{R} \times C^2$ and C^2), in contradiction to the fact that $t_1 < s+T$ is an upper bound of M. □

7 A Variational Equation

This section prepares the proof of differentiability properties of the solution operators $G_2(t,\cdot)$ in the subsequent Sects. 8 and 9. We consider a map $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ open, with the properties (g0)–(g3). For $\psi \in X_{2*}$ and $\chi \in T_{e,\psi}X_2 \subset C^1$ we study the IVP

$$v'(t) = D_e g_1(\partial x_t^\psi, x_t^\psi)(\partial v_t, v_t) \tag{15}$$

$$v_0 = \chi. \tag{16}$$

A solution is a continuously differentiable function $v : [-h,t_*) \to \mathbb{R}^n$, $0 < t_* \le t_\psi$, which satisfies Eqs. (16) and (15) for $0 < t < t_*$. We also consider solutions on compact intervals $[-h,t_*]$ with $0 < t_* < t_\psi$. For a solution Eq. (15) holds at $t=0$, too, and we have $v_t \in T_{e,G_2(t,\psi)}X_2$ for all $t \ge 0$ in its domain.

Proposition 7.1. *For every $\psi \in X_{2,*}$ and for each $\chi \in T_{e,\psi}X_2$ there is a solution $v^{\psi,\chi} : [-h, t_\psi) \to \mathbb{R}^n$ of the IVP (15) and (16) so that any other solution is a restriction thereof. The evolution operators*

$$T_{e,\psi}X_2 \ni \chi \mapsto v_t^{\psi,\chi} \in T_{e,G_2(t,\psi)}X_2,\ 0 \le t < t_\psi,$$

are linear.

Proof. 1. (Local existence) Let $\psi \in X_{2*}$, $x = x^\psi$, $\chi \in T_{e,\psi}X_2$. For $(\partial\psi, \psi) \in W_1$ choose $\Delta \in (0,h)$ and a neighbourhood $N_1 \subset W_1$ of $(\partial\psi, \psi)$ in $C^1 \times C^1$ and $r > 0$ according to Corollary 2.8. Choose $T \in (0,\Delta) \cap (0, t_\psi)$ so that for all $t \in [0,T]$, $(\partial x_t, x_t) \in N_1$. The map

$$A : [0,T] \times C \ni (t,\phi) \mapsto D_e g_1(\partial x_t, x_t)(\partial \chi_t^d, \phi) \in \mathbb{R}^n$$

satisfies

$$A(t,\phi) = L(t,\phi) + f(t)$$

with $L : [0,T] \times C \to \mathbb{R}^n$ given by

$$D_e g_1(\partial x_t, x_t)(0,\phi)$$

and $f : [0,T] \to \mathbb{R}^n$ given by

$$f(t) = D_e g_1(\partial x_t, x_t)(\partial \chi_t^d, 0).$$

L and f are continuous and each $L(t,\cdot) : C \to \mathbb{R}^n$, $0 \le t \le T$, is linear. The IVP

$$v'(t) = A(t,v_t) = L(t,v_t) + f(t)$$
$$v_0 = \chi$$

has a continuous solution $v : [-h,T] \to \mathbb{R}^n$ (see Appendix, Proposition 11.4). There is a right derivative $v'(0+)$ of v at $t = 0$, and $v'(0+) = A(0,\chi)$. As $v_0 = \chi$ is continuously differentiable with left derivative

$$v'(0-) = \chi'(0) = D_e g_1(\partial\psi, \psi)(\partial\chi, \chi) = A(0,\chi) = v'(0+)$$

we see that v is continuously differentiable. For $0 \le t \le T$ and $-h \le s \le -\Delta$ we have $-h \le t+s \le 0$, and thereby

$$\partial \chi_t^d(s) = \chi'(t+s) = v'(t+s) = \partial v_t(s).$$

Using Corollary 2.8 we infer

$$v'(t) = A(t, v_t) = D_e g_1(\partial x_t, x_t)(\partial \chi^d_t, v_t) = D_e g_1(\partial x_t, x_t)(\partial v_t, v_t)$$

for $0 \le t \le T$.

2. (Uniqueness) Suppose $v : [-h, t_*) \to \mathbb{R}^n$ and $w : [-h, t_*) \to \mathbb{R}^n$ are solutions of the IVP (15) and (16), $v_0 = \chi = w_0$ and $v(t) \ne w(t)$ for some $t \in (0, t_*)$. For $s = \inf\{t \in [0, t_*) : v(t) \ne w(t)\}$ we get $0 \le s < t_*$ and $v(t) = w(t)$ for $-h \le t \le s$. Proposition 2.6 and continuity combined yield $c \ge 0$ and $\delta > 0$ so that for $s \le t < s + \delta$ we have

$$|D_e g_1(\partial x_t, x_t)|_{L_c(C \times C, \mathbb{R}^n)} \le c.$$

Choose $\Delta \in (0, h)$ and a neighbourhood $N_1 \subset W_1$ of $(\partial x_s, x_s)$ in $C^1 \times C^1$ and $r > 0$ according to Corollary 2.8. Choose $T > 0$ with $T < \min\{\Delta, \delta, t_* - s\}$ so that for all $t \in (s, s+T)$, $(\partial x_t, x_t) \in N_1$. For such t we infer

$$\begin{aligned} |v(t) - w(t)| &= |v(s) - w(s) + \int_s^t D_e g_1(\partial x_u, x_u)((\partial v_u, v_u) - (\partial w_u, w_u))du| \\ &= |0 + \int_s^t D_e g_1(\partial x_u, x_u)((\partial v_u, v_u) - (\partial v_u, w_u))du| \\ &\quad \text{(due to Corollary 2.8; for } s \le u \le t \text{ and } -h \le a \le -\Delta, \\ &\quad u + a \le s, \text{ hence } \partial v_u(a) = v'(u+a) = w'(u+a) = \partial w_u(a)) \\ &\le \int_s^t |D_e g_1(\partial x_u, x_u)((\partial v_u, v_u) - (\partial v_u, w_u))|du \\ &\le c \max_{s \le u \le t} |v_u - w_u|(t-s) \\ &\le c \max_{s \le u \le t} |v(u) - w(u)|(t-s) \\ &\quad \text{(as } v(u) = w(u) \text{ for } -h \le u \le s). \end{aligned}$$

For $t < s + \frac{1}{2c}$ the preceding estimate implies a contradiction to the definition of s.

3. (Maximal solutions) Let

$$t_{\psi,\chi} = \sup\{t \in (0, t_\psi) : \text{There is a solution } v : [-h, t) \to \mathbb{R}^n \text{ of (15) and (16)}\}.$$

Then $0 < t_{\psi,\chi} \le t_\psi$. Define $v^{\psi,\chi} : [-h, t_{\psi,\chi}) \to \mathbb{R}^n$ by $v^{\psi,\chi}(s) = v(s)$, with a solution $v : [-h, t) \to \mathbb{R}^n$, $s < t$, of the IVP (15) and (16). Then $v^{\psi,\chi}$ is a solution such that any other solution of the IVP (15) and (16) which is defined on an interval of the form $[-h, t)$, $0 < t \le t_\psi$, is a restriction of $v^{\psi,\chi}$. We show $t_{\psi,\chi} = t_\psi$. Set $v = v^{\psi,\chi}$ and $s = t_{\psi,\chi}$ and suppose $s < t_\psi$. Choose $\Delta \in (0, h)$ and a neighbourhood $N_1 \subset W_1$ of $(\partial x_s, x_s)$ in $C^1 \times C^1$ and $r > 0$ according to Corollary 2.8. We may assume that $\Delta > 0$ and a further number $T \in (0, \Delta)$ are so small that $0 < s-T$, $s-T+\Delta < t_\psi$ and for $s-T \le t \le s-T+\Delta$ we have $(\partial x_t, x_t) \in N_1$. Let $\rho = v_{s-T} \in C^1$ and consider the continuous map

$$A : [0,\Delta] \times C \ni (t,\phi) \mapsto D_e g_1(\partial x_{s-T+t}, x_{s-T+t})(\partial \rho_t^d, \phi) \in \mathbb{R}^n.$$

As in part 1, Proposition 11.4 yields a (continuous) solution $w : [-h,\Delta] \to \mathbb{R}^n$ of the IVP

$$v'(t) = A(t, v_t),$$
$$v_0 = \rho = v_{s-T}.$$

For $0 \le t \le \Delta$ and $-h \le u \le -\Delta$ we have $-h \le t+u \le 0$, hence

$$\partial \rho_t^d(u) = (\rho^d)'(t+u) = (v_{s-T}^d)'(t+u) = (v_{s-T})'(t+u)$$
$$= (w_0)'(t+u) = w'(t+u) = \partial w_t(u).$$

Using Corollary 2.8 we infer

$$w'(t) = A(t, w_t) = D_e g_1(\partial x_{s-T+t}, x_{s-T+t})(\partial \rho_t^d, w_t)$$
$$= D_e g_1(\partial x_{s-T+t}, x_{s-T+t})(\partial w_t, w_t) \text{ for } 0 \le t \le \Delta.$$

The map $\bar{v} : [-h, s-T+\Delta] \to \mathbb{R}^n$ given by $\bar{v}(t) = v(t)$ for $-h \le t \le s-T$ and $\bar{v}(t) = w(t-(s-T))$ for $s-T < t \le s-T+\Delta$ is continuously differentiable and is a solution to the IVP (15) and (16), in contradiction to the fact that any solution of this IVP is a restriction of $v = v^{\psi,\chi}$.

4. The assertion about linearity now follows by means of uniqueness of solutions to the IVP and linearity of the maps $D_e g_1(\partial x_t, x_t) : C \times C \to \mathbb{R}^n$, $0 \le t < t_\psi$.

□

Proposition 7.2 (A bound for evolution operators, locally uniform on Ω_2). *Let $\psi \in X_{2*}$ and $T \in (0, t_\psi)$ be given. There exist $\varepsilon > 0$ and $c > 0$ such that for all $\phi \in X_{2*}$ with $|\phi - \psi|_2 < \varepsilon$ we have $T < t_\phi$ and*

$$|v_t^{\phi,\chi}|_1 \le c|\chi|_1 \textit{ for all } \chi \in T_{e,\phi}X_2 \textit{ and } t \in [0,T].$$

Proof. 1. (An estimate for small t) Let $\psi \in X_{2*}$. Using that Ω_2 is open, that G_2 is continuous, and Proposition 2.6 and Corollary 2.8, we find $t(\psi) > 0$, $\varepsilon = \varepsilon(\psi) > 0$ and $c = c(\psi) \ge 0$ with the following properties: For every $\phi \in X_{2*}$ with $|\phi - \psi|_2 < \varepsilon$ we have $t(\psi) < t_\phi$ and

$$|D_e g_1(\partial x_t^\phi, x_t^\phi)|_{L_c(C \times C, \mathbb{R}^n)} \le c \text{ for } 0 \le t \le t(\psi),$$

and for all $(\rho,\beta),(\eta,\beta)$ in $C\times C$ with $\rho(s)=\eta(s)$ for $-h\le s\le -\varepsilon$, and for all $t\in[0,t(\psi)]$,

$$D_e g_1(\partial x_t^{\phi},x_t^{\phi})(\rho,\beta)=D_e g_1(\partial x_t^{\phi},x_t^{\phi})(\eta,\beta). \tag{17}$$

We may assume $t(\psi)\le\varepsilon(\psi)$. Then $-h\le t+u\le 0$ for $t\in[0,t(\psi)]$ and for $-h\le u\le -\varepsilon$. It follows that for $\phi\in X_{2*}$ with $|\phi-\psi|_2<\varepsilon$, for all $\chi\in T_{e,\phi}X_2$, and for such t and u the function $v=v^{\phi,\chi}$ satisfies

$$\partial v_t(u)=v'(t+u)=\chi'(t+u)=(\chi^d)'(t+u)=\partial\chi_t^d(u).$$

Using this and Eq. (17) we infer

$$\begin{aligned}|v'(t)| &= |D_e g_1(\partial x_t^{\phi},x_t^{\phi})(\partial v_t,v_t)|=|D_e g_1(\partial x_t^{\phi},x_t^{\phi})(\partial\chi_t^d,v_t)|\\ &\le c(|\partial\chi_t^d|+|v_t|)\le c(|\chi|_1+|v_t|).\end{aligned} \tag{18}$$

Consequently,

$$\begin{aligned}|v(t)| &= |\chi(0)+\int_0^t v'(s)ds|\le|\chi(0)|+c\int_0^t(|\chi|_1+|v_s|)ds\\ &\le (1+c\varepsilon)|\chi|_1+c\int_0^t|v_s|ds.\end{aligned}$$

For $-h\le u\le 0$ we consider the cases $t+u\le 0$ and $0<t+u$ and obtain in either one that

$$|v(t+u)|\le(1+c\varepsilon)|\chi|_1+c\int_0^t|v_s|ds,$$

hence

$$|v_t|\le(1+c\varepsilon)|\chi|_1+c\int_0^t|v_s|ds.$$

Gronwall's lemma yields

$$|v_t|\le(1+c\varepsilon)|\chi|_1e^{ct},$$

for $0\le t\le t(\psi)$, $\phi\in X_{2*}$ with $|\phi-\psi|_2<\varepsilon$, and $\chi\in T_{e,\psi}X_2$. Using this and the estimate (18) we obtain for $0\le t\le t(\psi)$ and for $-h\le u\le 0$ with $0\le t+u$ that

$$\begin{aligned}|v'(t+u)| &\le c(|\chi|_1+|v_{t+u}|)\le c\left(|\chi|_1+(1+c\varepsilon)|\chi|_1e^{c(t+u)}\right)\\ &\le |\chi|_1\left(c+(1+c\varepsilon)e^{ct}\right).\end{aligned}$$

This holds also in case $t+u<0$. It follows that

$$|v_t|_1 = |\partial v_t| + |v_t| \\ \leq |\chi|_1 \left(c + (1+c\varepsilon)e^{ct} + (1+c\varepsilon)e^{ct}\right),$$

for $0 \leq t \leq t(\psi)$, $\phi \in X_{2*}$ with $|\phi - \psi|_2 < \varepsilon$, and $\chi \in T_{e,\psi}X_2$.

2. Consider the set A of all $t \in (0, t_\psi)$ such that there exist $\varepsilon_t > 0$ and $c_t > 0$ so that $\varepsilon_t < t_\phi$ and

$$|v_s^{\phi,\chi}|_1 \leq c_t |\chi|_1$$

for all $\phi \in X_{2*}$ with $|\phi - \psi|_2 < \varepsilon_t$, for all $\chi \in T_{e,\phi}X_2$, and for all $s \in [0,t]$. Part 1 guarantees that A is not empty. In order to prove the assertion we show that $s = \sup A > 0$ equals t_ψ. We argue by contradiction and assume $s < t_\psi$. Let $x = x^\psi$ and $\psi^* = x_s$.

2.1. According to part 1 there exist $t^* > 0$, $\varepsilon^* > 0$ and $c^* \geq 0$ so that for all

$$\phi^* \in N^* = \{\eta \in X_{2*} : |\eta - \psi^*|_2 < \varepsilon^*\}$$

we have $t^* < t_{\phi^*}$, and for all $\chi^* \in T_{e,\phi^*}X_2$ and $t \in [0, t^*]$,

$$|v_t^{\phi^*,\chi^*}|_1 \leq c^* |\chi^*|_1.$$

The definition of s and continuity combined show that there exists $s_1 \in (s - t^*, s) \cap A$ with $x_{s_1} \in N^*$. Now we use $s_1 \in A$ and the continuity of G_2 and find $\varepsilon_{s_1} > 0$ and $c_{s_1} \geq 0$ with the following properties: For each

$$\phi \in N_1 = \{\eta \in X_{2*} : |\eta - \psi|_2 < \varepsilon_{s_1}\},$$

$$s_1 < t_\phi \text{ and } G_2(s_1, \phi) \in N^*,$$

and for all $\chi \in T_{e,\phi}X_2$ and $s \in [0, s_1]$,

$$|v_s^{\phi,\chi}|_1 \leq c_{s_1} |\chi|_1.$$

In order to arrive at a contradiction we deduce in the sequel that $s_1 + t^* \in A$.

2.2. Let $\phi \in N_1$ be given. Set $\phi_1 = G_2(s_1, \phi) \in N^*$. For $s_1 \leq t < s_1 + t_{\phi_1}$ we get $(t - s_1, \phi_1) \in \Omega_2$. Using this and $(t - s_1, \phi_1) = (t - s_1, G_2(s_1, \phi))$ we find

$$(t, \phi) \in \Omega_2 \text{ and } x_t^\phi = x_{t-s_1}^{\phi_1}.$$

It follows that

$$s_1 + t^* < s_1 + t_{\phi_1} \leq t_\phi.$$

2.3. Consider $\chi \in T_{e,\phi}X_2$. Then

$$\chi_1 = v_{s_1}^{\phi,\chi} \in T_{e,\phi_1}X_2 \subset C^1.$$

Set $v^* = v^{\phi_1,\chi_1}$. Using $\phi_1 \in N^*$ and $\chi_1 \in T_{e,\phi_1}X_2$ we infer

$$|v_t^*|_1 \le c^*|\chi_1|_1 \text{ for } 0 \le t \le t^*.$$

2.4. Consider the function $v : [-h, s_1 + t_{\phi_1}) \to \mathbb{R}^n$ given by

$$v(t) = v^{\phi,\chi}(t) \text{ for } -h \le t \le s_1,$$
$$v(t) = v^*(t - s_1) \text{ for } s_1 < t < s_1 + t_{\phi_1}.$$

v is continuously differentiable because on each of the intervals $[-h, s_1]$, $[s_1 - h, s_1 + t_{\phi_1})$ it coincides with a continuously differentiable function, due to the equations

$$v^{\phi,\chi}(t) = v^{\phi,\chi}(s_1 + t - s_1) = v_{s_1}^{\phi,\chi}(t - s_1) = \chi_1(t - s_1) = v^*(t - s_1),$$

for $s_1 - h \le t \le s_1$. For $s_1 < t < s_1 + t_{\phi_1}$ and for $-h \le u \le 0$ with $t + u \le s_1$ the preceding equations yield

$$v_{t-s_1}^*(u) = v^*(t - s_1 + u) = v^*(t + u - s_1) = v^{\phi,\chi}(t + u) = v(t + u) = v_t(u).$$

For $s_1 < t < s_1 + t_{\phi_1}$ and for $-h \le u \le 0$ with $s_1 < t + u$ the definition of v gives

$$v_{t-s_1}^*(u) = v^*(t - s_1 + u) = v^*(t + u - s_1) = v(t + u) = v_t(u).$$

It follows that

$$v_{t-s_1}^* = v_t \text{ for } s_1 < t < s_1 + t_{\phi_1}.$$

For $s_1 < t < s_1 + t_{\phi_1}$ the preceding equation and the result of part 2.2 combined yield

$$\begin{aligned} v'(t) &= (v^*)'(t - s_1) = (v^{\phi_1,\chi_1})'(t - s_1) \\ &= D_e g_1(\partial x_{t-s_1}^{\phi_1}, x_{t-s_1}^{\phi_1})(\partial v_{t-s_1}^{\phi_1,\chi_1}, v_{t-s_1}^{\phi_1,\chi_1}) = D_e g_1(\partial x_t^{\phi}, x_t^{\phi})(\partial v_t, v_t), \end{aligned}$$

and we see that v is a solution to the IVP (15) and (16). Moreover, for $0 \le t \le s_1$ we have

$$|v_t|_1 = |v_t^{\phi,\chi}|_1 \le c_{s_1}|\chi|_1,$$

and for $s_1 \le t \le s_1 + t^*$ ($< s_1 + t_{\phi_1} \le t_\phi$, see part 2.2),

$$|v_t|_1 = |v^*_{t-s_1}|_1 = |v^{\phi_1,\chi_1}_{t-s_1}|_1$$
$$\le c^*|\chi_1|_1 = c^*|v^{\phi,\chi}_{s_1}|_1 \le c^* c_{s_1}|\chi|_1.$$

Altogether, we obtain $s_1 + t^* \in A$, which contradicts the fact that $s < s_1 + t^*$ is an upper bound for A. □

8 A Differentiability Property of Solution Operators

In this section we assume that $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ open, satisfies (g0)–(g4). For $t > 0$ we consider the set

$$\Omega_{2,t} = \{\psi \in X_{2*} : t < t_\psi\},$$

which is open in X_{2*} with respect to the topology given by C^2, and the continuous map

$$G_{2,t} : \Omega_{2,t} \ni \psi \mapsto G_2(t,\psi) \in X_{2*}.$$

Observe that for $0 \le s \le t$,

$$\Omega_{2,t} \subset \Omega_{2,s}.$$

In order to discuss differentiability of the solution operators $G_{2,t}$ we have to look at compositions with smooth maps into $X_{2*} \subset X_2 \subset C^2$. Recall the inclusion map $j : C^2 \to C^1$.

Proposition 8.1. *Let $t > 0$ and let a continuously differentiable map $p : Q \to C^2$, Q an open subset of a Banach space B, be given, with $p(Q) \subset \Omega_{2,t}$. The map $j \circ G_{2,t} \circ p$ is differentiable, and for every $q \in Q$ and $b \in B$,*

$$D(j \circ G_{2,t} \circ p)(q)b = v_t^{p(q),Dp(q)b}.$$

Proof. 1. Notice that for each $q \in Q$ and $b \in B$,

$$Dp(q)b \in T_{p(q)}X_2 \subset T_{e,p(q)}X_2.$$

Let $q \in Q$ be given. Set $\psi = p(q)$ and $x = x^\psi$. In the sequel we use the following conventions: For $\overline{\psi} \in X_{2*}$, $y = x^{\overline{\psi}}$, and for $\chi \in T_\psi X_2 \subset C^2 \cap T_{e,\psi}X_2$, $v = v^{\psi,\chi}$. There exist a neighbourhood $V \subset X_{2*}$ of ψ (with respect to the topology on X_{2*} given by C^2) so that $[0,t] \times V \subset \Omega_2$, and constants $c \ge 0$ and $\Delta \in (0,h) \cap (0,t]$ with the following five properties.

(i) For all $\overline{\psi} \in V$, all $s \in [0,t]$, and all $\theta \in [0,1]$,

$$(\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)] \in W_1. \tag{19}$$

This can be achieved using the continuity of G_2 and compactness of $[0,t]$.

(ii) For all $\overline{\psi} \in V$, all $s \in [0,t]$, and all $\theta \in [0,1]$,

$$|D_e g_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)])|_{L_c(C \times C, \mathbb{R}^n)} \leq c. \tag{20}$$

The estimate (20) is obtained from Proposition 2.6 in combination with the compactness of $[0,t]$ and the continuity of G_2.

(iii) For all $\overline{\psi} \in V$ and for $u \leq s \leq u + \Delta$ with $s \leq t$,

$$\begin{aligned} & D_e g_1(\partial x_s, x_s)(\partial(y_s - x_s - v_s), y_s - x_s - v_s) \\ & = D_e g_1(\partial x_s, x_s)((\partial(y_u - x_u - v_u))^c_{s-u}, y_s - x_s - v_s). \end{aligned} \tag{21}$$

The preceding property is due to an application of Corollary 2.8 to the compact subset $K = \{(\partial x_s, x_s) \in C^1 \times C^1 : 0 \leq s \leq t\} \subset W_1$ of $C^1 \times C^1$. With Δ given by Corollary 2.8, we have

$$\begin{aligned} (\partial(y_u - x_u - v_u))^c_{s-u}(a) &= (\partial(y_u - x_u - v_u))^c(s - u + a) \\ &= \partial(y_u - x_u - v_u)(s - u + a) \text{ (since } s - u + a \leq 0) \\ &= (y - x - v)'(u + s - u + a) = \partial(y_s - x_s - v_s)(a) \end{aligned}$$

for $u \leq s \leq t$ with $s \leq u + \Delta$ and for all $a \in [-h, -\Delta]$. Now Corollary 2.8 yields Eq. (21).

(iv) For all $\overline{\psi} \in V$, for $0 \leq s \leq t$, and for $0 \leq \theta \leq 1$,

$$\begin{aligned} |\{Dg_1((\partial x_s, x_s) \,+\, \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]) - Dg_1(\partial x_s, x_s)\} \\ (\partial(y_s - x_s), 0)| \\ \leq c|\partial\partial(y_s - x_s)||y_s - x_s|. \end{aligned} \tag{22}$$

This property is due to an application of Proposition 2.7 (ii) to the compact subset $K \subset W_1$ of $C^1 \times C^1$, with V so small that for all $s \in [0,t]$ and all $\theta \in [0,1]$ the pair

$$(\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]$$

belongs to the neighbourhood N_1 of K which is given by Proposition 2.7 (ii), and

$$\begin{aligned} & |(\partial x_s, x_s) \,+\, \theta[(\partial y_s, y_s) - (\partial x_s, x_s)] - (\partial x_s, x_s)|_{C^1 \times C^1} \\ & \leq |(\partial y_s, y_s) - (\partial x_s, x_s)|_{C^1 \times C^1} < r \end{aligned}$$

with the number $r > 0$ given by Proposition 2.7 (ii).

Finally, in view of Proposition 7.2 we can achieve that

(v) For all $\chi \in T_\psi X_2$ and for $0 \le s \le t$,

$$|v_s|_1 \le c|\chi|_1. \tag{23}$$

2. For $\overline{\psi} \in V$, $\chi \in T_\psi X_2 \subset C^2 \cap T_{e,\psi}X_2$, and $0 \le u \le s \le t$ with $s \le u + \Delta$ we obtain the following estimate.

$$|y'(s) - x'(s) - v'(s)| = |g(\partial y_s, y_s) - g(\partial x_s, x_s) - D_e g_1(\partial x_s, x_s)(\partial v_s, v_s)|$$

$$= |\int_0^1 \{Dg_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]) - D_e g_1(\partial x_s, x_s)\}$$

$$[(\partial y_s, y_s) - (\partial x_s, x_s)]d\theta + D_e g_1(\partial x_s, x_s)(\partial(y_s - x_s - v_s), y_s - x_s - v_s)|$$

(with (19) and continuous differentiability of g_1)

$$= |\int_0^1 \{Dg_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]) - Dg_1(\partial x_s, x_s)\}$$

$$((\partial y_s, y_s) - (\partial x_s, x_s))d\theta + D_e g_1(\partial x_s, x_s)((\partial(y_u - x_u - v_u))^c_{s-u}, y_s - x_s - v_s)|$$

(with (21))

$$= |\int_0^1 \{Dg_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]) - Dg_1(\partial x_s, x_s)\}$$

$$(\partial(y_s - x_s), 0)d\theta$$

$$+\int_0^1 \{Dg_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]) - Dg_1(\partial x_s, x_s)\}(0, y_s - x_s)d\theta$$

$$+D_e g_1(\partial x_s, x_s)((\partial(y_u - x_u - v_u))^c_{s-u}, y_s - x_s - v_s)|$$

$$\le c|\partial\partial(y_s - x_s)||y_s - x_s|$$

$$+|\int_0^1 \{D_e g_1((\partial x_s, x_s)+\theta[(\partial y_s, y_s)-(\partial x_s, x_s)])-D_e g_1(\partial x_s, x_s)\}(0, y_s-x_s-v_s)d\theta|$$

$$+|\int_0^1 \{Dg_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]) - Dg_1(\partial x_s, x_s)\}(0, v_s)d\theta|$$

$$+c(|(\partial(y_u - x_u - v_u))^c_{s-u}| + |y_s - x_s - v_s|)$$

(with (22) and (20))

$$\le c|\partial\partial(y_s - x_s)||y_s - x_s| + 2c|y_s - x_s - v_s|$$

$$+\max_{0\le s\le t, 0\le\theta\le 1} |Dg_1((\partial x_s, x_s)+\theta[(\partial y_s, y_s)-(\partial x_s, x_s)])-Dg_1(\partial x_s, x_s)|_{L_c(C^1\times C^1, \mathbb{R}^n)}|v_s|_1$$

$$+c(|\partial(y_u - x_u - v_u)| + |y_s - x_s - v_s|)$$

(with (20))

$$\leq c|\partial\partial(y_s - x_s)||y_s - x_s - v_s| + c|\partial\partial(y_s - x_s)||v_s| + 2c|y_s - x_s - v_s|$$

$$+ \max_{0\leq s\leq t, 0\leq\theta\leq 1} |Dg_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]) - Dg_1(\partial x_s, x_s)|_{L_c(C^1\times C^1,\mathbb{R}^n)}|v_s|_1$$

$$+c(|\partial(y_u - x_u - v_u)| + |y_s - x_s - v_s|)$$

$$\leq \Big(\max_{0\leq s\leq t, 0\leq\theta\leq 1} |Dg_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)]) - Dg_1(\partial x_s, x_s)|_{L_c(C^1\times C^1,\mathbb{R}^n)}$$

$$+c|\partial\partial(y_s - x_s)|\Big)|v_s|_1 + c|\partial(y_u - x_u - v_u)| + (c|\partial\partial(y_s - x_s)| + 3c)|y_s - x_s - v_s|.$$

It is convenient to set

$$c_1(\overline{\psi}) = \max_{0\leq s\leq t} |\partial\partial(y_s - x_s)|$$

and

$$c_2(\overline{\psi}) = \max_{0\leq s\leq t, 0\leq\theta\leq 1} |Dg_1((\partial x_s, x_s) + \theta[(\partial y_s, y_s) - (\partial x_s, x_s)])$$
$$-Dg_1(\partial x_s, x_s)|_{L_c(C^1\times C^1,\mathbb{R}^n)}.$$

Then the previous estimate becomes

$$\begin{aligned} |y'(s) - x'(s) - v'(s)| &\leq (c_2(\overline{\psi}) + c\,c_1(\overline{\psi}))|v_s|_1 \\ &\quad + c|\partial(y_u - x_u - v_u)| + (c\,c_1(\overline{\psi}) + 3c)|y_s - x_s - v_s| \\ &\leq (c_2(\overline{\psi}) + c\,c_1(\overline{\psi}))c|\chi|_1 \\ &\quad + c|\partial(y_u - x_u - v_u)| + (c\,c_1(\overline{\psi}) + 3c)|y_s - x_s - v_s| \\ &\quad \text{(with(23))}, \end{aligned} \tag{24}$$

for $\overline{\psi} \in V$, $\chi \in T_\psi X_2 \subset C^2 \cap T_{e,\psi} X_2$, and $0 \leq u \leq s \leq t$ with $s \leq u + \Delta$.

3. We have $c_1(\overline{\psi}) \to 0$ as $\overline{\psi} \to \psi$ in C^2 since G_2 is uniformly continuous on the compact set $[0,t] \times \{\psi\} \subset \Omega_2$ in the sense that, given $\varepsilon > 0$, there exists $\delta > 0$ so that for all $(r, \overline{\psi}) \in \Omega_2$ and for all $(s, \psi) \in [0,t] \times \{\psi\}$ with $|r - t| + |\overline{\psi} - \psi|_2 < \delta$,

$$|G_2(r, \overline{\psi}) - G_2(s, \psi)|_2 < \varepsilon.$$

4. We also have $c_2(\overline{\psi}) \to 0$ as $\overline{\psi} \to \psi$ in C^2. This follows from the uniform continuity of G_2 on the compact set $[0,t] \times \{\psi\} \subset \Omega_2$ (see part 3) in combination with the uniform continuity of Dg_1 on the compact subset $\{(\partial x_s, x_s) \in C^1 \times C^1 : 0 \leq s \leq t\} \subset W^1$ of $C^1 \times C^1$ (in the same sense as in part 3).

5. Next we use the estimate (24) in order to obtain an estimate of $|y_s - x_s - v_s|_1$, for $u, s, \overline{\psi}, \chi$ as in part 2. Notice that for $a \in [-h, 0]$ with $s + a \le u$,

$$|(y_s - x_s - v_s)(a)| = |y(s+a) - x(s+a) - v(s+a)| \le |y_u - x_u - v_u|$$

while in case $u < s + a$,

$$\begin{aligned} |(y_s - x_s - v_s)(a)| &= |y(s+a) - x(s+a) - v(s+a)| \\ &= |y(u) - x(u) - v(u) + \int_u^{s+a} (y'(r) - x'(r) - v'(r))dr| \\ &\le |y_u - x_u - v_u| + \int_u^s |y'(r) - x'(r) - v'(r)|dr. \end{aligned}$$

It follows that

$$|y_s - x_s - v_s| \le |y_u - x_u - v_u| + \int_u^s |y'(r) - x'(r) - v'(r)|dr. \tag{25}$$

We insert this into the estimate (24). An application of Gronwall's lemma yields

$$\begin{aligned} |y'(s) - x'(s) - v'(s)| \le &\{c(c\,c_1(\overline{\psi}) + c_2(\overline{\psi}))|\chi|_1 \\ &+ (c\,c_1(\overline{\psi}) + 4c)|y_u - x_u - v_u|_1\}e^{(s-u)(c\,c_1(\overline{\psi})+3c)} \end{aligned}$$

for $u \le s \le \min\{t, u + \Delta\}$, and $\overline{\psi}, \chi$ as in part 2. It is convenient to introduce

$$\begin{aligned} c_3(\overline{\psi}) &= c(c\,c_1(\overline{\psi}) + c_2(\overline{\psi})), \\ c_4(\overline{\psi}) &= c\,c_1(\overline{\psi}) + 4c, \end{aligned}$$

for $\overline{\psi} \in V$. From the previous estimate we obtain

$$|y'(s) - x'(s) - v'(s)| \le \{c_3(\overline{\psi})|\chi|_1 + c_4(\overline{\psi})|y_u - x_u - v_u|_1\}e^{t\,c_4(\overline{\psi})} \tag{26}$$

for $u \le s \le \min\{t, u + \Delta\}$, and $\overline{\psi}, \chi$ as before. For the same $u, s, \overline{\psi}, \chi$ and for $a \in [-h, 0]$ with $s + a \le u$ we have

$$|y'(s+a) - x'(s+a) - v'(s+a)| \le |y_u - x_u - v_u|_1$$

while in case $u < s + a$ the estimate (26) yields

$$|y'(s+a) - x'(s+a) - v'(s+a)| \le \{c_3(\overline{\psi})|\chi|_1 + c_4(\overline{\psi})|y_u - x_u - v_u|_1\}e^{t\,c_4(\overline{\psi})}.$$

Combining both cases we infer

$$|\partial(y_s - x_s - v_s)| \le \{c_3(\overline{\psi})|\chi|_1 + (1 + c_4(\overline{\psi}))|y_u - x_u - v_u|_1\}e^{t\,c_4(\overline{\psi})}. \qquad (27)$$

From (25) and (26) we get

$$\begin{aligned} |y_s - x_s - v_s| &\le |y_u - x_u - v_u| + \int_u^s |y'(r) - x'(r) - v'(r)|dr \\ &\le |y_u - x_u - v_u| + (s-u)\left(\{c_3(\overline{\psi})|\chi|_1 + c_4(\overline{\psi})|y_u - x_u - v_u|_1\}e^{t\,c_4(\overline{\psi})}\right). \end{aligned}$$

Adding this and the estimate (27) we arrive at

$$\begin{aligned} |y_s - x_s - v_s|_1 &\le |y_u - x_u - v_u| \qquad (28)\\ &\quad + (s-u)(\{c_3(\overline{\psi})|\chi|_1 + c_4(\overline{\psi})|y_u - x_u - v_u|_1\}e^{t\,c_4(\overline{\psi})}) \\ &\quad + \{c_3(\overline{\psi})|\chi|_1 + (1 + c_4(\overline{\psi}))|y_u - x_u - v_u|_1\}e^{t\,c_4(\overline{\psi})} \\ &\le \left((t+1)c_3(\overline{\psi})e^{t\,c_4(\overline{\psi})}\right)|\chi|_1 \\ &\quad + \left(1 + (1 + c_4(\overline{\psi}) + t\,c_4(\overline{\psi}))e^{t\,c_4(\overline{\psi})}\right)|y_u - x_u - v_u|_1 \end{aligned}$$

for $\overline{\psi} \in V$, $\chi \in T_\psi X_2 \subset C^2 \cap T_{e,\psi}X_2$, and $0 \le u \le s \le t$ with $s \le u + \Delta$.

6. We proceed to an estimate of $|y_t - x_t - v_t|_1$, for $\overline{\psi} \in V$, $\chi \in T_\psi X_2$. For $\overline{\psi} \in V$, set

$$A_1(\overline{\psi}) = (t+1)c_3(\overline{\psi})e^{t\,c_4(\overline{\psi})}$$

and

$$A_2(\overline{\psi}) = 1 + (1 + c_4(\overline{\psi}) + t\,c_4(\overline{\psi}))e^{t\,c_4(\overline{\psi})}.$$

Then

$$A_1(\overline{\psi}) \to 0 \text{ and } A_2(\overline{\psi}) \to 1 + (1 + 4c + 4tc)e^{4tc} \text{ as } \overline{\psi} \to \psi \text{ in } C^2. \qquad (29)$$

Choose an integer $J > 0$ so that $\frac{t}{J} \le \Delta$. For every $k \in \{1, \ldots, J\}$ and for all $\overline{\psi} \in V$ and $\chi \in T_\psi X_2$ the estimate (28) yields

$$|y_{k\frac{t}{J}} - x_{k\frac{t}{J}} - v_{k\frac{t}{J}}|_1 \le A_1(\overline{\psi})|\chi|_1 + A_2(\overline{\psi})|y_{(k-1)\frac{t}{J}} - x_{(k-1)\frac{t}{J}} - v_{(k-1)\frac{t}{J}}|_1.$$

Using induction we get

$$|y_t - x_t - v_t|_1 \le A_1(\overline{\psi})|\chi|_1 \sum_{k=0}^{J-1} A_2(\overline{\psi})^k + A_2(\overline{\psi})^J |y_0 - x_0 - v_0|_1$$

$$= A_1(\overline{\psi})|\chi|_1 \sum_{k=0}^{J-1} A_2(\overline{\psi})^k + A_2(\overline{\psi})^J |\overline{\psi} - \psi - \chi|_1$$

for $\overline{\psi} \in V$, $\chi \in T_\psi X_2$.

7. There is a neighbourhood $Q_1 \subset Q$ of q in B with $p(Q_1) \subset V$. For $\overline{q} \in Q_1$ set

$$\overline{\psi} = p(\overline{q}) \text{ and } \chi = Dp(q)(\overline{q} - q) \in T_\psi X_2.$$

It follows that for every $\overline{q} \in Q_1$ we have
$|(j \circ G_{2,t} \circ p)(\overline{q}) - (j \circ G_{2,t} \circ p)(q) - v_t^{p(q),Dp(q)(\overline{q}-q)}|_1$

$$\le A_1(p(\overline{q})) \sum_{k=0}^{J-1} A_2(p(\overline{q}))^k |Dp(q)(\overline{q} - q)|_1 + A_2(p(\overline{q}))^J |p(\overline{q}) - p(q) - Dp(q)(\overline{q} - q)|_1$$

$$\le A_1(p(\overline{q})) \sum_{k=0}^{J-1} A_2(p(\overline{q}))^k |Dp(q)||(\overline{q} - q)| + A_2(p(\overline{q}))^J |p(\overline{q}) - p(q) - Dp(q)(\overline{q} - q)|_1.$$

Now it is obvious how to complete the proof using the differentiability of p at q and the relations (29).

□

9 More on the Variational Equation

In this section we assume that $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ open, satisfies (g0)–(g5). The results below will be used in Sect. 10 about points of continuity of derivatives. We begin with a technical proposition.

Proposition 9.1. *Let $\psi \in X_{2*}$ be given, and $0 < T < t_\psi$, $L \ge 0$. There exist $\Delta \in (0,h) \cap (0,T]$ and $c \ge 0$ such that for $\chi \in T_{e,\psi} X_2$ with*

$$|\chi|_1 \le L$$

and for $t \le T$ and $0 \le u \le s \le t \le u + \Delta$ the functions $x = x^\psi$ and $v = v^{\psi,\chi}$ have the following properties:

$$(\partial x_s, x_s),\ (\partial((x_u)^{dd})_{s-u}, x_s),\ \text{and}\ (\partial((x_u)^{dd})_{t-u}, x_t)$$

belong to W_1, and

$$D_e g_1(\partial x_s, x_s)(\partial v_s, v_s) = D_e g_1(\partial((x_u)^{dd})_{s-u}, x_s)(\partial((v_u)^d)_{s-u}, v_s), \tag{30}$$

$$\begin{aligned}
&|[D_e g_1(\partial((x_u)^{dd})_{s-u}, x_s) - D_e g_1(\partial((x_u)^{dd})_{t-u}, x_t)](\partial((v_u)^d)_{t-u}, v_t)| \\
&\quad \le c|(\partial((x_u)^{dd})_{s-u}, x_s) - (\partial((x_u)^{dd})_{t-u}, x_t)|_{C^1 \times C^1} \\
&\qquad (Lip(\partial((v_u)^d)_{t-u}) + Lip(\partial\partial((x_u)^{dd})_{s-u}) + 1).
\end{aligned} \tag{31}$$

Proof. 1. Let $\psi \in X_{2*}$, $x = x^\psi$, $0 < T < t_\psi$. The set

$$K = \{(\partial x_t, x_t) \in C^1 \times C^1 : 0 \le t \le T\} \subset W_1$$

is a compact subset of $C^1 \times C^1$. Choose $\Delta \in (0,h)$, a neighbourhood $N_1 \subset W_1$ of K in $C^1 \times C^1$, $r > 0$, and $c \ge 0$ so that the assertions of Proposition 2.7 (iii) and of Corollary 2.8 hold. There exists $r_1 \in (0,r)$ so that all $(\phi, \eta) \in C^1 \times C^1$ with $|\phi - \partial x_t|_1 + |\eta - x_t|_1 < r_1$ for some $t \in [0,T]$ belong to N_1. As the maps $[0,T] \ni t \mapsto x_t \in C^2$ and

$$[0,T] \times [0,T] \ni (s,u) \mapsto ((x_u)^{dd})_s \in C^2$$

are uniformly continuous (see Proposition 2.4 (iii)) there exists $\Delta_* \in (0,\Delta)$ so that for all (u,s), (w,t) in $[0,T] \times [0,T]$ with $|u-w| + |s-t| < \Delta_*$ we have

$$|x_s - x_t|_2 < \frac{r_1}{2} \text{ and } |((x_u)^{dd})_s - ((x_w)^{dd})_t|_2 < \frac{r_1}{2}.$$

2. Let $L \ge 0$. Choose a constant $c_* \ge 0$ according to Proposition 7.2, so that for all $\chi \in T_{e,\psi}X_2$ with $|\chi|_1 \le L$ and for all $t \in [0,T]$ we have

$$|v_t^{\psi,\chi}|_1 \le c_* L.$$

3. Let $\chi \in T_{e,\psi}X_2$ with $|\chi|_1 \le L$ be given. Set $v = v^{\psi,\chi}$. Consider $t \in [0,T]$ and $0 \le u \le s \le t \le u + \Delta_*$. Then

$$|\partial((x_u)^{dd})_{s-u} - \partial x_u|_1 = |\partial((x_u)^{dd})_{s-u} - \partial((x_u)^{dd})_0|_1 < \frac{r_1}{2},$$

and $|x_s - x_u|_1 < \frac{r_1}{2}$, hence

$$|\partial((x_u)^{dd})_{s-u} - \partial x_u|_1 + |x_s - x_u|_1 < r_1,$$

and thereby $(\partial((x_u)^{dd})_{s-u}, x_s) \in N_1$. We deduce Eq. (30): Observe

$$|\partial((x_u)^{dd})_{s-u} - \partial x_s|_1 \le |\partial((x_u)^{dd})_{s-u} - \partial x_u|_1 + |\partial x_u - \partial x_s|_1 < r_1.$$

For $-h \le w \le -\Delta$ we have $-h \le s-u+w \le \Delta_* - \Delta < 0$, hence

$$\begin{aligned}
\partial((x_u)^{dd})_{s-u}(w) &= ((x_u)^{dd})'(s-u+w) = (x_u)'(s-u+w) \\
&= x'(u+s-u+w) = x'(s+w) = \partial x_s(w), \\
\partial((v_u)^{d})_{s-u}(w) &= ((v_u)^{d})'(s-u+w) = (v_u)'(s-u+w) \\
&= v'(u+s-u+w) = \partial v_s(w).
\end{aligned}$$

Now Corollary 2.8 yields Eq. (30).

4. Proof of the estimate (31). As in part 3,

$$(\partial((x_u)^{dd})_{s-u}, x_s) \in N_1 \text{ and } (\partial((x_u)^{dd})_{t-u}, x_t) \in N_1.$$

Moreover,

$$\begin{aligned}
&|(\partial((x_u)^{dd})_{s-u}, x_s) \;-\; (\partial((x_u)^{dd})_{t-u}, x_t)|_{C^1 \times C^1} \\
&\qquad = |\partial((x_u)^{dd})_{s-u} - \partial((x_u)^{dd})_{t-u}|_1 + |x_s - x_t|_1 < r_1 < r.
\end{aligned}$$

In case $v_t \neq 0$ Proposition 2.7 (iii) yields

$$\begin{aligned}
&\frac{|[D_e g_1(\partial((x_u)^{dd})_{s-u}, x_s) - D_e g_1(\partial((x_u)^{dd})_{t-u}, x_t)](\partial((v_u)^{d})_{t-u}, v_t)|}{|(\partial((v_u)^{d})_{t-u}, v_t)|_{C \times C}} \\
&\quad \le c|(\partial((x_u)^{dd})_{s-u}, x_s) - (\partial((x_u)^{dd})_{t-u}, x_t)|_{C^1 \times C^1} \\
&\qquad \left(Lip\left(\frac{1}{|(\partial((v_u)^{d})_{t-u}, v_t)|_{C \times C}} \partial((v_u)^{d})_{t-u} \right) + Lip(\partial\partial((x_u)^{dd})_{s-u}) + 1 \right).
\end{aligned}$$

Multiplication with

$$|(\partial((v_u)^{d})_{t-u}, v_t)|_{C \times C} = |\partial((v_u)^{d})_{t-u}| + |v_t| \le |\partial v_u| + |v_t| \le 2c_* L$$

yields

$$\begin{aligned}
&|[D_e g_1(\partial((x_u)^{dd})_{s-u}, x_s) - D_e g_1(\partial((x_u)^{dd})_{t-u}, x_t)](\partial((v_u)^{d})_{t-u}, v_t)| \\
&\quad \le c|(\partial((x_u)^{dd})_{s-u}, x_s) - (\partial((x_u)^{dd})_{t-u}, x_t)|_{C^1 \times C^1} (Lip(\partial((v_u)^{d})_{t-u}) \\
&\qquad + 2c_* L(Lip(\partial\partial((x_u)^{dd})_{s-u}) + 1)),
\end{aligned}$$

from which the estimate (31) follows. □

Proposition 9.2. *Let $\psi \in X_{2*}$ with $Lip(\partial\partial\psi) < \infty$ be given.*

(i) Then $\partial\psi \in T_{e,\psi}X_2$, and $(x^{\psi})'' = (v^{\psi,\partial\psi})'$ is locally Lipschitz continuous.

(ii) For every $T \in (0,t_\psi)$ and for every $L \geq 0$ there exists $c = c(\psi,T,L) \geq 0$ such that for each $\chi \in T_{e,\psi}X_2$ with

$$|\chi|_1 + Lip(\partial\chi) \leq L$$

we have

$$Lip((v^{\psi,\chi})'|[0,T]) \;\leq\; c.$$

Proof. 1. Let $\psi \in X_{2*}$ be given and set $x = x^\psi$. The definition of X_{2*} yields $\partial\psi \in T_{e,\psi}X_2$, and from $x_t \in X_{2*}$ for $0 \leq t < t_\psi$ we get $x' = v^{\psi,\partial\psi}$. Assume in addition $Lip(\partial\partial\psi) < \infty$. Choose $L_* \geq 0$ with

$$|\partial\psi|_1 + Lip(\partial\partial\psi) \leq L_* < \infty.$$

Let $L \geq L_*$ and $0 < T < t_\psi$. Choose $\Delta \in (0,h) \cap (0,T]$ and $c \geq 0$ according to Proposition 9.1. Choose $J \in \mathbb{N}$ with $\frac{T}{J} \leq \Delta$. The set

$$M = \left\{(u,s) \in [0,T] \times [0,T] : 0 \leq u \leq s \leq u + \frac{T}{J}\right\}$$

is compact, and the map

$$A : M \ni (u,s) \mapsto (\partial((x_u)^{dd})_{s-u}, x_s) \in C^1 \times C^1$$

is continuous, due to Proposition 2.4 (iii) and to the continuity of G_2. Proposition 9.1 shows that the compact set $A(M)$ belongs to W_1. Due to Proposition 2.6 there exists $c_e \geq 0$ with

$$|D_e g_1(\partial((x_u)^{dd})_{s-u}, x_s)|_{L_c(C\times C),\mathbb{R}^n)} \leq c_e \text{ for all } (u,s) \in M.$$

Choose a constant $c_T \geq 0$ according to Proposition 7.2, so that

$$|v_t^{\psi,\chi}|_1 \leq c_T|\chi|_1 \text{ for all } \chi \in T_{e,\psi}X_2 \text{ and } t \in [0,T].$$

We may assume that in addition we have

$$\max_{-h\leq w\leq T}|x'(w)| + \max_{-h\leq w\leq T}|x''(w)| \leq c_T.$$

2. Now consider $\chi \in T_{e,\psi}X_2$ with $|\chi|_1 + Lip(\partial\chi) \leq L$. Notice that the following applies in particular to $\chi = \partial\psi$. Let $v = v^{\psi,\chi}$. For

$$k \in \{1,\ldots,J\},\; u = (k-1)\frac{T}{J}, \text{ and } s \leq t \text{ in } \left[(k-1)\frac{T}{J}, k\frac{T}{J}\right] \subset [0,T]$$

we obtain

$$|v'(s) - v'(t)| = |D_e g_1(\partial x_s, x_s)(\partial v_s, v_s) - D_e g_1(\partial x_t, x_t)(\partial v_t, v_t)|$$

$$= |D_e g_1(\partial((x_u)^{dd})_{s-u}, x_s)(\partial((v_u)^d)_{s-u}, v_s) - D_e g_1(\partial((x_u)^{dd})_{t-u}, x_t)(\partial((v_u)^d)_{t-u}, v_t)|$$

(see Eq. (30))

$$\le |D_e g_1(\partial((x_u)^{dd})_{s-u}, x_s)[(\partial((v_u)^d)_{s-u}, v_s) - (\partial((v_u)^d)_{t-u}, v_t)]|$$

$$+|[D_e g_1(\partial((x_u)^{dd})_{s-u}, x_s) - D_e g_1(\partial((x_u)^{dd})_{t-u}, x_t)](\partial((v_u)^d)_{t-u}, v_t)|$$

$$\le c_e(|\partial((v_u)^d)_{s-u} - \partial((v_u)^d)_{t-u}| + |v_s - v_t|)$$

$$+c|(\partial((x_u)^{dd})_{s-u}, x_s) - (\partial((x_u)^{dd})_{t-u}, x_t)|_{C^1 \times C^1}$$

$$(Lip(\partial((v_u)^d)_{t-u}) + Lip(\partial\partial((x_u)^{dd})_{s-u}) + 1)$$

(with (31))

$$\le c_e(Lip(((v_u)^d)')|s-t| + \max_{-h \le w \le T} |v'(w)||s-t|)$$

$$+c(|\partial((x_u)^{dd})_{s-u} - \partial((x_u)^{dd})_{t-u}|_1 + |x_s - x_t|_1)(Lip(\partial((v_u)^d)_{t-u}) + Lip(\partial\partial x_u) + 1)$$

(with Proposition 2.4 (iii))

$$\le c_e(Lip(\partial v_u) + c_T L)|s-t|$$

$$+c(Lip(((x_u)^{dd})')|s-t| + Lip(((x_u)^{dd})'')|s-t|$$

$$+ \max_{-h \le w \le T} |x'(w)||s-t| + \max_{-h \le w \le T} |x''(w)||s-t|)(Lip(\partial v_u) + Lip(\partial\partial x_u) + 1)$$

(with Proposition 2.4 (ii))

$$\le |s-t|(c_e Lip(\partial v_u) + c_e c_T L + c(c_T L + Lip(\partial\partial x_u) + c_T))(Lip(\partial v_u) + Lip(\partial\partial x_u) + 1)$$

(with

$$Lip(((x_u)^{dd})') \le \sup_{-h \le w} |((x_u)^{dd})''(w)| \le |\partial\partial x_u| \le c_T$$

and

$$Lip(((x_u)^{dd})'') \le Lip(\partial\partial x_u).)$$

It follows that

$$Lip\left(v'|\left[(k-1)\frac{T}{J}, k\frac{T}{J}\right]\right) \le c_e Lip(\partial v_u) + c_e c_T L + c(2c_T + Lip(\partial\partial x_u))$$

$$(Lip(\partial v_u) + Lip(\partial\partial x_u) + 1).$$

3. The previous estimate, the relations $Lip(\partial\partial\,\psi) \le L_* \le L$ and $Lip(\partial\,\chi) \le L$, and an induction argument combined yield both assertions (ii)and (i). □

Corollary 9.3. *Let $\psi \in X_{2*}$ with $Lip(\partial\partial\,\psi) < \infty$ be given, and let $0 < T < t_\psi$, $L \ge 0$. Then the closure of the set*

$$M_{\psi,T,L} = \{(\partial\,x_t^\psi, x_t^\psi, \partial\,v_t^{\psi,\chi}, v_t^{\psi,\chi}) \in C^1 \times C^1 \times C \times C : \\ 0 \le t \le T,\ \chi \in T_{e,\psi}X_2, |\chi|_1 + Lip(\partial\,\chi) \le L\}$$

in $C^1 \times C^1 \times C \times C$ is compact and contained in $W_1 \times C \times C$.

Proof. Let ψ, T, L be given as in the corollary. Set $x = x^\psi$. Due to the continuity of the map $[0,T] \ni t \mapsto x_t \in C^2$ the set

$$M_{\psi,T} = \{(\partial\,x_t, x_t) \in C^1 \times C^1 : 0 \le t \le T\} \subset W_1$$

is a compact subset of $C^1 \times C^1$. We have

$$M_{\psi,T,L} \subset M_{\psi,T} \times M_0 \times M_1$$

where

$$M_0 = \{v_t^{\psi,\chi} \in C : 0 \le t \le T, \chi \in T_{e,\psi}X_2, |\chi|_1 + Lip(\partial\,\chi) \le L\}$$

and

$$M_1 = \{\partial\,v_t^{\psi,\chi} \in C : 0 \le t \le T, \chi \in T_{e,\psi}X_2, |\chi|_1 + Lip(\partial\,\chi) \le L\}.$$

In order to complete the proof we show that both sets M_0 and M_1 have compact closure in C. Proposition 7.2 implies that M_0 and M_1 are bounded subsets of C. The boundedness of M_1 yields that M_0 is equicontinuous. Proposition 9.2 yields that also M_1 is equicontinuous. The Theorem of Ascoli-Arzèla then guarantees that M_0 and M_1 have compact closures in C. □

10 Points of Continuity of Derivatives

This section deals with continuity of the derivative obtained in Proposition 8.1. For a map $g : C \times C^1 \supset W \to \mathbb{R}^n$, $W \subset C \times C^1$ open, which satisfies (g0)–(g5) we have the following result.

Proposition 10.1. *Let $t > 0$ and let a continuously differentiable map $p : Q \to C^2$, Q an open subset of a Banach space B, be given, with $p(Q) \subset \Omega_{2,t}$. The map*

$$D(j \circ G_{2,t} \circ p) : Q \to L_c(B, C^1)$$

is continuous at all points $q \in Q$ with $Lip(\partial\partial\,p(q)) < \infty$.

Proof. 1. Let t, B, Q, p, q be given as in the proposition. Set $\psi = p(q)$, $x = x^{\psi}$. In the sequel we use the following abbreviations: For $b \in B$ with $|b| \le 1$, $\chi = Dp(q)b$ and $v = v^{\psi,\chi}$. For $\overline{q} \in Q$, $\overline{\psi} = p(\overline{q})$ and $\overline{x} = x^{\overline{\psi}}$, and for $b \in B$ with $|b| \le 1$, $\overline{\chi} = Dp(\overline{q})b$ and $\overline{v} = v^{\overline{\psi},\overline{\chi}}$. Then

$$|[D(j \circ G_{2,t} \circ p)(\overline{q}) - D(j \circ G_{2,t} \circ p)(q)]b|_1 = |\overline{v}_t - v_t|_1,$$

and we have to show that for every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $b \in B$ with $|b| \le 1$ and for all $\overline{q} \in Q$ with $|\overline{q} - q| \le \delta$ we have

$$|\overline{v}_t - v_t|_1 \le \varepsilon.$$

2. For every $b \in B$ with $|b| \le 1$ we have $|\chi|_2 \le |Dp(q)|_{L_c(B,C^2)}$, and consequently

$$|\chi|_1 + Lip(\partial \chi) \le |\chi|_1 + |\partial\partial\chi| = |\chi|_2 \le |Dp(q)|_{L_c(B,C^2)}.$$

This estimate and Corollary 9.3 combined yield that the closure of the set

$$M = \{(\partial x_s, x_s, \partial v_s, v_s) \in C^1 \times C^1 \times C \times C : 0 \le s \le t, v = v^{\psi,\chi}, \chi = Dp(q)b, |b| \le 1\}$$

in $C^1 \times C^1 \times C \times C$ is compact and contained in $W_1 \times C \times C$. Let $\varepsilon > 0$. As the map

$$W_1 \times C \times C \ni (\hat{\phi}, \hat{\psi}, \hat{\chi}, \hat{\rho}) \mapsto D_e g_1(\hat{\phi}, \hat{\psi})(\hat{\chi}, \hat{\rho}) \in \mathbb{R}^n$$

is uniformly continuous on $cl\,M$ there exists $\delta > 0$ so that for all $(\hat{\phi}, \hat{\psi}, \hat{\chi}, \hat{\rho}) \in M$ and for all $(\phi^*, \psi^*, \chi^*, \rho^*) \in W_1 \times C \times C$ with

$$|(\phi^*, \psi^*, \chi^*, \rho^*) - (\hat{\phi}, \hat{\psi}, \hat{\chi}, \hat{\rho})|_{C^1 \times C^1 \times C \times C} \le \delta$$

we have

$$|D_e g_1(\phi^*, \psi^*)(\chi^*, \rho^*) - D_e g_1(\hat{\phi}, \hat{\psi})(\hat{\chi}, \hat{\rho})| \le \varepsilon.$$

The continuity of G_2 and a compactness argument show that, given $\delta > 0$, there exists $\delta_* > 0$ so that for all $\psi^* \in X_{2*}$ with $|\psi^* - \psi|_2 < \delta_*$ we have $t < t_{\psi^*}$ and for all $s \in [0, t]$, with $x^* = x^{\psi^*}$, $(\partial x_s^*, x_s^*) \in W_1$ and

$$|(\partial x_s^*, x_s^*) - (\partial x_s, x_s)|_{C^1 \times C^1} \le \delta.$$

For such ψ^* and s and for all $b \in B$ with $|b| \le 1$ we obtain

$$|[D_e g_1(\partial x_s^*, x_s^*) - D_e g_1(\partial x_s, x_s)](\partial v_s, v_s)| \le \varepsilon.$$

Next, there exists $\delta_{**} > 0$ such that for all $\overline{q} \in Q$ with $|\overline{q} - q| < \delta_{**}$ we have

$$|\overline{\psi} - \psi|_2 < \delta_*.$$

For such $\overline{q}$, for all $s \in [0,t]$, and for all $b \in B$ with $|b| \le 1$ we arrive at

$$|[D_e g_1(\partial \overline{x}_s, \overline{x}_s) - D_e g_1(\partial x_s, x_s)](\partial v_s, v_s)| \le \varepsilon.$$

3. The subset

$$K = \{(\partial x_s, x_s) \in C^1 \times C^1 : 0 \le s \le t\}$$

of W_1 is compact in $C^1 \times C^1$. Choose $\Delta \in (0,h)$, a neighbourhood $N_1 \subset W_1$ of K in $C^1 \times C^1$, and $r > 0$ according to Corollary 2.8. The continuity of G_2 and a compactness argument show that there exists $\varepsilon_0 > 0$ so that for all $\psi^* \in X_{2*}$ with $|\psi^* - \psi|_2 < \varepsilon_0$ we have $t < t_{\psi^*}$ and for all $s \in [0,t]$, with $x^* = x^{\psi^*}$, $(\partial x_s^*, x_s^*) \in N_1$. Using Corollary 2.8 we infer

$$D_e g_1(\partial x_s^*, x_s^*)(\rho_1, \eta) = D_e g_1(\partial x_s^*, x_s^*)(\rho_2, \eta)$$

for all $\psi^* \in X_{2*}$ with $|\psi^* - \psi|_2 < \varepsilon_0$, all $s \in [0,t]$, and all ρ_1, ρ_2, η in C with $\rho_1(u) = \rho_2(u)$ for $-h \le u \le -\Delta$.

4. Choose $\delta_0 > 0$ so that for all $\overline{q} \in B$ with $|\overline{q} - q| \le \delta_0$ we have $\overline{q} \in Q$ and $|p(\overline{q}) - p(q)|_2 < \varepsilon_0$. In addition we may assume that there is $c \ge 1$ so that

$$|D_e g_1(\partial \overline{x}_s, \overline{x}_s)|_{L_c(C \times C, \mathbb{R}^n)} \le c$$

for all $\overline{q} \in B$ with $|\overline{q} - q| \le \delta_0$ and all $s \in [0,t]$, because $D_e g_1$ is locally bounded (Proposition 2.6), and G_2 and p are continuous, and $[0,t]$ is compact. For $0 < \delta \le \delta_0$ we set

$$a_\delta = \sup_{|b| \le 1, |\overline{q} - q| \le \delta, 0 \le s \le t} |[D_e g_1(\partial \overline{x}_s, \overline{x}_s) - D_e g_1(\partial x_s, x_s)](\partial v_s, v_s)| \le \infty.$$

Part 2 yields

$$a_\delta \to 0 \text{ as } \delta \searrow 0.$$

Choose an integer $N > 0$ with $\frac{t}{N} \le \Delta$ and set $s_\nu = \nu \frac{t}{N}$ for $\nu = 0, 1, \ldots, N$. In the sequel we derive an estimate of $|\overline{v}_{s_\nu} - v_{s_\nu}|_1$ by a linear combination of $|\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1$ and a_δ, for $\overline{q} \in B$ with $|\overline{q} - q| \le \delta \le \delta_0$ and for $b \in B$ with $|b| \le 1$.

4.1. We begin with estimates of $|\overline{v}'(s) - v'(s)|$ for such $\overline{q}$ and b, and for

$$s_{\nu-1} - h \le s \le s_\nu \text{ with } \nu \in \{1, \ldots, N\}.$$

In case $s_{\nu-1} - h \le s \le s_{\nu-1}$ we have

$$|\overline{v}'(s) - v'(s)| \le |\partial \overline{v}_{s_{\nu-1}} - \partial v_{s_{\nu-1}}|. \tag{32}$$

In case $s_{\nu-1} < s \leq s_\nu$ we have

$$\begin{aligned} |\overline{v}'(s) - v'(s)| &= |D_e g_1(\partial \overline{x}_s, \overline{x}_s)(\partial \overline{v}_s, \overline{v}_s) - D_e g_1(\partial x_s, x_s)(\partial v_s, v_s)| \\ &\leq |D_e g_1(\partial \overline{x}_s, \overline{x}_s)[(\partial \overline{v}_s, \overline{v}_s) - (\partial v_s, v_s)]| \\ &\quad + |[D_e g_1(\partial \overline{x}_s, \overline{x}_s) - D_e g_1(\partial x_s, x_s)](\partial v_s, v_s)|. \end{aligned} \tag{33}$$

For each $u \in [-h, -\Delta]$,

$$s + u \leq s_{\nu-1},$$

hence $s - s_{\nu-1} + u \leq 0$, and thereby

$$\partial \overline{v}_s(u) = \overline{v}'(s+u) = \overline{v}'(s_{\nu-1} + s - s_{\nu-1} + u) = \partial(((\overline{v}_{s_{\nu-1}})^d)_{s-s_{\nu-1}})(u).$$

The preceding equation holds in particular for v in place of $\overline{v}$. Using both equations and part 3 we infer that the term in (33) equals

$$\begin{aligned} &|D_e g_1(\partial \overline{x}_s, \overline{x}_s)[(\partial(((\overline{v}_{s_{\nu-1}})^d)_{s-s_{\nu-1}}), \overline{v}_s) - (\partial(((v_{s_{\nu-1}})^d)_{s-s_{\nu-1}}), v_s)]| \\ &\quad + |[D_e g_1(\partial \overline{x}_s, \overline{x}_s) - D_e g_1(\partial x_s, x_s)](\partial v_s, v_s)| \\ &\leq c(|\partial(((\overline{v}_{s_{\nu-1}})^d)_{s-s_{\nu-1}}) - \partial(((v_{s_{\nu-1}})^d)_{s-s_{\nu-1}})| + |\overline{v}_s - v_s|) + a_\delta \\ &\leq c(|\partial \overline{v}_{s_{\nu-1}} - \partial v_{s_{\nu-1}}| + |\overline{v}_s - v_s|) + a_\delta \end{aligned}$$

(see Proposition 2.4). Consider the term $|\overline{v}_s - v_s|$. For $-h \leq u \leq 0$ and $s + u \leq s_{\nu-1}$ we have

$$|(\overline{v}_s - v_s)(u)| \leq |\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|$$

while in case $s_{\nu-1} \leq s + u$,

$$\begin{aligned} |(\overline{v}_s - v_s)(u)| &\leq |\overline{v}(s_{\nu-1}) - v(s_{\nu-1})| + \int_{s_{\nu-1}}^{s+u} |\overline{v}'(w) - v'(w)| dw \\ &\leq |\overline{v}(s_{\nu-1}) - v(s_{\nu-1})| + \int_{s_{\nu-1}}^{s} |\overline{v}'(w) - v'(w)| dw. \end{aligned}$$

Considering both cases we infer

$$|\overline{v}_s - v_s| \leq |\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}| + \int_{s_{\nu-1}}^{s} |\overline{v}'(w) - v'(w)| dw.$$

We return to the estimate of $|\overline{v}'(s) - v'(s)|$ and obtain

$$|\overline{v}'(s) - v'(s)| \leq c(|\partial \overline{v}_{s_{\nu-1}} - \partial v_{s_{\nu-1}}| + |\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}| + \int_{s_{\nu-1}}^{s} |\overline{v}'(w) - v'(w)| dw) + a_\delta$$

$$= c|\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1 + a_\delta + c \int_{s_{\nu-1}}^{s} |\overline{v}'(w) - v'(w)| dw.$$

Now Gronwall's lemma yields, for $s_{\nu-1} \le s \le s_\nu$,

$$|\overline{v}'(s) - v'(s)| \le (c|\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1 + a_\delta)e^{c(s-s_{\nu-1})}$$
$$\le (c|\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1 + a_\delta)e^{c\frac{t}{N}}.$$

4.2. Using the preceding estimate and (32) and $1 \le c$ we arrive at

$$|\partial\,\overline{v}_{s_\nu} - \partial\, v_{s_\nu}| \le (c|\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1 + a_\delta)e^{c\frac{t}{N}}. \tag{34}$$

For $-h \le u \le 0$ and $s_{\nu-1} \le s_\nu + u$ the estimate (34) and integration combined yield

$$|\overline{v}_{s_\nu}(u) - v_{s_\nu}(u)| \le |\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}| + \frac{t}{N}(c|\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1 + a_\delta)e^{c\frac{t}{N}}.$$

This holds for $s_\nu + u < s_{\nu-1}$ as well. It follows that

$$|\overline{v}_{s_\nu} - v_{s_\nu}| \le |\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}| + \frac{t}{N}(c|\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1 + a_\delta)e^{c\frac{t}{N}},$$

and addition of (34) yields

$$|\overline{v}_{s_\nu} - v_{s_\nu}|_1 \le |\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1 \left(1 + \frac{ct}{N}e^{\frac{ct}{N}} + c\,e^{\frac{ct}{N}}\right) + a_\delta \left(\frac{t}{N}e^{\frac{ct}{N}} + e^{\frac{ct}{N}}\right).$$

With

$$c_0 = \left(1 + \frac{ct}{N}e^{\frac{ct}{N}} + c\,e^{\frac{ct}{N}}\right) > 1$$

we get

$$|\overline{v}_{s_\nu} - v_{s_\nu}|_1 \le c_0|\overline{v}_{s_{\nu-1}} - v_{s_{\nu-1}}|_1 + c_0 a_\delta \tag{35}$$

for all $\nu \in \{1,\dots,N\}$, $\delta \in (0,\delta_0]$, $\overline{q} \in B$ with $|\overline{q} - q| \le \delta$, and $b \in B$ with $|b| \le 1$. For such δ, $\overline{q}$, and b it follows by iteration that

$$|\overline{v}_t - v_t|_1 = |\overline{v}_{s_N} - v_{s_N}|_1 \le c_0^N|\overline{v}_0 - v_0|_1 + a_\delta \sum_{\nu=1}^{N} c_0^\nu$$
$$= c_0^N|[Dp(\overline{q}) - Dp(q)]b|_1 + a_\delta \sum_{\nu=1}^{N} c_0^\nu$$
$$\le c_0^N|Dp(\overline{q}) - Dp(q)|_{L_c(B,C^2)} + a_\delta \sum_{\nu=1}^{N} c_0^\nu,$$

and it becomes obvious that given $\varepsilon > 0$ there exists $\delta \in (0, \delta_0]$ so that for all $\overline{q} \in B$ with $|\overline{q} - q| \leq \delta$ we have

$$\begin{aligned} \varepsilon &\geq \sup_{|b|\leq 1} |\overline{v}_t - v_t|_1 \\ &= \sup_{|b|\leq 1} |[D(j \circ G_{2,t} \circ p)(\overline{q}) - D(j \circ G_{2,t} \circ p)(q)]b|_1 \\ &= |D(j \circ G_{2,t} \circ p)(\overline{q}) - D(j \circ G_{2,t} \circ p)(q)|_{L_c(B,C^1)}. \end{aligned}$$ □

11 Appendix on Inhomogeneous Linear Delay Differential Equations

Let $T > 0$ and a continuous map $L : [0,T] \times C \to \mathbb{R}^n$ be given, with all maps $L(t,\cdot) : C \ni \phi \mapsto L(t,\phi) \in \mathbb{R}^n$, $0 \leq t \leq T$, linear. Let $f : [0,T] \to \mathbb{R}^n$ be continuous. A solution to the IVP

$$y'(t) = L(t, y_t) + f(t), \tag{36}$$

$$y_0 = \phi \in C, \tag{37}$$

is a continuous map $y : [-h,T] \to \mathbb{R}^n$ with $y_0 = \phi$ whose restriction to $(0,T]$ is differentiable and satisfies Eq. (36) for $0 < t \leq T$. We also consider solutions on smaller intervals $[-h,t]$ and $[-h,t)$ with $0 < t \leq T$. It is easy to see that each solution has a right derivative at $t = 0$, and that Eq. (36) holds at $t = 0$ with this right derivative.

Notice that the assumptions on L are slightly weaker than those in the chapters of the monographs [3,9] which deal with the IVP (36), (37). In the sequel we construct solutions. Similar results for the homogeneous case are contained in [23].

Proposition 11.1. *There exists $c \geq 0$ with $|L(t,\phi)| \leq c|\phi|$ for $0 \leq t \leq T$ and $\phi \in C$.*

Proof. For every $t \in [0,T]$, $L(t,0) = 0$, and continuity implies that there exists $\varepsilon_t > 0$ with $|L(t,\phi)| \leq 1$ for all $\phi \in C$ with $|\phi| < \varepsilon_t$. A compactness argument yields $\varepsilon > 0$ with $|L(t,\phi)| \leq 1$ for all $t \in [0,T]$ and for all $\phi \in C$ with $|\phi| < \varepsilon$. Set $c = \frac{1}{\varepsilon}$ and use

$$\left| L\left(t, \frac{\varepsilon}{|\phi|}\phi\right)\right| \leq 1 \text{ for } 0 \neq \phi \in C.$$ □

Let $t_0 = \min\left\{\frac{1}{2c}, T\right\}$.

Proposition 11.2. *For every $\phi \in C$ there is a solution $y : [-h,t_0] \to \mathbb{R}^n$ of the IVP (36), (37).*

Proof. Let C_0 denote the Banach space of continuous maps $[-h,t_0] \to \mathbb{R}^n$, with the norm given by $|y|_0 = \max_{-h \leq t \leq t_0} |y(t)|$. Let $\phi \in C$ be given. The subset $M = \{y \in$

$C_0 : y_0 = \phi\} \neq \emptyset$ is closed. For $y \in M$, define a map $A(y) : [-h,t_0] \to \mathbb{R}^n$ by

$$A(y)(t) = \phi(t) \text{ for } -h \le t \le 0$$
$$A(y)(t) = \phi(0) + \int_0^t (L(s,y_s) + f(s))ds \text{ for } 0 < t \le t_0.$$

As the map $[0,t_0] \ni s \mapsto y_s \in C$ is continuous we get that $A(y)$ is continuous, and $A(y) \in M$. For y,z in M and for all $t \in [0,t_0]$ we have

$$|A(y)(t) - A(z)(t)| \le \int_0^t |L(s,y_s) - L(s,z_s)|ds = \int_0^t |L(s,y_s - z_s)|ds$$
$$\le t\,c \max_{0\le s\le t} |y_s - z_s| \le \frac{1}{2} \max_{-h\le u\le t_0} |y(u) - z(u)| = \frac{1}{2}|y-z|_0,$$

and we see that A defines a contraction M. The fixed point equation

$$y(t) = A(y)(t) = \phi(0) + \int_0^t (L(s,y_s) + f(s))ds \text{ for } 0 < t \le t_0$$

shows that its fixed point is differentiable and satisfies $y'(t) = L(t,y_t)$ for $0 < t \le t_0$. □

Proposition 11.3. *Let $\phi \in C$ be given. Any two solutions of the IVP (36), (37) coincide on the intersection of their domains of definition.*

Proof. 1. Suppose $y : [-h,t) \to \mathbb{R}^n$, $0 < t \le T$, and $z : [-h,t) \to \mathbb{R}^n$ are solutions of Eq. (36) with $y_0 = \phi = z_0$, and $y(s) \neq z(s)$ for some $s \in (0,t)$. For

$$t_i = \inf\{u \in (0,t) : y(u) \neq z(u)\}$$

we have $0 \le t_i < s$. By continuity, $y(u) = z(u)$ for $-h \le u \le t_i$. For $t_i < u < \min\left\{t_i + \frac{1}{2c}, s\right\}$ we deduce

$$|y(u) - z(u)| = \left|\int_{t_i}^u L(v, y_v - z_v)dv\right|$$
$$\le (u - t_i)c \max_{t_i\le v\le u} |y_v - z_v| \le (u - t_i)c \max_{-h\le v\le u} |y(v) - z(v)|$$
$$= (u - t_i)c \max_{t_i\le v\le u} |y(v) - z(v)| \le \frac{1}{2} \max_{t_i\le v\le u} |y(v) - z(v)|,$$

which yields $y(u) - z(u) = 0$ for $t_i < u < \min\left\{t_i + \frac{1}{2c}, s\right\}$. This implies a contradiction to the definition of t_i.

2. The proof in case the intersection of the domains of y and z is a compact interval is the same. □

For $\phi \in C$ we set

$$t_\phi = \sup\{t \in (0,T] : \text{There is a solution } y^t : [-h,t] \to \mathbb{R}^n \text{ of the IVP (36) and (37)}\}$$

and define $y^{\phi *} : [-h,t_\phi) \to \mathbb{R}^n$ by $y^{\phi *}(s) = y^t(s)$ with $s < t < t_\phi$. Then $y^{\phi *}$ is a solution of the IVP (36) and (37) so that any other solution of the same IVP is a restriction of $y^{\phi *}$.

Proposition 11.4. *For every $\phi \in C$ there is a unique solution $y : [-h,T] \to \mathbb{R}^n$ of the IVP (36) and (37). In case $f(t) = 0$ for $0 \le t \le T$ we have*

$$|y_t| \le |\phi| e^{ct} \textit{ for all } t \in [0,T].$$

Proof. Let $m = \max_{0 \le t \le T} |f(t)|$. Let $\phi \in C$ be given and set $y = y^{\phi *}$. For $0 \le t < t_\phi$ integration of Eq. (36) yields

$$|y(t)| \le |\phi(0)| + \int_0^t |L(s,y_s) + f(s)| ds \le |\phi(0)| + c \int_0^t |y_s| ds + mT.$$

Using this and the estimate $|y(t+u)| \le |\phi|$ for $-h \le t+u \le 0$ we infer

$$|y_t| \le |\phi| + mT + c \int_0^t |y_s| ds \text{ for } 0 \le t < t_\phi,$$

and Gronwall's lemma yields

$$|y_t| \le (|\phi| + mT) e^{ct} \text{ for } 0 \le t < t_\phi.$$

Using Eq. (36) we deduce

$$|y'(t)| \le c(|\phi| + mT) e^{cT} + m \text{ for } 0 \le t < t_\phi.$$

This implies Lipschitz continuity of $y|[0,t_\phi)$. It follows that y has a continuation y^ϕ to $[-h,t_\phi]$ which is Lipschitz continuous. The continuity of L and f and Eq. (36) combined then show that also y' has a limit at $t = t_\phi$. It follows that y^ϕ is differentiable for $0 < t \le t_\phi$ and satisfies Eq. (36) for these t. Proof of $t_\phi = T$: Suppose $t_\phi < T$. Consider $L_1 : [0, T - t_\phi] \times C \to \mathbb{R}^n$ given by $L_1(t,\chi) = L(t_\phi + t, \chi)$ and $f_1 : [0, T - t_\phi] \to \mathbb{R}^n$ given by $f_1(t) = f(t_\phi + t)$. There is a solution $z : [-h, \varepsilon] \to \mathbb{R}^n$, $\varepsilon > 0$, of the IVP

$$z'(t) = L_1(t, z_t) + f_1(t), \; z_0 = y^\phi_{t_\phi}.$$

Then the equation $y^*(t) = z(t - t_\phi)$ for $t_\phi \le t < t_\phi + \varepsilon$ defines a continuation $y^* : [-h, t_\phi + \varepsilon] \to \mathbb{R}^n$ of $y = y^{\phi *}$, which contradicts the fact that any solution of the IVP (36), (37) is a restriction of $y^{\phi *}$.

Finally, in case $m = 0$ continuity implies that the desired estimate is valid for all $t \in [0, t_\phi] = [0, T]$. $\square$

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Threshold Dynamics of Scalar Linear Periodic Delay-Differential Equations

Yuming Chen and Jianhong Wu

This paper is dedicated to Professor George Sell on the occasion of his 70th birthday.

Abstract We consider the scalar linear periodic delay-differential equation $\dot{x}(t) = -x(t) + ag(t)x(t-1)$, where $g : [0,\infty) \to (0,\infty)$ is continuous and periodic with the minimal period $\omega > 0$. We show that there exists a positive a^+ such that the zero solution is stable if $a \in (0, a^+)$ and unstable if $a > a^+$. Examples and preliminary analysis suggest the challenge in obtaining analogous results when $a < 0$.

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1 Introduction

In [1], Chen et al. studied the issue of desynchronization in large-scale delayed neural networks by considering a ring network of identical neurons described by the system of delay-differential equations

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$$\dot{x}_i(t) = -\mu x_i(t) - \frac{1}{2}[f(x_{i-1}(t-r)) + f(x_{i+1}(t-r))], \tag{1}$$

where $i\,(\mathrm{mod}\, n)$ for a given positive integer n, μ, and r are positive real constants and the activation function $f : \mathbb{R} \to \mathbb{R}$ is bounded and C^1-smooth with $f(0) = 0$ and $f'(x) > 0$ for all $x \in \mathbb{R}$. A solution $x = (x_1, \ldots, x_n)^T : [-r, \infty) \to \mathbb{R}^n$ of (1) is said to be *synchronous* if $x_1(t) = \cdots = x_n(t)$ for all $t \in [-r, \infty)$ and *asynchronous* if otherwise. A necessary and sufficient condition for the stability of a synchronous periodic solution $p^s = (p, \ldots, p)^T : \mathbb{R} \to \mathbb{R}^n$ of (1) is that the zero solution of each scalar linear periodic delay-differential equation

$$\dot{u}(t) = -\mu u(t) + b_k(t)u(t-r) \tag{2}$$

is stable, where $b_k(t) = -f'(p(t-r))\cos\frac{2k\pi}{n}$ and $k = 0,\ 1,\ \ldots,\ n-1$ (see [1, Lemma 1]). It was shown that the zero solution of (2) is unstable when $k = \lfloor \frac{n}{2} \rfloor$ and n is large enough. Therefore, the large-scale and the delayed inhibition combined lead to desynchronization in the considered network of neurons. It is natural to ask the critical size of the size/scale of the network for such a desynchronization to occur. This leads to the issue of the stability of the following linear scalar periodic delay-differential equation:

$$\dot{x}(t) = -x(t) + ag(t)x(t-1), \tag{3}$$

where $g : [0, \infty) \to (0, \infty)$ is continuous and periodic with the minimal period $\omega > 0$. Without loss of generality, we assume $\omega > 1$.

The issue becomes trivial if the delay is absent. Namely, for the periodic ordinary differential equation

$$\dot{x}(t) = -x(t) + ag(t)x(t), \tag{4}$$

the general solution to (4) is

$$x(t) = x(0)\mathrm{e}^{-t + a\int_0^t g(s)\,\mathrm{d}s}.$$

It follows that the zero solution of (4) is asymptotically stable if and only if $-\omega + a\int_0^{\omega} g(s)\,\mathrm{d}s < 0$, or $a < \frac{\omega}{\int_0^{\omega} g(s)\,\mathrm{d}s}$.

Here we are considering if a similar result holds for the periodic delay-differential equation (3). The remaining part of this chapter is organized as follows. In Sect. 2, we introduce the monodromy operator for (3) and the discrete Lyapunov functionals. It is shown that the monodromy operator is continuous in a with respect to the operator norm. Then, we show that, for the case where $a > 0$, (3) has the threshold dynamics. Namely, there exists an $a^+ > 0$ such that the zero solution of (3) is stable if $a \in (0, a^+)$ and unstable if $a > a^+$. This result is an analog of that for (4). As for the case where $a < 0$, our analysis and example in Sect. 4 indicate that an analogous result as that for (4) may not hold. In contract, we expect the threshold dynamics as follows: there exists an $a^- < 0$ such that the zero solution of (4) is stable if $a \in (a^-, 0)$ and unstable if $a < a^-$.

We refer the readers to [3, 8–10] for related studies on the existence and stability of periodic solutions to scalar delay-differential equations.

2 Preliminaries

First, we introduce the monodromy operator.

Let $C = C([-1,0],\mathbb{R})$. Define $\|\phi\| = \max_{\theta\in[-1,0]} |\phi(\theta)|$. Then, $(C, \|\cdot\|)$ is a Banach space, which is taken as the phase space for (3). Obviously, for each $\phi \in C$, there exists a unique continuous mapping $x^{\phi,a} : [-1,\infty) \to \mathbb{R}$ such that $x_0^{\phi,a} = \phi$ and $x^{\phi,a}$ satisfies (3) for $t > 0$, where $x_t^{\phi,a} \in C$ for $t \geq 0$ is defined by $x_t^{\phi,a}(\theta) = x^{\phi,a}(t+\theta)$ for $\theta \in [-1,0]$. In fact, for any $\phi \in C$ and $a \in \mathbb{R}$, it follows from the variation-of-constant formula that

$$x^{\phi,a}(t) = \mathrm{e}^{-t}\phi(0) + a\int_0^t \mathrm{e}^{-(t-s)} g(s) x^{\phi,a}(s-1)\,\mathrm{d}s \qquad \text{for } t \geq 0. \tag{5}$$

The period map $M_a : C \ni \phi \mapsto x_\omega^{\phi,a} \in C$ is called the monodromy operator. It is well known that M_a is linear, continuous, and compact. Let σ_a be the spectrum set of M_a. Note that the spectrum of M_a is the spectrum of its complexification. That is, M_a has been complexified to $C_{\mathbb{C}} = C([-1,0],\mathbb{C})$ by $M_a(\phi) = M_a(\mathrm{Re}(\phi)) + iM_a(\mathrm{Im}(\phi))$ for $\phi \in C_{\mathbb{C}}$. For the sake of simplification, we use the same notation for this complexification. Note that $\|\cdot\|$ is naturally extended to $C_{\mathbb{C}}$ and (5) still holds for $\phi \in C_{\mathbb{C}}$. Every point $\lambda \in \sigma_a \setminus \{0\}$ is an eigenvalue of finite multiplicity and is isolated in σ_a. These eigenvalues in $\sigma_a \setminus \{0\}$ are called Floquet multipliers. We know that the zero solution of (3) is stable if $\sigma_a \subseteq \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and is unstable if $\sigma_a \cap \{\zeta \in \mathbb{C} : |\zeta| > 1\} \neq \emptyset$.

Let $M_g = \max_{t\in[0,\omega]} g(t)$ and $m_g = \min_{t\in[0,\omega]} g(t)$.

Lemma 2.1. *M_a is continuous in a with respect to the operator norm.*

Proof. For given $\overline{a} \in \mathbb{R}$, it follows from (5) that there exists an $M \geq 1$ such that

$$|x^{\phi,a}(t)| \leq M \qquad \text{for } t \in [-1,\omega],\ a \in [\overline{a}-1, \overline{a}+1] \text{ and } \phi \in B_1,$$

where $B_1 = \{\phi \in C_{\mathbb{C}} : \|\phi\| = 1\}$. Thus, for $t \in [0,\omega]$ and $\phi \in B_1$ and a, $b \in [\overline{a}-1, \overline{a}+1]$, we obtain from (5) that

$$\begin{aligned}
&|x^{\phi,a}(t) - x^{\phi,b}(t)| \\
&\quad = \left| a\int_0^t \mathrm{e}^{-(t-s)} g(s) x^{\phi,a}(s-1)\,\mathrm{d}s - b\int_0^t \mathrm{e}^{-(t-s)} g(s) x^{\phi,b}(s-1)\,\mathrm{d}s \right| \\
&\quad \leq \left| (a-b)\int_0^t \mathrm{e}^{-(t-s)} g(s) x^{\phi,a}(s-1)\,\mathrm{d}s \right| \\
&\qquad + |b|\int_0^t \mathrm{e}^{-(t-s)} g(s) |x^{\phi,a}(s-1) - x^{\phi,b}(s-1)|\,\mathrm{d}s \\
&\quad \leq |a-b| M M_g + \overline{a}^* M_g \int_0^t \|x_s^{\phi,a} - x_s^{\phi,b}\|\,\mathrm{d}s,
\end{aligned}$$

where $\overline{a}^* = \max\{|\overline{a}-1|, |\overline{a}+1|\}$. That is,

$$\|x_t^{\phi,a} - x_t^{\phi,b}\| \leq |a-b|MM_g + \overline{a}^* M_g \int_0^t \|x_s^{\phi,a} - x_s^{\phi,b}\| \,\mathrm{d}s \qquad \text{for } t \in [0,\omega].$$

Applying Gronwall's inequality, we obtain

$$\|x_t^{\phi,a} - x_t^{\phi,b}\| \leq |a-b|MM_g \mathrm{e}^{\overline{a}^* M_g t} \qquad \text{for } t \in [0,\omega] \text{ and } \phi \in B_1. \tag{6}$$

Now, for any $\varepsilon > 0$, let $\delta \in \left(0, \min\left\{1, \frac{\varepsilon e^{-\overline{a}^* M_g \omega}}{MM_g}\right\}\right)$. Then, for $a \in \mathbb{R}$ such that $|a - \overline{a}| < \delta$, it follows from (6) that

$$\|M_a(\phi) - M_{\overline{a}}(\phi)\| = \|x_\omega^{\phi,a} - x_\omega^{\phi,\overline{a}}\| < \varepsilon \qquad \text{for } \phi \in B_1.$$

This implies the continuity of M_a at $\overline{a}$ with respect to the operator norm. By the arbitrariness of $\overline{a}$, M_a is continuous in a with respect to the operator norm. This completes the proof.

In the proof of our main results, we need the discrete Lyapunov functional $V^{\pm}$ introduced by Mallet-Paret and Sell [7]. Let $K = \{\phi \in C : \phi(\theta) \geq 0 \text{ for } \theta \in [-1,0]\}$. For $\phi \in C \setminus \{0\}$, define $\mathrm{sc}(\phi)$, the number of sign change of ϕ, as follows. If $\phi \notin K \cup (-K)$, then

$$\mathrm{sc}(\phi) = \sup\left\{k \geq 0 : \begin{array}{l} \text{there exist } \theta^i \in [-1,0] \text{ for } 0 \leq i \leq k \text{ with} \\ \theta^{i-1} < \theta^i \text{ and } \phi(\theta^{i-1})\phi(\theta^i) < 0 \text{ for } 1 \leq i \leq k \end{array}\right\};$$

if $\phi \in K \cup (-K)$, then $\mathrm{sc}(\phi) = 0$. Then, we define $V^+ : C \setminus \{0\} \to \{0, 2, 4, \cdots\}$ by

$$V^+(\phi) = \begin{cases} \mathrm{sc}(\phi) & \text{if } \mathrm{sc}(\phi) \text{ is even or infinity} \\ \mathrm{sc}(\phi) + 1 & \text{if } \mathrm{sc}(\phi) \text{ is odd} \end{cases}$$

and define $V^- : C \setminus \{0\} \to \{1, 3, 5, \cdots\}$ by

$$V^-(\phi) = \begin{cases} \mathrm{sc}(\phi) & \text{if } \mathrm{sc}(\phi) \text{ is odd or infinity,} \\ \mathrm{sc}(\phi) + 1 & \text{if } \mathrm{sc}(\phi) \text{ is even.} \end{cases}$$

When $a > 0$, $V^+(x_t^{\phi,a})$ is nonincreasing in t, and hence V^+ is called a discrete Lyapunov functional for (3) when $a > 0$. Similarly, V^- is called a discrete Lyapunov functional for (3) when $a < 0$. We refer to Mallet-Paret and Sell [7] for more properties of $V^{\pm}$.

3 Threshold Dynamics in the Case Where $a > 0$

First, we show that the zero solution of (3) is stable for small a and unstable for $a > 0$ large enough.

Define $h : [0,\infty) \to \mathbb{R}$ by $h(t) = \frac{e^{-\omega+1}+te^{t\omega+1}}{1+t}$. Then, $h(0) = e^{1-\omega} < 1$ and $h(t) \to \infty$ as $t \to \infty$. Moreover,

$$h'(t) = \frac{(e^{t\omega+1} - e^{-\omega+1}) + t\omega e^{t\omega+1} + t^2\omega e^{t\omega+1}}{(1+t)^2} > 0 \qquad \text{for } t > 0.$$

It follows that there exists a unique $t_0 \in (0,\infty)$ such that

$$h(t) \begin{cases} < 1 & \text{if } t \in (0,t_0), \\ = 1 & \text{if } t = t_0, \\ > 1 & \text{if } t > t_0. \end{cases} \tag{7}$$

Define $a_0 = \frac{t_0}{M_g}$.

Lemma 3.1. *The zero solution of* (3) *is stable for* $a \in (-a_0, a_0)$.

Proof. For given $\phi \in C_{\mathbb{C}}$ and $a \in (-a_0,a_0)$, it follows from (5) that

$$|x^{\phi,a}(t)| \le e^{-t}\|\phi\| + |a|M_g \int_0^t \|x_s^{\phi,a}\|\,ds \qquad \text{for } t \ge 0.$$

Thus,

$$\|x_t^{\phi,a}\| \le e^{-t+1}\|\phi\| + |a|M_g \int_0^t \|x_s^{\phi,a}\|\,ds \qquad \text{for } t \ge 0.$$

Using Gronwall's inequality, we get

$$\begin{aligned}\|x_t^{\phi,a}\| &\le e^{-t+1}\|\phi\| + |a|M_g \int_0^t e^{|a|M_g(t-s)+1-s}\|\phi\|\,ds \\ &= \frac{e^{1-t} + |a|M_g e^{|a|M_g t+1}}{1+|a|M_g}\|\phi\|.\end{aligned}$$

In particular,

$$\|x_\omega^{\phi,a}\| \le h(|a|M_g)\|\phi\|.$$

Note that $|a|M_g < t_0$. It follows from (7) that $h(|a|M_g) < 1$. By definition, $\sigma_a \subseteq \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ for $a \in (-a_0,a_0)$, namely, the zero solution of (3) is stable for $a \in (-a_0,a_0)$. This completes the proof.

Lemma 3.2. *Suppose that* $a > \frac{e^{\omega}-1}{m_g(e-1)}$. *Then, the zero solution of* (3) *is unstable.*

Proof. Let $\phi_0 \equiv 1 \in C$. It is easy to see that if $a > 0$, then $x^{\phi_0,a}(t) > 0$ by (5). It also follows from (5) that

$$\begin{aligned} x^{\phi_0,a}(\omega) &\geq \mathrm{e}^{-\omega} + am_g \int_0^1 \mathrm{e}^{-(\omega-s)}\,\mathrm{d}s \\ &= \mathrm{e}^{-\omega} + am_g[\mathrm{e}^{-\omega-1} - \mathrm{e}^{-\omega}] \\ &= \mathrm{e}^{-\omega}[1 + am_g(\mathrm{e}-1)]. \end{aligned}$$

If $a > \frac{\mathrm{e}^{\omega}-1}{m_g(e-1)}$, then $x^{\phi_0,a}(\omega) > 1$. Note that $\|x_\omega^{\phi_0,a}\| \geq x^{\phi_0,a}(\omega)$. By definition, there exists a $\sigma \in \sigma_a$ such that $|\sigma| > 1$. This implies that the zero solution of (3) is unstable if $a > \frac{\mathrm{e}^{\omega}-1}{m_g(e-1)}$, and hence the proof is complete.

Now, we are ready to prove the main result of this section.

Theorem 3.3. *There exists an $a^+ \in (0,\infty)$ such that the zero solution of* (3) *is stable (respectively, unstable) if $a \in (0,a^+)$ (respectively, $a \in (a^+,\infty)$).*

Proof. We divide the proof into several steps.

Step 1. *There exist $\lambda_a > 0$ and $\phi_a \in \overset{\circ}{K} = \{\phi \in C : \phi(\theta) > 0$ for $\theta \in [-1,0]\}$ such that* $M_a\phi_a = \lambda_a\phi_a$ (see Proposition VII.1 (i) of Krisztin, Walther, and Wu [6]). Obviously, $\lambda_a \in \sigma_a$.

Step 2. *If $\lambda \in \sigma_a$, then $|\lambda| \leq \lambda_a$.* By way of contradiction, assume that there exists a $\lambda^* \in \sigma_a$ such that $|\lambda^*| > \lambda_a$. We distinct two cases to get contradictions. First, we assume that λ^* is real. Let ϕ^* be a real eigenvector associated with λ^*. Then, ϕ^* and ϕ_a are linearly independent since $\lambda^* \neq \lambda_a$. By Theorem 3.1 of Mallet-Paret and Sell [7], $V^+(\phi^*) \leq V^+(\phi_a)$. Note that $V^+(\phi_a) = 0$. Thus, $V^+(\phi^*) = 0$. Then, the realized generalized eigenspace associated with $\{\lambda_a, \lambda^*\}$ is at least two-dimensional, a contradiction to Theorem 3.1 (b) of Mallet-Paret and Sell [7]. Second, we assume that λ^* is complex. Let $u^* + iv^*$ be an eigenvector associated with λ^*, where both u^* and v^* are real. Then, u^* and v^* are linearly independent. Again, we have $V^+(u^*) = V^+(v^*) = 0$, which gives us a contradiction as in the first case. In conclusion, in any case, we have got a contradiction, and hence $|\lambda| \leq \lambda_a$.

Step 3. *λ_a is strictly increasing in a.* The proof given here is similar to that of Lemma 1 in Huang [5]. Let $0 < a_1 < a_2$. Then, $x_\omega^{\phi_1,a_1} = \lambda_{a_1}\phi_{a_1}$. Using (5), we compare $x^{\phi_{a_1},a_1}$ and $x^{\phi_{a_1},a_2}$ consecutively on $[0,1]$, $[1,2]$, ..., to obtain

$$x^{\phi_{a_1},a_2}(t) > x^{\phi_{a_1},a_1}(t) \qquad \text{for } t \in [0,\infty).$$

In particular,

$$x_\omega^{\phi_{a_1},a_2} > x_\omega^{\phi_{a_1},a_1} = \lambda_{a_1}\phi_{a_1},$$

which implies that there exists a $\lambda \in \sigma_{a_2}$ such that $|\lambda| > \lambda_{a_1}$. This, combined with the result of STEP 2, implies that $\lambda_{a_1} < \lambda_{a_2}$. That is, λ_a is strictly increasing in a.

Step 4. Let

$$S = \{\alpha \in (0,\infty) : \text{ the zero solution of (3) is stable for } a \in (0,\alpha)\}.$$

Then, by Lemmas 3.1 and 3.2, $a_0 \in S$ and $S \subseteq \left(0, \frac{e^{\omega}-1}{m_g(\omega-1)}\right]$. Let

$$a^+ = \sup_{\alpha \in S} \alpha.$$

Obviously, $a^+ \in (0,\infty)$. Moreover, $\lambda_{a^+} = 1$ by the definition of a^+ and Lemma 2.1. Then, the theorem follows directly from the result in STEP 3. This completes the proof.

It follows from the proof of Theorem 3.3 that $a^+ \le \frac{\mathrm{e}^{\omega}-1}{m_g(e-1)}$. The following result gives a lower bound of a^+.

Proposition 3.4. $a^+ \ge \frac{1}{M_g}$.

Proof. Let S be defined as in the proof of Theorem 3.3. It suffices to show that $\frac{1}{M_g} \in S$. To achieve this, we first show that the zero solution of

$$\dot{x}(t) = -x(t) + bx(t-1) \tag{8}$$

is asymptotically stable for $b \in \left(-\frac{\sqrt{4+\pi^2}}{2}, 1\right)$. In fact, the zero solution of (8) is asymptotically stable if and only if all the roots of $(\lambda+1)\mathrm{e}^{\lambda} - b = 0$ have negative real parts, and this is true by Theorem A.5 of Hale [4]. For the sake of completeness, we give the detail here. We know that the characteristic of (8) is $(\lambda+1)\mathrm{e}^{\lambda} - b = 0$, all of whose roots having negative real parts if and only if

$$1 - b > 0, \tag{9}$$

$$-b < \rho \sin\rho - \cos\rho, \tag{10}$$

where $\rho \in (0,\pi)$ satisfies $\rho = -\tan\rho$. Since $b < 1$, inequality (9) is obviously true. Note that $\rho \in (\frac{\pi}{2},\pi)$ and

$$\rho \sin\rho - \cos\rho = -\tan\rho \sin\rho - \cos\rho = -\frac{1}{\cos\rho} = \sqrt{1+\tan^2\rho} = \sqrt{1+\rho^2}.$$

It follows that $\rho \sin\rho - \cos\rho \in \left(\frac{\sqrt{4+\pi^2}}{2}, \sqrt{1+\pi^2}\right)$. Since $b > -\frac{\sqrt{4+\pi^2}}{2}$, the inequality (10) also holds. Therefore, for $b \in \left(-\frac{\sqrt{4+\pi^2}}{2}, 1\right)$, the zero solution of (8) is asymptotically stable.

Now, for any $\beta \in (0,1)$, by the variation-of-constant formula, for any $\phi \in C$, the unique solution $x^{\phi} : [-1,\infty) \to \mathbb{R}$ of (8) satisfies

$$x^{\phi}(t) = \mathrm{e}^{-t}\phi(0) + \beta \int_0^t \mathrm{e}^{-(t-s)} x^{\phi}(s-1)\,\mathrm{d}s. \tag{11}$$

Thus, for $a \in \left(0, \frac{\beta}{M_g}\right]$, comparing $x^{\phi_a,a}$ and x^{ϕ_a} consecutively on $[0,1]$, $[1,2]$, ..., we know from (5) and (11) that

$$x^{\phi_a,a}(t) \le x^{\phi}(t) \quad \text{for } t \in [-1,\infty).$$

In particular,

$$0 < x^{\phi_a,a}(n\omega) = \lambda_a^n \phi_a(0) \le x^{\phi_a}(n\omega) \qquad \text{for } n \in \mathbb{N}. \tag{12}$$

Note that $\lim_{t\to\infty} x^{\phi_a}(t) = 0$. This, combined with (12), implies that $\lambda_a < 1$. Therefore, for $a \in \left(0, \frac{\beta}{M_g}\right]$, the zero solution of (3) is stable by the result of STEP 3 in the proof of Theorem 3.3. Since $\beta \in (0,1)$ is arbitrary, we know that for $a \in \left(0, \frac{1}{M_g}\right)$ the zero solution of (3) is stable. This completes the proof.

4 Discussion for the Case Where $a < 0$

In Sect. 3, we obtained the threshold dynamics of (3) in the case where $a > 0$ (see Theorem 3.3). It is natural to ask whether a similar result holds for the case where $a < 0$. We do not have a definite answer here, although results below suggest that threshold dynamics holds for the case where $a < 0$.

The first result indicates that the case where $a < 0$ is *more stable* than the case where $a > 0$.

Proposition 4.1. *Let* $a > 0$. *Then,* $\max\{|\lambda| : \lambda \in \sigma_{-a}\} < \lambda_a$.

Proof. We shall use the following simple *coupling* technique (first used in the work of Chen, Krisztin, and Wu [2] and then used in Chen, Huang, and Wu [1]): $(X,Y) : \mathbb{R} \to \mathbb{C}^2$ is a solution of the decoupled system

$$\begin{cases} \dot{X}(t) = -X(t) - ag(t)X(t-1) \\ \dot{Y}(t) = -Y(t) + ag(t)Y(t-1) \end{cases}$$

if and only if $(U,V) : \mathbb{R} \to \mathbb{C}^2$ given by

$$U(t) = X(t) + Y(t) \quad \text{and} \quad V(t) = -X(t) + Y(t)$$

is a solution of the coupled system

$$\begin{cases} \dot{U}(t) = -U(t) + ag(t)V(t-1), \\ \dot{V}(t) = -V(t) + ag(t)U(t-1). \end{cases} \tag{13}$$

Let $F_a : C([-1,0],\mathbb{C}^2) \to C([-1,0],\mathbb{C}^2)$ be the monodromy operator of (13). Namely,

$$F_a(\psi)(\theta) = (U^\psi(\omega+\theta), V^\psi(\omega+\theta)), \qquad \theta \in [-1,0],\ \psi \in C([-1,0],\mathbb{C}^2),$$

where (U^ψ, V^ψ) is the solution of (13) with $(U^\psi, V^\psi)|_{[-1,0]} = \psi$. Let Σ_a denote the spectrum of F_a. Then,

$$\Sigma_a = \sigma_a \cup \sigma_{-a} \qquad \text{for } a > 0$$

(for a proof see [2, Proposition 4.2]). Moreover, $\max\{|\lambda| : \lambda \in \sigma_{-a}\} < |\lambda_a|$ (the argument is similar to that in Sect. 4 of [2]), and this completes the proof.

In the first section, we mentioned that the zero solution of the periodic ordinary differential equation (4) is always stable for $a < 0$. The following result shows that this is no longer the case for the periodic delay-differential equation (3).

Proposition 4.2. *Let $g(t) = 2 + \cos(\pi t)$. If $a < -\frac{(1+e)(1+\pi^2)}{2\pi^2 e + 2 - 3 - 2\pi^2}$, then the zero solution of (3) is unstable.*

Proof. Take $\phi_0 \equiv 1 \in C$. Then,

$$x^{\phi_0,a}(t) = \mathrm{e}^{-t}\phi_0(0) + a\int_0^t \mathrm{e}^{-(t-s)} g(s) x^{\phi_0,a}(s-1)\,\mathrm{d}s.$$

In particular,

$$\begin{aligned} x^{\phi_0,a}(1) &= \mathrm{e}^{-1} + a\int_0^1 \mathrm{e}^{-(1-s)}(2+\cos(\pi s))\,\mathrm{d}s \\ &= \mathrm{e}^{-1} + a\mathrm{e}^{-1}\frac{2\pi^2\mathrm{e} + \mathrm{e} - 3 - 2\pi^2}{1+\pi^2}. \end{aligned}$$

Since $a < -\frac{(1+\mathrm{e})(1+\pi^2)}{2\pi^2\mathrm{e}+\mathrm{e}-3-2\pi^2)}$, we have $x^{\phi_0,a}(1) < -1$. It follows that

$$\|x_\omega^{\phi_0,a}\| = \|x_2^{\phi_0,a}\| \geq |x^{\phi_0,a}(1)| > 1.$$

This implies that there exists a $\sigma \in \sigma_a$ such that $|\sigma| > 1$, and hence the zero solution of (3) is unstable.

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Differential Equations with Random Delay

S. Siegmund and T.S. Doan

This paper is dedicated to George Sell

Abstract The Multiplicative Ergodic Theorem by Oseledets on Lyapunov spectrum and Oseledets subspaces is extended to linear random differential equations with random delay, using a recent result by Lian and Lu. Random differential equations with bounded delay are discussed as an example.

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1 Introduction

Delays in difference and in differential equations are used for mathematical modeling in many applications for the description of evolutions which incorporate influences of events from the past, e.g. in biological models when traditional pointwise modeling assumptions are replaced by more realistic distributed assumptions.

In contrast to ordinary differential equations the state space of a differential equation with random delay is an infinite dimensional space. As a consequence established tools for the analysis of ordinary differential equations do not directly apply to delay equations. Based on recent work of Lian and Lu [5] the first step

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towards a general theory of difference equations incorporating random delays which are not assumed to be bounded is established in Crauel, Doan and Siegmund [2]. In this paper, we extend this work to differential equations with random delays:

$$\dot{x}(t) = A(\theta_t \omega)x(t) + B(\theta_t \omega)x(t - r(\theta_t \omega)), \tag{1}$$

where $(\theta_t)_{t\in\mathbb{R}}$ is an ergodic flow on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $A(\cdot), B(\cdot)$ and $r(\cdot)$ are random coefficients and delay, respectively.

The paper is organized as follows: in Sect. 2 we introduce a class of exponentially weighted state spaces for (1). After that existence and uniqueness of solutions of initial value problems is proved. We end up this section with the definition of random dynamical systems generated by differential equations with random delay. Section 3 is devoted to show the multiplicative ergodic theorem for random dynamical systems with random delay. The Lyapunov exponents are shown to be independent of the weight factor chosen for the state space. A special case, differential equations with random bounded delay, is investigated in Sect. 4.

To fix notation, for $a < b$ we denote by $\mathrm{C}([a,b],\mathbb{R}^d)$ the Banach space of all continuous functions $\mathbf{f} : [a,b] \to \mathbb{R}^d$ together with sup norm, i.e. $\|\mathbf{f}\| = \sup_{t\in[a,b]} |\mathbf{f}(t)|$. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ denote by $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ the set of all measurable functions $f : \Omega \to \mathbb{R}$ satisfying the integrability condition $\int_\Omega |f(\omega)|\, d\mathbb{P}(\omega) < \infty$. Let $\mathcal{L}^1(\mathbb{P})$ denote the space of all measurable maps $A : \Omega \to \mathbb{R}^{d\times d}$ such that $\|A(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

2 Differential Equations with Random Delay

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\theta_t)_{t\in\mathbb{R}} : \Omega \to \Omega$ an ergodic flow which preserves the probability measure $\mathbb{P}$ and which has a measurable inverse, and let $r : \Omega \to \mathbb{R}^+$ be a measurable map. We consider the random linear differential equation with random delay

$$\dot{x}(t) = A(\theta_t \omega)x(t) + B(\theta_t \omega)x(t - r(\theta_t \omega)) \qquad \text{for } t \geq 0, \tag{2}$$

where $A, B \in \mathcal{L}^1(\mathbb{P})$. In order to introduce a random dynamical system generated by (2) we first need to construct an appropriate state space. Since the delay map r is in general unbounded, an initial value for (2) is a continuous function $\mathbf{x} : (-\infty, 0] \to \mathbb{R}^d$. A corresponding integrated version of (2) for $t \geq 0$ is given by

$$\varphi(t,\omega)\mathbf{x} = \mathbf{x}(0) + \int_0^t A(\theta_s\omega)\varphi(s,\omega)\mathbf{x} + B(\theta_s\omega)\varphi(s - r(\theta_s\omega),\omega)\mathbf{x}\, \mathrm{d}s, \tag{3}$$

with the convention that $\varphi(s,\omega)\mathbf{x} = \mathbf{x}(s)$ for all $s \leq 0$. If (3) holds, we say that $t \mapsto \varphi(t,\omega)\mathbf{x} =: \varphi_\omega(t,\mathbf{x})$ is a *solution of equation* (2) *starting at* 0 *in* $\mathbf{x}$.

Since in the unbounded delay case the initial data is always part of the solution, some kind of regularity must be imposed from the beginning (see, e.g. Hale and Kato [3] and Hino et al. [4]). This leads us to work with a canonical phase space

$$\mathbf{X}_\gamma := \left\{ \mathbf{x} \in \mathrm{C}\big((-\infty,0],\mathbb{R}^d\big) : \lim_{t\to-\infty} \mathrm{e}^{\gamma t}\mathbf{x}(t) \quad \text{exists} \right\},$$

$$\|\mathbf{x}\|_\gamma := \sup_{t\in(-\infty,0]} \mathrm{e}^{\gamma t}|\mathbf{x}(t)|.$$

Throughout this paper we assume $\gamma > 0$ and consider (2) on the state space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$. It is easy to see that $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$ is a Banach space. The following lemma ensures the separability of the space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$.

Lemma 2.1. *For $\gamma > 0$ the space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$ is separable.*

Proof. The set $\mathbb{Q}^d$ of all vectors in $\mathbb{R}^d$ whose components are rational is dense in $\mathbb{R}^d$. For each $N \in \mathbb{N}$ we consider the Banach space $\mathrm{C}([-N,0],\mathbb{R}^d)$ together with the sup norm $\|\cdot\|$, i.e.

$$\|f\|_\infty = \sup_{t\in[-N,0]} |f(t)| \qquad \text{for all } f \in \mathrm{C}([-N,0],\mathbb{R}^d).$$

It is well known that $\mathrm{C}([-N,0],\mathbb{R}^d)$ is a separable Banach space, see e.g. Willard [7]. Consequently, there exists a countable set

$$A_N := \left\{ f_1^{(N)}, f_2^{(N)}, \dots \right\}, \qquad f_1^{(N)}, f_2^{(N)}, \dots \in \mathrm{C}([-N,0],\mathbb{R}^d),$$

which is dense in $(\mathrm{C}([-N,0],\mathbb{R}^d), \|\cdot\|_\infty)$. For each function $f_k^{(N)}$, $v \in \mathbb{Q}^d$ and $p \in \mathbb{Q}^+$ we defined the extended function $\widetilde{f}_{k,v,p}^{(N)} : (-\infty,0] \to \mathbb{R}^d$ by

$$\widetilde{f}_{k,v,p}^{(N)}(t) := \begin{cases} f_k^{(N)}(t), & t \in [-N,0], \\ \left(\frac{t+N}{p}+1\right) f_k^{(N)}(-N) - \frac{N+t}{p}\, e^{(N+p)\gamma} v, & t \in [-N-p,-N), \\ \mathrm{e}^{-\gamma t} v & t \in (-\infty,-N-p). \end{cases} \tag{4}$$

Obviously, for all $k \in \mathbb{N}, v \in \mathbb{Q}^d$ and $p \in \mathbb{Q}^+$ the function $\widetilde{f}_{k,v,p}^{(N)}$ is continuous and

$$\lim_{t\to-\infty} \mathrm{e}^{\gamma t} \widetilde{f}_{k,v,p}^{(N)}(t) = v,$$

which implies that $\widetilde{f}_{k,v,p}^{(N)} \in \mathbf{X}_\gamma$. Define

$$\widetilde{A}_N := \bigcup_{(v,p)\in\mathbb{Q}^d\times\mathbb{Q}^+} \left\{ \widetilde{f}_{1,v,p}^{(N)}, \widetilde{f}_{2,v,p}^{(N)}, \dots \right\} \qquad \text{for all } N \in \mathbb{N}.$$

To prove the separability of the Banach space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$, it is sufficient to show that

$$\bigcup_{N\in\mathbb{N}} \widetilde{A}_N \qquad \text{is dense in } (\mathbf{X}_\gamma, \|\cdot\|_\gamma). \tag{5}$$

For a given $\mathbf{x} \in \mathbf{X}_\gamma$, set $u := \lim_{t\to-\infty} e^{\gamma t}\mathbf{x}(t)$. Hence, for an arbitrary $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|e^{\gamma t}\mathbf{x}(t) - u| \le \frac{\varepsilon}{8} \qquad \text{for all } t \le -N. \tag{6}$$

Since $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$, it follows that there exists $v \in \mathbb{Q}^d$ such that $|v-u| \le \frac{\varepsilon}{8}$. On the other hand, due to the denseness of A_N in the space $C([-N,0],\mathbb{R}^d)$ there exists $k \in \mathbb{N}$ such that

$$\sup_{t\in[-N,0]} |f_k^{(N)}(t) - \mathbf{x}(t)| < \frac{\varepsilon}{8}. \tag{7}$$

Direct estimates yield that

$$\lim_{p\to 0} \sup_{t\in[-N-p,-N]} e^{\gamma t}\left|f_k^{(N)}(-N) - \mathbf{x}(t)\right| = e^{-\gamma N}\left|f_k^{(N)}(-N) - \mathbf{x}(-N)\right| \le e^{-\gamma N}\frac{\varepsilon}{8}$$

and

$$\begin{aligned}
&\lim_{p\to 0} \sup_{t\in[-N-p,-N]} e^{\gamma t}\left|f_k^{(N)}(-N) - e^{(N+p)\gamma}v\right| \\
&= \left|e^{-\gamma N} f_k^{(N)}(-N) - v\right| \\
&\le e^{-\gamma N}\frac{\varepsilon}{8} + \left|e^{-\gamma N}\mathbf{x}(-N) - u\right| + |u-v| \ \le \ \frac{3\varepsilon}{8}.
\end{aligned}$$

As a consequence, there exists $p \in \mathbb{Q}^+$ such that for all $t \in [-N-p,-N]$ we have

$$e^{\gamma t}\left|f_k^{(N)}(-N) - \mathbf{x}(t)\right| \le \frac{\varepsilon}{3}, \quad e^{\gamma t}\left|f_k^{(N)}(-N) - e^{(N+p)\gamma}v\right| \le \frac{\varepsilon}{2}. \tag{8}$$

We now estimate $\|\widetilde{f}_{k,v,p}^{(N)} - \mathbf{x}\|_\gamma$. By (4) and (7)

$$e^{\gamma t}\left|\widetilde{f}_{k,v,p}^{(N)}(t) - \mathbf{x}(t)\right| = e^{\gamma t}|f_k^{(N)}(t) - \mathbf{x}(t)| \le |f_k^{(N)}(t) - \mathbf{x}(t)| \le \frac{\varepsilon}{8}$$

holds for all $t \in [-N,0]$. For all $t \in (-\infty,-N-p]$, by (4) we have

$$e^{\gamma t}\left|\widetilde{f}_{k,v,p}^{(N)}(t) - \mathbf{x}(t)\right| = e^{\gamma t}|e^{-\gamma t}v - \mathbf{x}(t)| \le |u-v| + |u - e^{\gamma t}\mathbf{x}(t)| \le \frac{\varepsilon}{4},$$

where we use (6) to obtain the last inequality. On the other hand, for all $t \in [-N-p, -N]$ by (4) we have

$$\begin{aligned} e^{\gamma t}\left|\widetilde{f}^{(N)}_{k,v,p}(t)-\mathbf{x}(t)\right| &= e^{\gamma t}\left|\left(\frac{N+t}{p}+1\right)f^{(N)}_{k}(-N)-\frac{N+t}{p}e^{(N+p)\gamma}v-\mathbf{x}(t)\right| \\ &\le e^{\gamma t}\left|f^{(N)}_{k}(-N)-\mathbf{x}(t)\right|+e^{\gamma t}\left|f^{(N)}_{k}(-N)-e^{(N+p)\gamma}v\right| \\ &\le \frac{\varepsilon}{3}+\frac{\varepsilon}{2}, \end{aligned}$$

where we use (8) to obtain the last inequality. Therefore, we have

$$\|\widetilde{f}^{(N)}_{k,v,p}-\mathbf{x}\|_{\gamma} = \sup_{t\in(-\infty,0]} e^{\gamma t}|\widetilde{f}^{(N)}_{k,v,p}(t)-\mathbf{x}(t)| \le \varepsilon,$$

which proves that $\widetilde{A}$ is dense in $\mathbf{X}_\gamma$ and the proof is complete.

In the following theorem, we give a sufficient condition for the existence and uniqueness of solutions of initial value problems of (2) on the state space $\mathbf{X}_\gamma$.

Theorem 2.2 (Existence and Uniqueness of Solutions). *Suppose that $A(\cdot)$, $B(\cdot)e^{\gamma(\cdot)} \in \mathcal{L}^1(\mathbb{P})$. Then there exists a measurable set $\widetilde{\Omega}$ of full measure such that for every $\omega \in \widetilde{\Omega}$ the following pathwise random delay differential equation*

$$\dot{x}(t) = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)), \tag{9}$$

with the initial condition $x(t) = \mathbf{x}(t)$ for all $t \in (-\infty, 0]$ for some $\mathbf{x} \in \mathbf{X}_\gamma$, has a unique solution on $\mathbb{R}$, denoted by $\varphi_\omega(\cdot, \mathbf{x})$. Furthermore, for a fixed $\mathbf{x} \in \mathbf{X}_\gamma$ and $T > 0$ the map $\widetilde{\Omega} \to \mathbb{R}^d$, defined by

$$\omega \mapsto \varphi_\omega(T, \mathbf{x}),$$

is measurable.

Proof. For convenience, we divide the proof into several steps.

Step 1. We define

$$\widetilde{\Omega} := \{\omega \in \Omega : t \mapsto \|A(\theta_t\omega)\| + \|B(\theta_t\omega)\|e^{\gamma(\theta_t\omega)} \text{ is locally integrable}\}.$$

It is easy to see that $\widetilde{\Omega}$ is a θ-invariant measurable set and $\mathbb{P}(\widetilde{\Omega}) = 1$ (see e.g. [1, Lemma 2.2.5]). We finish this step by showing that for all $0 < a < b$ and measurable functions $f : \widetilde{\Omega} \to \mathbb{R}^d$ the following function

$$\omega \mapsto \int_a^b A(\theta_s\omega)f(\omega)\,ds \qquad \text{is measurable for all } t \ge 0. \tag{10}$$

Since

$$\int_a^b |A(\theta_s\omega)v| \, \mathrm{d}s \le |v| \int_a^b \|A(\theta_s\omega)\| \, \mathrm{d}s < \infty \quad \text{for all } v \in \mathbb{R}^d,$$

it follows that the map

$$\omega \mapsto \int_a^b A(\theta_s\omega)v \, \mathrm{d}s \qquad \text{is measurable for all } v \in \mathbb{R}^d.$$

By approximating f by a sequence of simple functions, (10) is proved.

Step 2. For a fixed $\omega \in \widetilde{\Omega}$ and $T \in \mathbb{R}^+$ we show that (9) has a unique solution on $[0,T]$ with the initial value $\mathbf{x} \in \mathbf{X}_\gamma$. Define

$$\mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d) := \{\mathbf{f} \in \mathrm{C}([0,T],\mathbb{R}^d) \; : \; \mathbf{f}(0) = \mathbf{x}(0)\}.$$

Obviously, $\mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$ is a closed subset of $\mathrm{C}([0,T],\mathbb{R}^d)$. Corresponding to each $\mathbf{f} \in \mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$ we define the continuous function $\widetilde{\mathbf{f}} : (-\infty, T] \to \mathbb{R}^d$ by

$$\widetilde{\mathbf{f}}(t) = \begin{cases} \mathbf{f}(t), & \text{if } t \ge 0, \\ \mathbf{x}(t), & \text{if } t \le 0. \end{cases} \tag{11}$$

By the definition of $\mathbf{X}_\gamma$ we have

$$\left|\widetilde{\mathbf{f}}(s - r(\theta_s\omega))\right| \le \max\left\{ \sup_{0 \le t \le T} |\mathbf{f}(t)|, \|\mathbf{x}\|_\gamma \mathrm{e}^{\gamma r(\theta_s\omega)} \right\} \qquad \text{for all } s \in [0,T].$$

Hence, by the definition of $\widetilde{\Omega}$ we obtain

$$\int_0^t |A(\theta_s\omega)\mathbf{f}(s)| \, \mathrm{d}s < \infty \quad \text{and} \quad \int_0^t |B(\theta_s\omega)\widetilde{\mathbf{f}}(s - r(\theta_s\omega))| \, \mathrm{d}s < \infty \quad \text{for all } t \in [0,T].$$

To solve (9) we define the operator $\mathbf{T}_\omega : \mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d) \to \mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$ by

$$\mathbf{T}_\omega\mathbf{f}(t) := \mathbf{f}(0) + \int_0^t A(\theta_s\omega)\mathbf{f}(s) \, \mathrm{d}s + \int_0^t B(\theta_s\omega)\widetilde{\mathbf{f}}(s - r(\theta_s\omega)) \, \mathrm{d}s \quad \text{for all } t \in [0,T]. \tag{12}$$

Clearly, $\mathbf{T}_\omega\mathbf{f}$ is a continuous function and $\mathbf{T}_\omega\mathbf{f}(0) = \mathbf{f}(0) = \mathbf{x}(0)$. Hence, $\mathbf{T}_\omega$ is well-defined. Let $\mathbf{f}, \mathbf{g} \in \mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$. We show that

$$|\mathbf{T}_\omega^n\mathbf{f}(t) - \mathbf{T}_\omega^n\mathbf{g}(t)| \le \frac{1}{n!} \left| \int_0^t \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\| \, \mathrm{d}s \right|^n \|\mathbf{f} - \mathbf{g}\| \tag{13}$$

for all $t \in [0,T], n \in \mathbb{N}$. Indeed, due to (12) we obtain

$$\begin{aligned}&|\mathbf{T}_\omega \mathbf{f}(t) - \mathbf{T}_\omega \mathbf{g}(t)|\\ &\quad = \int_0^t A(\theta_s\omega)(\mathbf{f}(s) - \mathbf{g}(s))\,\mathrm{d}s + \int_0^t B(\theta_s\omega)(\widetilde{\mathbf{f}}(s - r(\theta_s\omega)) - \widetilde{\mathbf{g}}(s - r(\theta_s\omega)))\,\mathrm{d}s.\end{aligned}$$

Using the fact that $\widetilde{\mathbf{f}}(t) = \widetilde{\mathbf{g}}(t)$ for all $t \le 0$ we have

$$|\mathbf{T}_\omega \mathbf{f}(t) - \mathbf{T}_\omega \mathbf{g}(t)| \le \int_0^t \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\,\mathrm{d}s \cdot \|\mathbf{f} - \mathbf{g}\|,$$

which proves inequality (13) for $n = 1$. Now assume that (13) is proven for some $n \in \mathbb{N}$. For $n+1$, using the proof for $n = 1$, we have

$$\begin{aligned}&\|\mathbf{T}_\omega^{n+1}\mathbf{f}(t) - \mathbf{T}_\omega^{n+1}\mathbf{g}(t)\|\\ &\quad = \int_0^t \|A(\theta_s\omega)\||\mathbf{T}_\omega^n\mathbf{f}(s) - \mathbf{T}_\omega^n\mathbf{g}(s)|\,\mathrm{d}s + \int_0^t \|B(\theta_s\omega)\||\widetilde{\mathbf{T}}_\omega^n\mathbf{f}(s) - \widetilde{\mathbf{T}}_\omega^n\mathbf{g}(s)|\,\mathrm{d}s\\ &\quad \le \int_0^t l(s)\cdot\frac{1}{n!}\left(\int_0^s l(u)\,du\right)^n \mathrm{d}s \cdot \|\mathbf{f} - \mathbf{g}\|,\end{aligned}$$

where $l(s) := \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|$. Together with the equality

$$\int_0^t l(s)\cdot\frac{1}{n!}\left(\int_0^s l(u)\,\mathrm{d}u\right)^n \mathrm{d}s = \frac{1}{(n+1)!}\left(\int_0^t l(s)\,\mathrm{d}s\right)^{n+1}$$

this implies (13) for $n+1$. Due to Step 1 we know that $\int_0^T \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\,\mathrm{d}s < \infty$. Therefore, there exists $N \in \mathbb{N}$ such that

$$K_N := \frac{1}{N!}\left|\int_0^T \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\,\mathrm{d}s\right|^N < 1,$$

which together with (13) implies that $\mathbf{T}_\omega^N$ is a contractive map from $\mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$ into itself. As an application of the Banach fixed point theorem, there exists a unique fixed point of $\mathbf{T}_\omega$ in $\mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$ denoted by $\mathbf{f}_\omega$. Since T can be chosen arbitrary, $\mathbf{f}_\omega$ can be extended to achieve a unique continuous function $\mathbf{f}_\omega : [0,\infty) \to \mathbb{R}^d$ such that $\mathbf{f}_\omega(0) = \mathbf{x}(0)$ and

$$\mathbf{f}_\omega(t) := \mathbf{f}_\omega(0) + \int_0^t A(\theta_s\omega)\mathbf{f}_\omega(s)\,\mathrm{d}s + \int_0^t B(\theta_s\omega)\widetilde{\mathbf{f}}_\omega(s - r(\theta_s\omega))\,\mathrm{d}s \quad \text{for all } t \in \mathbb{R}_+.$$

In other words, (9) has a unique solution for each $\omega \in \widetilde{\Omega}$.

Step 3. It remains to show the measurability of the map $\widetilde{\Omega} \to \mathbb{R}^d$ defined by

$$\omega \mapsto \varphi_\omega(T,\mathbf{x}),$$

where $\mathbf{x} \in \mathbf{X}_\gamma$ and $T > 0$ are fixed and $\varphi_\omega(\cdot, \mathbf{x})$ is the solution of (9) with the initial value $\mathbf{x}$. Choose and fix $\mathbf{f} \in C_{\mathbf{x}}([0,T], \mathbb{R}^d)$. Define a sequence of functions $g_n : [0,T] \times \widetilde{\Omega} \to \mathbb{R}^d$ by

$$g_n(t,\omega) = \mathbf{T}^n_\omega \mathbf{f}(t) \qquad \text{for all } (t,\omega) \in [0,T] \times \widetilde{\Omega}.$$

By (12), we have

$$g_{n+1}(t,\omega) = \mathbf{f}(0) + \int_0^t A(\theta_s\omega) g_n(s,\omega)\, \mathrm{d}s + \int_0^t B(\theta_s\omega)\widetilde{g}_n(s - r(\theta_s\omega), \omega)\, \mathrm{d}s. \quad (14)$$

On the other hand, as is proved in Step 2, we have

$$\varphi_\omega(T, \mathbf{x}) = \lim_{n\to\infty} \mathbf{T}^n_\omega \mathbf{f}(T) = \lim_{n\to\infty} g_n(T,\omega) \quad \text{for all } \omega \in \widetilde{\Omega}.$$

Therefore, it is sufficient to show the measurability of the mappings $g_n(t,\cdot) : \widetilde{\Omega} \to \mathbb{R}^d$ for all $t \in [0,T], n \in \mathbb{N}$. We will prove this fact by induction. Clearly, the statement holds for $n = 0$. Suppose that for some $n \in \mathbb{N}$ and arbitrary $t \in [0,T]$ the function $g_n(t,\cdot) : \widetilde{\Omega} \to \mathbb{R}^d$ is measurable. Define $g^k_n : [0,t) \times \widetilde{\Omega} \to \mathbb{R}^d$ by

$$g^k_n(s,\omega) = \sum_{i=0}^{k-1} \chi_{[\frac{it}{k}, \frac{(i+1)t}{k})}(s) g_n\left(\frac{it}{k}, \omega\right) \qquad \text{for all } (s,\omega) \in [0,t) \times \widetilde{\Omega}.$$

Using the fact that $g(\cdot,\omega) : [0,t] \to \mathbb{R}^d$ is a continuous function we derive that

$$\lim_{k\to\infty} \int_0^t A(\theta_s\omega) g^k_n(s,\omega)\, \mathrm{d}s = \int_0^t A(\theta_s\omega) g(s,\omega)\, \mathrm{d}s \qquad \text{for all } \omega \in \widetilde{\Omega}.$$

As a consequence, by using Step 1, the mapping

$$\omega \mapsto \int_0^t A(\theta_s\omega) g_n(s,\omega)\, \mathrm{d}s$$

is $\mathcal{F}, \mathcal{B}(\mathbb{R}^d)$-measurable for all $t \in [0,T]$. On the other hand, due to the measurability of the mapping $(s,\omega) \mapsto r(\theta_s\omega)$ there is a sequence of simple functions from $[0,t] \times \Omega$ to $\mathbb{R}$ converging pointwise to r. Using similar arguments as above, we also obtain the measurability of the map

$$\omega \mapsto \int_0^t B(\theta_s\omega)\widetilde{g}(s - r(\theta_s\omega), \omega)\, \mathrm{d}s.$$

Hence, the mapping $g_n(t,\cdot)$ is measurable for all $t \in [0,T]$ and $n \in \mathbb{N}$. This completes the proof.

Remark 2.3. Since we can choose a θ-invariant set $\widetilde{\Omega}$ with full measure, we can assume from now on w.l.o.g. that the statements in Theorem 2.2 hold on Ω.

Now we are in a position to define the random dynamical system on $\mathbf{X}_\gamma$ which is generated by (2).

Definition 2.4. Let $A, B \in \mathcal{L}^1(\mathbb{P})$ and $r : \Omega \to \mathbb{R}^+$ be a random delay satisfying that $B(\cdot)e^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P})$. Consider a random differential equation with random delay

$$\dot{x}(t) = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)) \qquad \text{for } t \geq 0. \tag{15}$$

The random dynamical system $\Phi : \mathbb{R}^+ \times \Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ defined by

$$\Phi(t,\omega)\mathbf{x}(s) := \begin{cases} \mathbf{x}(t+s), & \text{if } t+s \leq 0, \\ \varphi_\omega(t+s,\mathbf{x}), & \text{if } t+s \geq 0, \end{cases}$$

for all $s \in \mathbb{R}^-$, where $\varphi_\omega(\cdot,\mathbf{x})$ is the unique solution of (15) with the initial value $\mathbf{x}$, is called the *random dynamical system* generated by (15).

Remark 2.5. From the unique existence of solutions of (15) we derive that $\Phi(t,\omega)$ is injective for all $t \in \mathbb{R}^+$. For $\omega \in \Omega, t \in \mathbb{R}^+$ and $\mathbf{x} \in \operatorname{im}\big(\Phi(t,\theta_{-t}\omega)\mathbf{X}_\gamma\big)$ due to the injectivity of $\Phi(t,\theta_{-t}\omega)$ there is a unique $\mathbf{y} \in \mathbf{X}_\gamma$, which is also denoted by $\Phi(-t,\omega)\mathbf{x}$, such that

$$\Phi(t,\theta_{-t}\omega)\mathbf{y} = \mathbf{x}.$$

Lemma 2.6 (Strong Measurability of Φ). *Let Φ be the random dynamical system generated by* (15)*. Then the mapping $\Phi(1,\cdot) : \Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ is strongly measurable, i.e., $\Phi(1,\cdot)\mathbf{x} : \Omega \to \mathbf{X}_\gamma$ is measurable for each $\mathbf{x} \in \mathbf{X}_\gamma$.*

Proof. It is sufficient to show that the set

$$A := \{\omega \in \Omega : \|\Phi(1,\omega)\mathbf{x} - \mathbf{y}\|_\gamma \leq \varepsilon\}$$

is measurable for all $\mathbf{y} \in \mathbf{X}_\gamma$ and $\varepsilon > 0$. By Definition 2.4 we can rewrite the set A as follows

$$\begin{aligned} A := \{\omega \in \Omega : \mathrm{e}^{\gamma s}|\varphi_\omega(s+1,\mathbf{x}) - \mathbf{y}(s)| \leq \varepsilon \text{ for all } s \in [-1,0], \\ \mathrm{e}^{\gamma s}|\mathbf{x}(s+1) - \mathbf{y}(s)| \leq \varepsilon \text{ for all } s \in (-\infty,-1)\}. \end{aligned}$$

Clearly, if the estimate

$$\mathrm{e}^{\gamma s}|\mathbf{x}(s+1) - \mathbf{y}(s)| \leq \varepsilon \qquad \text{for all } s \in (-\infty,-1) \tag{16}$$

does not hold then $A = \emptyset$ and hence A is measurable. Therefore, it remains to deal with the case that (16) holds. Using continuity of $\mathbf{y}$ and $\varphi_\omega(\cdot, \mathbf{x})$, we obtain

$$A = \bigcap_{s \in \mathbb{Q} \cap [-1,0]} \{\omega \in \Omega : \mathrm{e}^{\gamma s} |\varphi_\omega(s+1, \mathbf{x}) - \mathbf{y}(s)| \leq \varepsilon\}.$$

According to Theorem 2.2, the set $\{\omega \in \Omega : \mathrm{e}^{\gamma s} |\varphi_\omega(s+1, \mathbf{x}) - \mathbf{y}(s)| \leq \varepsilon\}$ is measurable for each $s \in \mathbb{R}^+$. Consequently, A is measurable and the proof is completed.

3 MET for Differential Equation with Random Delay

So far we have proved the existence of the random dynamical system Φ generated by random differential equations with random delay

$$\dot{x} = A(\theta_t \omega) x(t) + B(\theta_t \omega) x(t - r(\theta_t \omega)), \tag{17}$$

where $A(\cdot), B(\cdot)\mathrm{e}^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P})$. Recall that Φ is said to satisfy the *integrability condition* provided that

$$\sup_{0 \leq t \leq 1} \log^+ \|\Phi(t, \cdot)\|_\gamma \quad \text{and} \quad \sup_{0 \leq t \leq 1} \log^+ \|\Phi(1-t, \theta_t \cdot)\|_\gamma \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}),$$

(see Lian and Lu [5]).

3.1 Integrability

The aim of this subsection is to show the integrability condition of the random dynamical system Φ generated by a differential equation (17) with random delay.

Lemma 3.1 (Sufficient Integrability Condition). *Let $A \in \mathcal{L}^1(\mathbb{P})$ and $r : \Omega \to \mathbb{R}^+$ be a random map such that $B(\cdot)e^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P})$. Denote by $\Phi : \mathbb{R} \times \Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ the random dynamical system generated by* (17)*. Then Φ satisfies the integrability condition, i.e.*

$$\sup_{0 \leq t \leq 1} \log^+ \|\Phi(t, \cdot)\|_\gamma \quad \textit{and} \quad \sup_{0 \leq t \leq 1} \log^+ \|\Phi(1-t, \theta_t \cdot)\|_\gamma \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Proof. For each $\omega \in \Omega$ let $\varphi_\omega(\cdot, \mathbf{x})$ be the solution of (17) starting at $t = 0$ with the initial value $\mathbf{x} \in \mathbf{X}_\gamma$. By Definition 2.4, we obtain

$$\begin{aligned}\|\Phi(t,\omega)\mathbf{x}\|_\gamma &= \max\left\{\sup_{s\in(-\infty,-t]} \mathrm{e}^{\gamma s}|\mathbf{x}(t+s)|,\ \sup_{s\in(-t,0]} \mathrm{e}^{\gamma s}|\varphi_\omega(t+s,\mathbf{x})|\right\}\\ &= \max\left\{\mathrm{e}^{-\gamma t}\|\mathbf{x}\|_\gamma,\ \sup_{s\in(0,t]} \mathrm{e}^{\gamma(s-t)}|\varphi_\omega(s,\mathbf{x})|\right\}.\end{aligned}$$

Therefore, the following inequalities

$$\sup_{0\le t\le 1} \log^+\|\Phi(t,\omega)\|_\gamma \le \sup_{0\le t\le 1} \log^+\|\varphi_\omega(t,\cdot)\| \tag{18}$$

and

$$\sup_{0\le t\le 1} \log^+\|\Phi(1-t,\theta_t\omega)\|_\gamma \le \sup_{0\le t\le 1, 0\le s\le 1-t} \log^+\|\varphi_{\theta_t\omega}(s,\cdot)\| \tag{19}$$

hold for all $\omega\in\Omega$. In what follows, we estimate $|\varphi_\omega(t,\mathbf{x})|$ for all $0\le t\le 1$. To simplify the notation, define

$$M_\omega := \left\{s\in\mathbb{R}_+ : s\ge r(\theta_s\omega)\right\}$$

and the operator $\mathbf{T}_\omega : \mathrm{C}_{\mathbf{x}}([0,1],\mathbb{R}^d)\to \mathrm{C}_{\mathbf{x}}([0,1],\mathbb{R}^d)$ by

$$\begin{aligned}\mathbf{T}_\omega\mathbf{f}(t) &= \mathbf{x}(0) + \int_{[0,t]\cap M_\omega} B(\theta_s\omega)\mathbf{f}(s-r(\theta_s\omega))\,\mathrm{d}s\\ &\quad + \int_0^t A(\theta_s\omega)\mathbf{f}(s)\,\mathrm{d}s + \int_{[0,t]\cap M_\omega^c} B(\theta_s\omega)\mathbf{x}(s-r(\theta_s\omega))\,\mathrm{d}s.\end{aligned}$$

By (3) we have that $\varphi_\omega(\cdot,\mathbf{x})$ is the unique fixed point of $\mathbf{T}_\omega$. Moreover, due to the contractiveness of $\mathbf{T}_\omega^N$ for some $N\in\mathbb{N}$ we obtain

$$\varphi_\omega(t,\mathbf{x}) = \lim_{n\to\infty}\mathbf{T}_\omega^n\mathbf{f}(t) \qquad \text{for all } \mathbf{f}\in\mathrm{C}_{\mathbf{x}}([0,1],\mathbb{R}^d), t\in[0,1]. \tag{20}$$

From the definition of $\mathbf{T}_\omega$, it is easy to see that

$$\begin{aligned}|\mathbf{T}_\omega\mathbf{f}(t)| &\le |\mathbf{x}(0)| + \int_{[0,t]\cap M_\omega} \|B(\theta_s\omega)\||\mathbf{f}(s-r(\theta_s\omega))|\,\mathrm{d}s\\ &\quad + \int_0^t \|A(\theta_s\omega)\||\mathbf{f}(s)| + \int_{[0,t]\cap M_\omega^c} \|B(\theta_s\omega)\|e^{\gamma(r(\theta_s\omega)-s)}\|\mathbf{x}\|_\gamma\,\mathrm{d}s.\end{aligned}$$

Consequently, the following inequality

$$\begin{aligned}|\mathbf{T}_\omega\mathbf{f}(t)| &\le k(\omega)\|\mathbf{x}\|_\gamma + \int_{[0,t]\cap M_\omega} \|B(\theta_s\omega)\||\mathbf{f}(s-r(\theta_s\omega))|\,\mathrm{d}s\\ &\quad + \int_0^t \|A(\theta_s\omega)\||\mathbf{f}(s)|\,\mathrm{d}s,\end{aligned}$$

where $k(\omega) := 1 + \int_0^1 \|B(\theta_s\omega)\| e^{\gamma(r(\theta_s\omega)-s)}\, ds$, holds for all $0 \le t \le 1$. A direct computation yields that the non-empty closed set

$$B_\omega := \left\{ \mathbf{f} \in \mathrm{C}_{\mathbf{x}}([0,1],\mathbb{R}^d) : \mathbf{f}(t) \le k(\omega)\|\mathbf{x}\|_\gamma \mathrm{e}^{\int_0^t \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\, ds} \text{ for all } 0 \le t \le 1 \right\}$$

is invariant under $\mathbf{T}_\omega$. Therefore, together with (20) we get

$$\|\varphi_\omega(t,\cdot)\| \le k(\omega)\mathrm{e}^{\int_0^t \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\, ds} \qquad \text{for all } 0 \le t \le 1,$$

which gives

$$\sup_{0\le t\le 1} \log^+ \|\varphi_\omega(t,\cdot)\| \le \log k(\omega) + \int_0^1 \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\, \mathrm{d}s \tag{21}$$

and

$$\sup_{0\le t\le 1, 0\le s\le 1-t} \log^+ \|\varphi_{\theta_t\omega}(s,\cdot)\| \le \sup_{0\le t\le 1} \log k(\theta_t\omega) + \int_0^1 \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\, \mathrm{d}s. \tag{22}$$

Using the inequality $\log(1+x) \le 1 + \log^+ x$ for $x \in \mathbb{R}_+$, we have

$$\sup_{0\le t\le 1} \log k(\theta_t\omega) \le 1 + \log^+ \int_0^2 \|B(\theta_s\omega)\| \mathrm{e}^{\gamma r(\theta_s\omega)}\, \mathrm{d}s. \tag{23}$$

By the Fubini theorem, we get

$$\int_\Omega \int_0^2 \|B(\theta_s\omega)\| \mathrm{e}^{\gamma r(\theta_s\omega)}\, \mathrm{d}s\, \mathrm{d}\mathbb{P} = \int_0^2 \int_\Omega \|B(\theta_s\omega)\| \mathrm{e}^{\gamma r(\theta_s\omega)}\, \mathrm{d}\mathbb{P}\, \mathrm{d}s.$$

On the other hand, for all $s \in [0,2]$

$$\int_\Omega \|B(\theta_s\omega)\| \mathrm{e}^{\gamma r(\theta_s\omega)}\, \mathrm{d}\mathbb{P}(\omega) = \int_\Omega \|B(\omega)\| \mathrm{e}^{\gamma r(\omega)}\, \mathrm{d}\mathbb{P}(\omega) < \infty.$$

Hence,

$$\int_0^2 \|B(\theta_s\cdot)\| \mathrm{e}^{\gamma r(\theta_s\cdot)}\, \mathrm{d}s \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}),$$

which together with (23) proves that $\sup_{0\le t\le 1} \log k(\theta_t\cdot) \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$. Therefore, by (18), (19) and (21), (22) we obtain

$$\sup_{0\le t\le 1} \log^+ \|\Phi(t,\cdot)\|_\gamma, \sup_{0\le t\le 1} \log^+ \|\Phi(1-t,\theta_t\cdot)\|_\gamma \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}).$$

This completes the proof.

3.2 Kuratowski Measure

Recall that for a subset $A \subset \mathbf{X}_\gamma$, the *Kuratowski measure of noncompactness* of A is defined by

$$\alpha(A) := \inf\{d : A \text{ has a finite cover by sets of diameter } d\}.$$

For a bounded linear map $L : \mathbf{X}_\gamma \to \mathbf{X}_\gamma$ we define

$$\|L\|_\alpha = \alpha(L(B_1(0))).$$

Let Φ be the linear cocycle defined as in Definition 2.4. We introduce the following quantity

$$l_\alpha(\Phi) := \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\|_\alpha,$$

and note that it is constant $\mathbb{P}$-a.s. due to the ergodicity of θ and the Kingman subadditive ergodic theorem (see e.g. Arnold [1, pp. 122], Ruelle [6, Appendix A]). To compute l_α of the random dynamical system Φ, we first prove the following preparatory Lemma.

Lemma 3.2. *(i) Let $T > 0$ and $a : [0,T] \to \mathbb{R}$ be an integrable function. Then for any $\varepsilon > 0$ there exists a partition $0 = t_0 < t_1 < \cdots < t_K = T$ such that*

$$\int_{t_i}^{t_{i+1}} |a(s)|\, \mathrm{d}s \le \varepsilon \qquad \text{for all } i = 0,\dots,K-1.$$

(ii) Let $T > 0$, $\omega \in \Omega$ and suppose that the function $t \mapsto \|A(\theta_t\omega)\| + \|B(\theta_t\omega)\|\mathrm{e}^{\gamma(\theta_t\omega)}$ is locally integrable. Define

$$A := \{\varphi_\omega(\cdot,\mathbf{x}) : [0,T] \to \mathbb{R}^d, \mathbf{x} \in B_1(0)\}.$$

Then $l_\alpha(A) = 0$, where A is considered as a subset of $\mathrm{C}([0,T],\mathbb{R}^d)$.

Proof. (i) The proof is straightforward by using the fact that the function

$$t \mapsto \int_0^t |a(s)|\, \mathrm{d}s$$

is continuous.

(ii) By the same arguments as in the proof of Lemma 3.1, the inequality

$$|\varphi_\omega(t,\mathbf{x})| \le \left(1 + \int_0^T \|B(\theta_s\omega)\|\mathrm{e}^{\gamma r(\theta_s\omega)}\, \mathrm{d}s\right) \mathrm{e}^{\int_0^T \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\, \mathrm{d}s}$$

holds for all $\mathbf{x} \in B_1(0)$ and $t \in [0,T]$. Then there exists $M > 0$ such that

$$|\varphi_\omega(t,\mathbf{x})| \le M \qquad \text{for all } t \in [0,T], \mathbf{x} \in B_1(0). \tag{24}$$

For any $t,s \in [0,T]$ with $t > s$ and $\mathbf{x} \in B_1(0)$, by using (3) we have

$$\begin{aligned}
&\varphi_\omega(t,\mathbf{x}) - \varphi_\omega(s,\mathbf{x}) \\
&\quad = \int_s^t A(\theta_u\omega)\varphi_\omega(u,\mathbf{x}) + B(\theta_u\omega)\varphi_\omega(u - r(\theta_u\omega),\mathbf{x})\,\mathrm{d}u \\
&\quad = \int_s^t A(\theta_u\omega)\varphi_\omega(u,\mathbf{x})\,\mathrm{d}u + \int_{[s,t]\cap M_\omega} B(\theta_u\omega)\varphi_\omega(u - r(\theta_u\omega),\mathbf{x})\,\mathrm{d}u \\
&\qquad + \int_{[s,t]\cap M_\omega^c} B(\theta_u\omega)\mathbf{x}(u - r(\theta_u\omega))\,\mathrm{d}u,
\end{aligned}$$

where $M_\omega := \{s \in \mathbb{R}_+ : s \ge r(\theta_s\omega)\}$. Together with (24) this implies that

$$|\varphi_\omega(t,\mathbf{x}) - \varphi_\omega(s,\mathbf{x})| \le M\int_s^t \|A(\theta_u\omega)\| + \|B(\theta_u\omega)\|\,\mathrm{d}u + \int_s^t \|B(\theta_u\omega)\| rme^{\gamma r(\theta_u\omega)}\mathrm{d}u$$

holds for all $\varphi_\omega(\cdot,\mathbf{x}) \in A$. Applying (i) to the right hand side of the estimate, we get for an arbitrary $\varepsilon > 0$ a partition $0 = t_0 < t_1 < \cdots < t_K = T$ such that

$$|f(t) - f(s)| \le \frac{\varepsilon}{3} \qquad \text{for all } f \in A, t_k \le t, s \le t_{k+1}, k = 0,\dots,K-1. \tag{25}$$

In the following, we first give a proof in the scalar case, i.e. $d = 1$. Choose and fix $N \in \mathbb{N}$ such that $\frac{M}{N} \le \frac{\varepsilon}{3}$. For each index $(i_1,\dots,i_K) \in \{-N,-N+1,\dots,N-1,N\}^K$, by writing each $t \in [0,T]$ uniquely as $t = \alpha t_k + \beta t_{k+1}$ for $k \in \{0,\dots,K-1\}$ and $\alpha,\beta \in [0,1]$ with $\alpha + \beta = 1$, we define a continuous function $f_{i_1,\dots,i_K} \in \mathrm{C}([0,T],\mathbb{R})$ by

$$f_{i_1,\dots,i_K}(\alpha t_k + \beta t_{k+1}) = \alpha\frac{i_k M}{N} + \beta\frac{i_{k+1}M}{N}.$$

Now we show that

$$A \subset \bigcup_{-N \le i_1,\dots,i_K \le N} B_\varepsilon(f_{i_1,\dots,i_K}). \tag{26}$$

By the definition of A and inequality (24) we have

$$-M \le f(t_k) \le M \qquad \text{for all } f \in A, k = 0,\dots,K-1.$$

which implies together with the inequality $\frac{M}{N} \le \frac{\varepsilon}{3}$ that for any $f \in A$ there exists an index $(i_1,\dots,i_K) \in \{-N,-N+1,\dots,N-1,N\}^K$ such that

$$\left|f(t_k) - \frac{i_k M}{N}\right| \le \frac{M}{N} \le \frac{\varepsilon}{3} \qquad \text{for all } k = 0,\dots,K-1.$$

Equivalently,

$$|f(t_k) - f_{i_1,\dots,i_K}(t_k)| \le \frac{\varepsilon}{3} \qquad \text{for all } k = 0,\dots,K-1. \tag{27}$$

For any $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ we get

$$\begin{aligned}
&|f(\alpha t_k + \beta t_{k+1}) - \alpha f(t_k) - \beta f(t_{k+1})| \\
&\quad \le \alpha |f(\alpha t_k + \beta t_{k+1}) - f(t_k)| + \beta |f(t_{k+1}) - f(\alpha t_k + \beta t_{k+1})| \le \frac{\varepsilon}{3},
\end{aligned}$$

where we use (25) to obtain the last estimate. This implies with (27) that

$$\begin{aligned}
&|f(\alpha t_k + \beta t_{k+1}) - f_{i_1,\dots,i_K}(\alpha t_k + \beta t_{k+1})| \\
&\quad \le \frac{\varepsilon}{3} + \alpha |f(t_k) - f_{i_1,\dots,i_K}(t_k)| + \beta |f(t_{k+1}) - f_{i_1,\dots,i_K}(t_{k+1})| \le \frac{2\varepsilon}{3}
\end{aligned}$$

for all $k = 0,\dots,K-1$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. This proves (26) and since ε can be chosen arbitrarily small, it follows that $l_\alpha(A) = 0$ in the case $d = 1$. Since each continuous function $f \in \mathrm{C}([0,T],\mathbb{R}^d)$ can be written as $f = (f_1,\dots,f_d)$, where $f_1,\dots,f_d$ are scalar continuous functions, the case $d \ge 2$ can be easily reduced to the scalar case and therefore we also obtain the conclusion in the general case. This completes the proof.

Lemma 3.3. *Let $\Phi : \mathbb{R} \times \Omega \times \mathbf{X}_\gamma \to \mathbf{X}_\gamma$ be the random dynamical system generated by* (17)*. Then*

$$l_\alpha = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\|_\alpha = -\gamma.$$

Proof. For convenience, throughout the proof we only deal with the max norm on $\mathbb{R}^d$, i.e. $|x| = \max_{1\le i\le d} |x_i|$ for all $x = (x_1,\dots,x_d)^{\mathrm{T}} \in \mathbb{R}^d$. We first obtain the inequality $l_\alpha \ge -\gamma$ by showing that

$$\alpha(\Phi(T,\omega)B_1(0)) \ge \mathrm{e}^{-\gamma(T+1)} \qquad \text{for all } T > 0. \tag{28}$$

Thereto, define the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \subset B_1(0)$ as follows

$$\mathbf{x}_n(t) = \begin{cases} 0, & \text{if } t \in (-n+1, 0], \\ \mathrm{e}^{\gamma(n-1)}(-n+1-t)(1,\dots,1)^{\mathrm{T}}, & \text{if } t \in (-n, -n+1], \\ \mathrm{e}^{\gamma(n-1)}(1,\dots,1)^{\mathrm{T}}, & \text{if } t \in (-\infty, -n]. \end{cases}$$

Obviously, the function $\mathbf{x}_n : (-\infty, 0] \to \mathbb{R}^d$ is continuous, and the relations

$$\lim_{t\to-\infty} \mathrm{e}^{\gamma t}\mathbf{x}_n(t) = 0, \qquad \sup_{t\in(-\infty,0]} \mathrm{e}^{\gamma t}|\mathbf{x}_n(t)| \le 1,$$

proves that $\mathbf{x}_n \in B_1(0)$ for all $n \in \mathbb{N}$. A straightforward computation yields that for all $m > n$ the following equality holds

$$\begin{aligned}\Phi(T,\omega)\mathbf{x}_m(-n-T) - \Phi(T,\omega)\mathbf{x}_n(-n-T) &= \mathbf{x}_m(-n) - \mathbf{x}_n(-n)\\ &= -\mathrm{e}^{\gamma(n-1)}(1,\dots,1)^{\mathrm{T}}.\end{aligned}$$

Thus,

$$\|\Phi(T,\omega)\mathbf{x}_m - \Phi(T,\omega)\mathbf{x}_n\|_\gamma \geq \mathrm{e}^{-\gamma(T+1)},$$

which proves (28). Hence,

$$l_\alpha(\Phi) = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\|_\alpha \geq -\gamma.$$

Therefore, it remains to show that

$$l_\alpha(\Phi) \leq -\gamma + \varepsilon \qquad \text{for all } 0 < \varepsilon < \gamma. \tag{29}$$

Choose and fix $T \geq \frac{\log 3}{\varepsilon}$. By definition of $\Phi(T,\omega)$ (see Definition 2.4), we have

$$\Phi(T,\omega)\mathbf{x}(t) = \begin{cases} \mathbf{x}(t+T), & \text{for all } t \in (-\infty,-T),\\ \varphi_\omega(t+T,\mathbf{x}), & \text{for all } t \in [-T,0], \end{cases}$$

for all $\mathbf{x} \in B_1(0)$. Therefore,

$$\Phi(T,\omega)\mathbf{x}(\cdot) \equiv \varphi_\omega(\cdot + T,\mathbf{x}) \quad \text{ on } [-T,0] \tag{30}$$

According to Lemma 3.2 (ii), there exist $\widetilde{f}_1,\dots,\widetilde{f}_n \in \mathrm{C}([0,T],\mathbb{R}^d)$ such that

$$\{\varphi_\omega(\cdot,\mathbf{x}) : [0,T] \to \mathbb{R}^d, \mathbf{x} \in B_1(0)\} \subset \bigcup_{k=1}^n B_{\mathrm{e}^{(-\gamma+\varepsilon)T}}(\widetilde{f}_k),$$

which implies with (30) that

$$\left\{\Phi(T,\omega)\mathbf{x}|_{[-T,0]}, \mathbf{x} \in B_1(0)\right\} \subset \bigcup_{k=1}^n B_{\mathrm{e}^{(-\gamma+\varepsilon)T}}(f_k), \tag{31}$$

where $f_k : [-T,0] \to \mathbb{R}^d$ is defined by $f_k(t) = \widetilde{f}_k(t+T)$. Define $\widehat{f}_k : (-\infty,0] \to \mathbb{R}^d$ by

$$\widehat{f}_k(t) := \begin{cases} f_k(t), & \text{if } t \in [-T,0],\\ f_k(-T), & \text{if } t \in (-\infty,-T). \end{cases}$$

We show that

$$\Phi(T,\omega)B_1(0) \subset \bigcup_{k=1}^{n} B_{\mathrm{e}^{(-\gamma+\varepsilon)T}}(\widehat{f}_k). \tag{32}$$

To prove this statement let $\mathbf{x} \in B_1(0)$. Using (31), there exists $k \in \{1,\dots,n\}$ such that

$$|\Phi(T,\omega)\mathbf{x}(t) - f_k(t)| \le \mathrm{e}^{(-\gamma+\varepsilon)T} \qquad \text{for all } t \in [-T,0].$$

In particular, $f_k(-T) \le 1 + \mathrm{e}^{(-\gamma+\varepsilon)T}$. On the other hand, for all $t \in (-\infty,-T]$ we get

$$\begin{aligned}
\mathrm{e}^{\gamma t}|\Phi(T,\omega)\mathbf{x}(t) - \widehat{f}_k(t)| &= \mathrm{e}^{\gamma t}|\mathbf{x}(t+T) - f_k(-T)| \\
&\le \mathrm{e}^{\gamma t}\left(\mathrm{e}^{-\gamma(t+T)} + 1 + e^{(-\gamma+\varepsilon)T}\right) \\
&\le 3\mathrm{e}^{-\gamma T},
\end{aligned}$$

which together with $T \ge \frac{\log 3}{\varepsilon}$ proves (32). Consequently, we have

$$\|\Phi(T,\omega)\|_\alpha \le \mathrm{e}^{(-\gamma+\varepsilon)T} \qquad \text{for all } T \ge \frac{\log 3}{\varepsilon},$$

which implies that

$$l_\alpha(\Phi) = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\|_\alpha \le -\gamma + \varepsilon,$$

proving (29) and the proof is completed.

3.3 Multiplicative Ergodic Theorem

We have proved so far that the random dynamical system generated by a differential equation with random delay fulfills all assumptions of the multiplicative ergodic theorem on Banach spaces (see Lian and Lu [5]). Therefore, we are now in a position to state the multiplicative ergodic theorem for differential equations with random delay.

Theorem 3.4 (Multiplicative Ergodic Theorem for Differential Equations with Random Delay). *Let $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ be an ergodic metric dynamical system and $A,B : \Omega \to \mathbb{R}^d$ and $r : \Omega \to \mathbb{R}_+$ be measurable functions satisfying that*

$$A(\cdot), B(\cdot)\mathrm{e}^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P}).$$

Denote by $\Phi : \mathbb{R}^+ \times \mathbf{X}_\gamma \to \mathbf{X}_\gamma$ the random dynamical system generated by the differential equation with random delay

$$\dot{x} = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)).$$

Then, there exists a θ-invariant subset $\widetilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \widetilde{\Omega}$ exactly one of the following statements holds

(I) $\kappa(\Phi) = -\gamma$

(II) There exists $k \in \mathbb{N}$, Lyapunov exponents $\lambda_1 > \cdots > \lambda_k > -\gamma$ *and a splitting into measurable* Oseledets spaces

$$\mathbf{X}_\gamma = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F(\omega)$$

with finite dimensional linear subspaces $E_j(\omega)$ and an infinite dimensional linear subspace $F(\omega)$ such that the following properties hold:

(i) Invariance: $\Phi(t,\omega)E_j(\omega) = E_j(\theta_t\omega)$ and $\Phi(t,\omega)F(\omega) \subset F(\theta_t\omega)$.

(ii) Lyapunov exponents:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}\|_\gamma = \lambda_j \quad \textit{for all } \mathbf{x} \in E_j(\omega) \setminus 0 \textit{ and } j = 1,\ldots,k.$$

(iii) Exponential Decay Rate on $F(\omega)$:

$$\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi(t,\omega)|_{F(\omega)}\|_\gamma \le -\gamma.$$

Moreover, for $\mathbf{x} \in F(\omega) \setminus 0$ such that $\Phi(t,\theta_{-t}\omega)^{-1}\mathbf{x} := \Phi(-t,\omega)\mathbf{x}$ exists for all $t \in \mathbb{R}^+$ we get

$$\liminf_{t\to+\infty} \frac{1}{t} \log \|\Phi(-t,\omega)\mathbf{x}\|_\gamma \ge \gamma.$$

(III) There exist infinitely many finite dimensional measurable subspaces $E_j(\omega)$, infinitely many infinite dimensional subspaces $F_j(\omega)$ and infinitely many Lyapunov exponents

$$\lambda_1 > \lambda_2 > \cdots > -\gamma \quad \textit{with} \quad \lim_{j\to+\infty} \lambda_j = -\gamma$$

such that the following properties hold:

(i) Invariance: $\Phi(t,\omega)E_j(\omega) = E_j(\theta_t\omega)$ and $\Phi(t,\omega)F_j(\omega) \subset F_j(\theta_t\omega)$.

(ii) Invariant Splitting

$$\mathbf{X}_\gamma = E_1(\omega) \oplus \cdots \oplus E_j(\omega) \oplus F_j(\omega) \quad \textit{and} \quad F_j(\omega) = E_{j+1}(\omega) \oplus F_{j+1}(\omega).$$

(iii) Lyapunov exponents:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}\|_\gamma = \lambda_j \quad \textit{for all } \mathbf{x} \in E_j(\omega) \setminus 0 \textit{ and } j = 1,\ldots,k.$$

(iv) Exponential Decay Rate on $F_j(\omega)$:

$$\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi(t,\omega)|_{F_j(\omega)}\|_\gamma = \lambda_{j+1}.$$

Moreover, for $\mathbf{x} \in F_j(\omega) \setminus 0$ such that $\Phi(t,\theta_{-t}\omega)^{-1}\mathbf{x} := \Phi(-t,\omega)\mathbf{x}$ exists for all $t \in \mathbb{R}^+$ we get

$$\liminf_{t\to+\infty} \frac{1}{t} \log \|\Phi(-t,\omega)\mathbf{x}\|_\gamma \geq -\lambda_{j+1}.$$

The following theorem shows that the Lyapunov exponents which exist by the Multiplicative Ergodic Theorem 3.4 are independent of the exponential weight factor $\gamma > 0$.

Theorem 3.5 (Lyapunov Exponents are Independent of Exponential Weight Factor). *Let $\gamma > 0$ and consider (15) on the state space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$. Assume that $\lambda > -\gamma$ is a Lyapunov exponent of (15), i.e. for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists $\mathbf{x}(\omega) \in \mathbf{X}_\gamma$ such that*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma = \lambda\ .$$

Then for every $\zeta > \gamma$ satisfying that $\mathrm{e}^{\zeta r(\cdot)}B(\cdot) \in \mathcal{L}^1(\mathbb{P})$ we have $\mathbf{x}(\omega) \in \mathbf{X}_\zeta$ and the number λ is also a Lyapunov exponent of (15) on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$. In particular,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta = \lambda\ . \tag{33}$$

Proof. Let $\mathbf{y} \in \mathbf{X}_\gamma$. From the definition of $\mathbf{X}_\gamma$ we obtain that $\lim_{t\to\infty} \mathrm{e}^{-\gamma t}\mathbf{y}(-t)$ exists. For $\zeta > \gamma$ it is easy to see that

$$\lim_{t\to\infty} \mathrm{e}^{-\zeta t}\mathbf{y}(-t) = \lim_{t\to\infty} \mathrm{e}^{(\gamma-\zeta)t}\mathrm{e}^{-\gamma t}\mathbf{y}(-t) = 0,$$

which implies that $\mathbf{y} \in \mathbf{X}_\zeta$. Furthermore, for any $\mathbf{y} \in \mathbf{X}_\gamma$ we have

$$\|\mathbf{y}\|_\zeta = \sup_{t\in[0,\,\infty)} \mathrm{e}^{-\zeta t}|\mathbf{y}(-t)| \leq \sup_{t\in[0,\,\infty)} \mathrm{e}^{-\gamma t}|\mathbf{y}(-t)| \leq \|\mathbf{y}\|_\gamma.$$

As a consequence, we get

$$\limsup_{t\to\infty} \frac{1}{t}\ \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \leq \lim_{t\to\infty} \frac{1}{t}\ \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma, \tag{34}$$

and

$$\liminf_{t\to-\infty} \frac{1}{t}\ \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \geq \lim_{t\to-\infty} \frac{1}{t}\ \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma. \tag{35}$$

In view of Theorem 3.4 we divide the proof into several cases.

Case 1: The linear cocycle Φ *on the state space* $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ *has finitely many Lyapunov exponents* . Let $-\zeta < \lambda_k < \lambda_{k-1} < \cdots < \lambda_1$ be the Lyapunov exponents of the linear cocycle Φ on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ and

$$\mathbf{X}_\zeta = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F(\omega)$$

the corresponding Oseledets splitting of Φ. We write $\mathbf{x}(\omega)$ in the following form

$$\mathbf{x}(\omega) = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k + \mathbf{x}_F,$$

where $\mathbf{x}_i \in E_i(\omega)$ and $\mathbf{x}_F \in F(\omega)$. For convenience, we divide the proof into several steps.

Step 4. We first show that $\mathbf{x}_F = 0$ by contradiction, i.e. we assume that $\mathbf{x}_F \neq 0$. In view of Theorem 3.4, we have

$$\limsup_{t\to-\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}_F\|_\zeta \leq -\zeta,$$

and for all $i \in \{1,\dots,k\}$ with $\mathbf{x}_i \neq 0$

$$\lim_{t\to-\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}_i\|_\zeta = \lambda_i.$$

Therefore, for any $\varepsilon \in (0, \frac{\lambda_k+\zeta}{4})$ there exists $T(\varepsilon) \in \mathbb{R}^+$ such that

$$\frac{1}{t}\, \log \|\Phi(t,\omega)\mathbf{x}_F\|_\zeta \leq -\zeta + \varepsilon \qquad \text{for all } t \leq -T(\varepsilon),$$

and for all $i \in \{1,\dots,k\}$ with $\mathbf{x}_i \neq 0$

$$\lambda_i - \frac{\varepsilon}{2} \leq \frac{1}{t}\, \log \|\Phi(t,\omega)\mathbf{x}_i\|_\zeta \leq \lambda_i + \frac{\varepsilon}{2} \qquad \text{for all } t \leq -T(\varepsilon).$$

Hence, for all $t \leq -T(\varepsilon)$ we have

$$\begin{aligned} \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta &= \Big\|\Phi(t,\omega)\mathbf{x}_F + \sum_{i\in\{1,\dots,k\},\, \mathbf{x}_i\neq 0} \Phi(t,\omega)\mathbf{x}_i\Big\|_\zeta \\ &\geq \|\Phi(t,\omega)\mathbf{x}_F\|_\zeta - \Big\|\sum_{i\in\{1,\dots,k\},\, \mathbf{x}_i\neq 0} \Phi(t,\omega)\mathbf{x}_i\Big\|_\zeta \\ &\geq \mathrm{e}^{t(-\zeta+\varepsilon)} - \sum_{i=1}^{k} \mathrm{e}^{t(\lambda_i-\varepsilon)}. \end{aligned}$$

Consequently,

$$\frac{1}{t}\log\|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \leq \frac{1}{t}\log\left(e^{t(-\zeta+\varepsilon)} - \sum_{i=1}^{k} \mathrm{e}^{t(\lambda_i-\varepsilon)}\right) \qquad \text{for all } t \leq -T(\varepsilon),$$

which implies that

$$\limsup_{t\to-\infty}\frac{1}{t}\log\|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \leq -\zeta+\varepsilon,$$

where we use the fact that $\lim_{t\to-\infty}\frac{1}{t}\log\left(\mathrm{e}^{ta}-\mathrm{e}^{tb}\right) = a$ provided that $a<b$, to obtain the last inequality. Since ε can be chosen arbitrarily small it follows together with (35) that

$$\lim_{t\to-\infty}\frac{1}{t}\ \log\|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma \leq -\zeta, \tag{36}$$

which contradicts the fact that

$$-\zeta < -\gamma < \lambda = \lim_{t\to-\infty}\frac{1}{t}\ \log\|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma.$$

Step 2. Define

$$i_{\min} := \min_{\mathbf{x}_i\neq 0} i, \qquad i_{\max} := \max_{\mathbf{x}_i\neq 0} i.$$

By the same argument as in Step 1, we obtain that

$$\lim_{t\to\infty}\frac{1}{t}\ \log\|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \geq \lambda_{i_{\min}},$$

$$\lim_{t\to-\infty}\frac{1}{t}\ \log\|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \leq \lambda_{i_{\max}},$$

which implies together with (34), (35), and the fact that $\lambda_{i_{\min}} \geq \lambda_{i_{\max}}$ that

$$\lim_{t\to\pm\infty}\frac{1}{t}\ \log\|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta = \lambda,$$

proving (33) and the proof in this case is completed.

Case 2: The linear cocycle Φ *on the state space* $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ *has infinitely many Lyapunov exponents* . Let $-\zeta < \cdots < \lambda_2 < \lambda_1$ with $\lim_{k\to\infty}\lambda_k = -\zeta$ be the Lyapunov exponents of the linear cocycle Φ on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ and

$$\mathbf{X}_\zeta = E_1(\omega)\oplus\cdots\oplus E_k(\omega)\oplus F_k(\omega)$$

the corresponding invariant splitting. We prove the fact that $\lambda \in \{\lambda_1, \lambda_2, \dots\}$ by contradiction, i.e. we assume that $\lambda_k \neq \lambda$ for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} \lambda_k = -\zeta$ it follows that there exists $k \in \mathbb{N}$ such that $\lambda_k < \lambda$. Set $k^* := \min\{k : \lambda_k < \lambda\}$. By using (35) and in view of Theorem 3.4, we obtain that $k^* > 1$. Hence, $\lambda_{k^*} < \lambda < \lambda_{k^*-1}$. We write $\mathbf{x}(\omega)$ in the following form

$$\mathbf{x}(\omega) = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_{k^*-1} + \mathbf{x}_F,$$

where $\mathbf{x}_i \in E_i(\omega), i = 1, \dots, k^* - 1$ and $\mathbf{x}_F \in F_{k^*-1}(\omega)$. Using a similar proof as in the Step 1 of Case 1, we also have $\mathbf{x}_F = 0$ and

$$\liminf_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \geq \lambda_{k^*-1},$$

which together with (34) contradicts the fact that $\lambda < \lambda_{k^*-1}$ and the proof is completed.

4 Differential Equations with Bounded Delay

The aim of this section is to investigate differential equations with bounded delay. If the delay is bounded we do not need all information for $t \in (-\infty, 0]$ in order to know the value of solutions in the future. As a consequence, there are several options to define a dynamical system generated by such an equation. Naturally, we can ask the question whether there are any relations between the Lyapunov exponents of these dynamical systems. Throughout this section we consider the following system

$$\dot{x}(t) = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)). \tag{37}$$

Assume that the random delay map r is bounded, i.e. there exists $M > 0$ such that

$$r(\omega) \leq M \qquad \text{for all } \omega \in \Omega$$

and $A, B \in \mathcal{L}^1(\mathbb{P})$. Due to the boundedness of the delay the initial values of (37) can be either in $\mathbf{X}_\gamma$ or in $\mathrm{C}([-M,0],\mathbb{R}^d)$. Using the same procedure to introduce and investigate the random dynamical system in $\mathbf{X}_\gamma$ generated by (37), we also obtain a random dynamical system in $\mathrm{C}([-M,0],\mathbb{R}^d)$ generated by (37) as follows:

Random Dynamical System on $\mathrm{C}([-M,0],\mathbb{R}^d)$: For each $\omega \in \Omega$ and an initial value $x \in \mathrm{C}([-M,0],\mathbb{R}^d)$, equation (37) has a unique solution denoted by $\psi_\omega(\cdot, x)$, i.e. the equality

$$\psi_\omega(t,x) = \int_0^t A(\theta_s\omega)\psi_\omega(s,x) + \int_0^t B(\theta_s\omega)\widetilde{\psi}_\omega(s-r(\theta_s\omega),x)\,\mathrm{d}s$$

holds for all $t \in \mathbb{R}_+$, where

$$\widetilde{\psi}_\omega(s-r(\theta_s\omega),x) = \begin{cases} x(s-r(\theta_s\omega)), & \text{if } s \le r(\theta_s\omega), \\ \psi_\omega(s-r(\theta_s\omega),x), & \text{otherwise.} \end{cases}$$

Based on the existence and uniqueness of solution of (37) we can define a random dynamical system $\Psi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(\mathrm{C}([-T,0],\mathbb{R}^d))$, where $\mathcal{L}(\mathrm{C}([-T,0],\mathbb{R}^d))$ denotes the space of all bounded linear operators from $\mathrm{C}([-T,0],\mathbb{R}^d)$ into itself, by

$$\Psi(t,\omega)x(s) = \begin{cases} \psi_\omega(t+s,x), & \text{if } t+s \ge 0, \\ x(t+s), & \text{otherwise,} \end{cases}$$

for all $s \in [-M,0]$.

Properties of Ψ: Along the lines of the proof of Theorem 2.2, Lemma 3.1 and Lemma 3.3 one can show that

- Ψ is strongly measurable.
- Ψ satisfies the integrability condition, i.e.

$$\sup_{0\le t\le 1} \log^+ \|\Psi(t,\cdot)\| \quad \text{and} \quad \sup_{0\le t\le 1} \log^+ \|\Psi(1-t,\theta_t\cdot)\| \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}).$$

- $l_\alpha(\Psi) = -\infty$.

The following theorem compares the Lyapunov exponents of Φ and Ψ and shows equality for those which are larger than the exponential weight factor $-\gamma$.

Theorem 4.1. *Let $\alpha_1 > \alpha_2 > \dots$ be the Lyapunov exponents of Φ and $\beta_1 > \beta_2 > \dots$ be the Lyapunov exponent of Ψ. Then*

$$\{\alpha_i\} = \{\beta_i : \beta_i > -\gamma\}.$$

Proof. ($\Rightarrow$) Suppose that λ is a Lyapunov exponent of Φ. Fix $\omega \in \Omega$ and let $\mathbf{x} \in \mathbf{X}_\gamma$ be a vector corresponding to this Lyapunov exponent, i.e.

$$\lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}\|_\gamma = \lambda.$$

Define $x \in \mathrm{C}([-M,0],\mathbb{R}^d)$ by

$$x(s) = \mathbf{x}(s) \qquad \text{for all } s \in [-M,0].$$

A direct computation yields that

$$\varphi_\omega(t,\mathbf{x}) = \psi_\omega(t,x) \qquad \text{for all } t \geq 0,$$

which leads

$$\Phi(t,\omega)\mathbf{x}(s) = \Psi(t,\omega)x(s) \qquad \text{for all } s \in [-M,0], t \geq M.$$

Consequently, for all $t \geq M$ we have

$$\frac{1}{t}\log\|\Phi(t,\omega)\mathbf{x}\|_\gamma \geq \frac{1}{t}\log(\mathrm{e}^{-\gamma M}\|\Psi(t,\omega)x\|),$$

proving that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|\Psi(t,\omega)x\| \leq \lambda.$$

To prove λ is a Lyapunov exponent of Ψ, by virtue of Theorem 3.4 it is sufficient to show that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|\Psi(t,\omega)x\| = \lambda.$$

Thereto, we assume the opposite, i.e. there exists $\varepsilon \in (0,\lambda+\gamma)$ and $T > 0$ such that

$$\|\Psi(t,\omega)x\| \leq \mathrm{e}^{(\lambda-\varepsilon)t} \qquad \text{for all } t \geq T.$$

Therefore,

$$|\varphi_\omega(t,\mathbf{x})| = |\psi_\omega(t,x)| \leq \mathrm{e}^{(\lambda-\varepsilon)t} \qquad \text{for all } t \geq T.$$

As a consequence, for all $t \geq T$ we have

$$\begin{aligned}
\|\Phi(t,\omega)\mathbf{x}\|_\gamma &= \max\left\{\sup_{-\infty<s\leq -t}\mathrm{e}^{\gamma s}|\Phi(t,\omega)\mathbf{x}(s)|, \sup_{-t\leq s\leq 0}\mathrm{e}^{\gamma s}|\Phi(t,\omega)\mathbf{x}(s)|\right\}\\
&= \max\left\{\mathrm{e}^{-\gamma t}\|\mathbf{x}\|_\gamma, \sup_{-t\leq s\leq 0}\mathrm{e}^{\gamma s}|\varphi_\omega(t+s,\mathbf{x})|\right\}\\
&\leq \max\left\{\mathrm{e}^{-\gamma t}\|\mathbf{x}\|_\gamma, \mathrm{e}^{-\gamma t}\sup_{0\leq s\leq T}\mathrm{e}^{\gamma s}|\varphi_\omega(s,\mathbf{x})|, \mathrm{e}^{(\lambda-\varepsilon)t}\right\}.
\end{aligned}$$

This implies together with $-\gamma \leq \lambda - \varepsilon$ that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|\Phi(t,\omega)\mathbf{x}\|_\gamma \leq \lambda - \varepsilon.$$

This is a contradiction and we get the desired conclusion.

($\Leftarrow$) Assume that $\beta > -\gamma$ is a Lyapunov exponent of Ψ and let $x \in \mathrm{C}([-M,0],\mathbb{R}^d)$

be a vector corresponding to β, i.e.

$$\lim_{t\to\infty} \frac{1}{t} \log \|\Psi(t,\omega)x\| = \beta.$$

Define $\mathbf{x} : (-\infty, 0] \to \mathbb{R}^d$ by

$$\mathbf{x}(s) = \begin{cases} x(s), & \text{if } s \in [-M, 0], \\ x(-M), & \text{otherwise.} \end{cases}$$

Using similar arguments as in the first part of the proof, we also have

$$\lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}\|_\gamma = \beta.$$

Therefore, β is a Lyapunov exponent of Φ and the proof is completed.

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Beyond Diffusion: Conditional Dispersal in Ecological Models

Chris Cosner

Abstract Reaction-diffusion models have been widely used to describe the dynamics of dispersing populations. However, many organisms disperse in ways that depend on environmental conditions or the densities of other populations. Those can include advection along environmental gradients and nonlinear diffusion, among other possibilities. In this paper I will give a survey of some models involving conditional dispersal and discuss its effects and evolution. The presence of conditional dispersal can strongly influence the equilibria of population models, for example by causing the population to concentrate at local maxima of resource density. The analysis of the evolutionary aspects of dispersal is typically based on a study of models for two competing populations that are ecologically identical except for their dispersal strategies. The models consist of Lotka-Volterra competition systems with some spatially varying coefficients and with diffusion, nonlinear diffusion, and/or advection terms that reflect the dispersal strategies of the competing populations. The evolutionary stability of dispersal strategies can be determined by analyzing the stability of single-species equilibria in such models. In the case of simple diffusion in spatially varying environments it has been known for some time that the slower diffuser will exclude the faster diffuser, but conditional dispersal can change that. In some cases a population whose dispersal strategy involves advection along environmental gradients has the advantage or can coexist with a population that simply diffuses. As is often the case in reaction-diffusion theory, many of the results depend on the analysis of eigenvalue problems for linearized models.

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1 Introduction

Traditionally, spatially explicit models for population dynamics describe dispersal in terms of diffusion. Let $u(x,t)$ represent a population density at location x at time t, where $(x,t) \in \Omega \times (0,\infty)$, $\Omega \subset \mathbb{R}^n$, where Ω is a bounded domain. A typical model for a single population dispersing through a closed environment would take the form

$$\begin{aligned} u_t &= \mu \Delta u + f(x,u)u \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{1}$$

Diffusive movement is random and has no relation to the growth rate $f(x,u)$. In that sense it is "unconditional." Possible alternative or additional types of dispersal include a) taxis (directed movement upward along environmental gradients) and b) kinesis or area-restricted search (change in diffusion rate because of environmental conditions). These sorts of dispersal can be "conditional" in that they may depend on environmental conditions, including perhaps the local population density. (This terminology is taken from McPeek and Holt [15] .) A model including taxis and kinesis in a closed environment could be given as

$$\begin{aligned} u_t = \nabla \cdot [\mu(x,u)\nabla u - \alpha u \nabla e(x,u)] + f(x,u)u \\ \text{in } \Omega \times (0,\infty), \\ \mu \frac{\partial u}{\partial n} - \alpha u \frac{\partial \mathrm{e}}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{2}$$

We will give a survey of some recent work on conditional dispersal based on taxis and kinesis in population models. We will always assume that the coefficients, nonlinearities, and the underlying domain in (2) and other models are smooth, the domain Ω is bounded, and the diffusion coefficient μ is positive. We will focus our attention on cases where the local population growth rate is logistic and the model can be rescaled into a form where $f(x,u) = m(x) - u$, and we will consider dispersal processes where $\mathrm{e}(x,u)$ is constructed from $m(x)$ and perhaps u. The mathematical analysis is motivated by the question "Which forms of dispersal confer an advantage relative to other forms?" To decide whether a strategy is advantageous, the usual approach has been to use the idea of evolutionarily stable strategies. A strategy is said to be evolutionarily stable if a population using that strategy cannot be invaded by a small population of mutants using another strategy. Mathematically, this typically means that in some type of spatial population model the equilibrium where all of the population is using a given strategy is locally stable in the usual sense of dynamical systems. This approach naturally requires formulating the spatial model as a dynamical or semidynamical system. In the case of reaction-diffusion models and their generalizations, the theory of infinite dimensional dynamical systems as developed by Sell and others is an essential part of the necessary mathematical background for such a formulation.

Some modeling approaches, specifically that introduced by Hastings [14], use a single equation for the density of an invading mutant as the basic model. Others use a system of two equations for competing species that are ecologically identical but use different dispersal strategies, that is, where the reaction terms in the two equations are the same but the dispersal terms differ. That approach was used in numerical studies of discrete time two-patch models by McPeek and Holt [15] and in a rigorous mathematical treatment of reaction diffusion models by Dockery et al. [13]. The mathematical analysis of competition models usually requires a good understanding of single species equilibria. Thus, we will first look at single species models and then turn to models for two competitors.

2 Single-Species Models

2.1 General Background

For single species models such as (2) the instability of $u = 0$ typically implies the persistence of the population; see for example [4]. Stability means that extinction is possible. Instability or stability are typically determined by the sign of the principal eigenvalue for the problem

$$\nabla \cdot [\mu(x,0)\nabla\psi - \alpha\psi\nabla e(x,0)] + f(x,0))\psi = \sigma\psi \quad \text{in } \Omega,$$
$$\mu(x,0)\frac{\partial\psi}{\partial n} - \alpha\psi\frac{\partial e(x,0)}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{3}$$

An interesting special case occurs if $\mathrm{e}(x,u)$ equals $m(x)$ or $m(x) - u$, since in those cases the advection term describes movement in the direction of the gradient of the local growth rate, perhaps modified to allow for the effects of crowding. In that case (3) becomes

$$\nabla \cdot [\mu(x,0)\nabla\psi - \alpha\psi\nabla m(x)] + m(x)\psi = \sigma\psi \quad \text{in } \Omega,$$
$$\mu\frac{\partial\psi}{\partial n} - \alpha\psi\frac{\partial m}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{4}$$

In this situation it might be expected that increasing the factor α on the advective term would tend to promote persistence, since the advection would tend to move individuals to regions where the growth rate is higher. That is sometimes true, specifically when α is large or Ω is convex, but somewhat surprisingly it is not true in general.

Theorem 2.1 ([3]). *If $m(x) > 0$ on some open set and α (the rate of movement up the gradient of the growth rate) is sufficiently large in (4) , then $u = 0$ is unstable .*

Theorem 2.2 ([12]). *For convex domains, increasing α from $\alpha = 0$ in (4) can make $u = 0$ unstable if it is stable but cannot make it stable if it is unstable, so increasing α from 0 favors persistence. This is not always true in general domains.*

Some mathematical notes:

- The substitution $\phi = \mathrm{e}^{-(\alpha/\mu)m(x)}\psi$ converts (4) into a variational form with Neumann boundary conditions which can be studied via classical methods.

- The principal eigenvalue in (4) can be seen to depend differentiably on α by the Implicit Function Theorem.

2.2 Evolution of Dispersal

A key background idea in connecting population models to evolutionary theory is the idea of evolutionarily stable (or unstable) strategies. A strategy is evolutionarily stable if a population using it cannot be invaded by any small population using a different strategy. Such strategies are the ones that can be expected to persist under the process of natural selection. A simple approach to modeling a situation where a small population is invading an established resident population was introduced by Hastings [14]. Suppose that u^* is a stable positive equilibrium of

$$\begin{aligned} u_t &= D\nabla \cdot [\mu(x)\nabla u] + f(x,u)u \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{5}$$

Interpret the equilibrium u^* as the density of a resident population. Let v denote the density of a small invading population. Assume that the invading population is so small that its effect on the resident population is negligible. Model the invading population with the equation

$$\begin{aligned} v_t = d\nabla \cdot [\mu(x)\nabla v] + f(x,u^*+v)v \\ \text{in } \Omega \times (0,\infty), \\ \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{6}$$

Theorem 2.3 ([14]). *If $f(x,u^*)$ is not identically 0 then the equilibrium $v \equiv 0$ in (6) is unstable (in the sense that v can increase when $v(x,0) > 0$ is small) if and only if $d < D$.*

The proof of this result is based on applying an eigenvalue comparison result to a linearized model analogous to (4). When Hastings' result applies there should be selection for lower rates of movement. Recall that Hastings' result requires that $f(x,u^*)$ not be identically 0. Integrating the equilibrium equation for (5) shows

$$\int_\Omega f(x,u^*)u^*\mathrm{d}x = 0.$$

It follows that $f(x,u^*)$ must be positive some places and negative others for Hastings' result to apply.

Biological interpretation: If $f(x,u^*)$ changes sign then the dispersal strategy embodied by the term $D\nabla\cdot[\mu(x)\nabla u]$ causes the population to overmatch the available resources in some places but undermatch them elsewhere. This suggests that dispersal strategies that tend to match population densities to resource levels might have an advantage over those that do not. In the case where $f(x,u) = m(x) - u$ and μ is constant, $u^* \to m(x)$ as $D \to 0$ (see for example [4, Sects. 3.5.3]) so a smaller diffusion rate allows a population to match resource levels more accurately.

3 Two-Species Models

3.1 Systems with Simple Diffusion

A limitation of Hastings' approach is that it does not explicitly model the interactions of the resident and invading populations. A reaction-diffusion system that does that was introduced by Dockery et al. [13] :

$$\begin{aligned} u_t &= \mu\Delta u + [m(x) - u - v]u \quad \text{in } \Omega\times(0,\infty), \\ v_t &= \nu\Delta v + [m(x) - u - v]v \quad \text{in } \Omega\times(0,\infty), \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega\times(0,\infty). \end{aligned} \tag{7}$$

In this model u and v represent densities of populations that are identical except for their dispersal strategies. Dockery et al. [13] proved

Theorem 3.1 ([13]). *If $\mu < \nu$ and (u,v) is a nonnegative solution of (7) with $u(x,0) > 0$ on an open set, then $v \to 0$ as $t \to \infty$.*

Again, the slower diffuser wins.

The system (7) has all the usual features of reaction-diffusion models for two competing species. Specifically, it is monotone with respect to the ordering

$$(u_1,v_1) \le (u_2,v_2) \Longleftrightarrow u_1 \le u_2 \text{ and } v_1 \ge v_2.$$

Hence, if the model has semi-trivial equilibria $(u^*,0)$ and $(0,v^*)$ which are both unstable it will predict coexistence (in the sense that there will exist a positive equilibrium and the dynamical model will be permanent), while if one is unstable and there are no positive equilibria then the competitor with the unstable equilibrium will be driven to extinction. If $m(x)$ is positive then the semi-trivial equilibria will exist and will be unique. The semi-trivial equilibrium $(u^*,0)$ will be unstable if the principal eigenvalue is positive in the problem

$$\nu\Delta\psi + (m(x) - u^*)\psi = \sigma\psi \quad \text{in } \Omega,$$
$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{8}$$

(Analogous results hold for similar systems where the dispersal terms are given by linear elliptic operators more general that the Laplacian.)

We will now sketch of analysis from [13]: The principal eigenvalue in the problem (8) is given by

$$\sigma_0 = sup\,\frac{-\nu\int_\Omega |\nabla\psi|^2 dx + \int_\Omega m(x)\psi^2 dx}{\int_\Omega \psi^2 dx}, \tag{9}$$

where the sup is taken over $W^{1,2}(\Omega)$.

Since $u^* > 0$ satisfies

$$\mu\Delta u^* + (m(x) - u^*)u^* = 0 \quad \text{in } \Omega,$$
$$\frac{\partial u^*}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{10}$$

the principal eigenvalue for (8) with $\nu = \mu$ is 0, with the eigenfunction being a multiple of u^*. In view of (9), it follows that the principal eigenvalue in (8) is decreasing in ν, so it is positive if $\nu < \mu$, and thus $(u^*,0)$ is unstable in that case. If $\nu > \mu$ the principal eigenvalue is negative and $(u^*,0)$ is stable. If $\nu \neq \mu$ the existence of a positive equilibrium (u^{**},v^{**}) can be ruled out on the basis of related eigenvalue comparisons. The remaining conclusions follow from the general features of models for two competitors.

3.2 Models with Taxis up Environmental Gradients

A model analogous to (7) but allowing for directed movement up the gradient of the local growth rate $m(x)$ is

$$u_t = \nabla\cdot[\mu\nabla u - \alpha u\nabla m] + [m(x) - u - v]u$$
$$v_t = \nabla\cdot[\nu\nabla v - \beta v\nabla m] + [m(x) - u - v]v$$

$$\text{in } \Omega \times (0,\infty),$$
$$\mu \frac{\partial u}{\partial n} - \alpha u \frac{\partial m}{\partial n} = \nu \frac{\partial v}{\partial n} - \beta v \frac{\partial m}{\partial n} = 0$$
$$\text{on } \partial\Omega \times (0,\infty). \tag{11}$$

This system still has all the usual features of reaction-diffusion models for two competing species. If $m(x)$ is positive (and in some situations where $m(x)$ changes sign) semi-trivial equilibria $(u^*,0)$ and $(0,v^*)$ will exist, and each will be unique if it exists. Again, the stability properties of the semi-trivial equilibria largely determine the model's predictions of whether the competitors can coexist or which competitor has the advantage. The semi-trivial equilibrium $(u^*,0)$ of (11) will be unstable if the principal eigenvalue is positive in the problem

$$\nabla \cdot [\nu \nabla \psi - \beta \psi \nabla m(x)] + (m(x) - u^*)\psi = \sigma \psi$$
$$\text{in } \Omega,$$
$$\nu \frac{\partial \psi}{\partial n} - \beta \psi \frac{\partial m}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{12}$$

Similarly, $(0,v^*)$ will be unstable if the principal eigenvalue is positive in the problem

$$\nabla \cdot [\mu \nabla \phi - \alpha \phi \nabla m(x)] + (m(x) - v^*)\phi = \tau \phi$$
$$\text{in } \Omega,$$
$$\mu \frac{\partial \phi}{\partial n} - \alpha \phi \frac{\partial m}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{13}$$

The effects of small amounts of taxis up the gradient of $m(x)$ when competitors are similar were studied in [6] by perturbation methods. Start with parameters $\mu = \nu = \mu_0, \alpha = 0, \beta = 0$ in (11) then maintain $\beta = 0$ but perturb the other parameters as

$$\mu = \mu_0 + s\mu_1 + o(s), \ \nu = \mu_0 + s\nu_1 + o(s),$$
$$\alpha = \alpha_1 s + o(s).$$

At $s = 0$ the equations for u and v are the same so that $u^* = v^* = \theta$ for some positive function θ, and the principal eigenvalues $\sigma_0(0)$ and $\tau_0(0)$ in (12) and (13) are both 0. We have the following:

Theorem 3.2 ([6]). *For small s,* $\sigma_0(s) = \sigma_1 s + o(s)$, $\tau_0(s) = \tau_1 s + o(s)$, *where* σ_1 *and* τ_1 *can be computed as*

$$\tau_1 = -\sigma_1$$
$$= \frac{(\nu_1 - \mu_1)\int_\Omega |\nabla\theta|^2 \mathrm{d}x + \alpha_1 \int_\Omega \theta \nabla\theta \cdot \nabla m \mathrm{d}x}{\int_\Omega \theta^2 \mathrm{d}x}. \tag{14}$$

If the underlying domain Ω is convex, the second integral in the numerator is positive. (This uses some classical PDE arguments.) If there is no taxis then the slower diffuser has the advantage. In particular, if $\mu_1 > \nu_1$ then with no taxis ($\alpha_1 = 0$) the second competitor can invade when the first is resident, but not vice-versa, so the second (more slowly diffusing) competitor has the advantage as in [13]. However, if $\alpha_1 > 0$ is large enough, then for small positive s the effects of taxis can overcome those of diffusion, so even if $\mu_1 > \nu_1$ the first competitor has the advantage.

This gives some insight about what happens with weak taxis (small α). What about strong taxis? That case was studied in [7]. A key point is to establish the behavior of the single species equilibrium u^* as $\alpha \to \infty$. The term u^* in the semi-trivial equilibrium $(u^*, 0)$ satisfies

$$\nabla \cdot [\mu \nabla u^* - \alpha u^* \nabla m(x)] + (m(x) - u^*)u^* = 0$$

$$\text{in } \Omega$$

$$\mu \frac{\partial u^*}{\partial n} - \alpha u^* \frac{\partial m}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{15}$$

Integrating the equation over Ω yields

$$\|u^*\|_2^2 = \int_\Omega |u^*|^2 \mathrm{d}x = \int_\Omega m(x)u^* \mathrm{d}x \leq \|m\|_2 \| \|u^*\|_2,$$

so that u^* is bounded in $L^2(\Omega)$ and hence in $L^1(\Omega)$ since Ω is bounded. Suppose that $\partial m/\partial n = 0$ on $\partial\Omega$. (This restriction can be removed.) Multiplying (15) by $m(x)$, integrating by parts, and dividing by α yields

$$\begin{aligned} 0 &= \mu \int_\Omega m \nabla \cdot \nabla u^* \mathrm{d}x - \alpha \int_\Omega m \nabla \cdot u^* \nabla m \mathrm{d}x + \int_\Omega m(m - u^*)u^* \mathrm{d}x \\ &= (\mu/\alpha) \int_\Omega u^* \Delta m \mathrm{d}x + \int_\Omega u^* |\nabla m|^2 \mathrm{d}x + (1/\alpha) \int_\Omega m(m - u^*)u^* \mathrm{d}x. \end{aligned}$$

It follows that as $\alpha \to \infty$,

$$\int_\Omega u^* |\nabla m|^2 \mathrm{d}x \to 0.$$

If the set of critical points of m has measure 0, then $u^* \to 0$ a.e. This suggests that u^* concentrates at critical points of m. That has been shown in certain cases [9]. In general, having $u^* \to 0$ a.e. can lead to coexistence.

Theorem 3.3 ([7]). *Suppose that $m(x)$ is positive, nonconstant, and has an isolated maximum at some point $x_0 \in \Omega$. For any fixed $\mu > 0$ the system (13) with $\beta = 0$, ν bounded below, and α sufficiently large is permanent.*

(Permanence implies the coexistence of the two populations in the model.)

Sketch of Proof. Recall that $(u^*,0)$ is unstable if $\sigma_0 > 0$ in (12), where in this case $\beta = 0$. Dividing (12) by ψ and integrating by parts yields

$$\nu \int_\Omega \frac{|\nabla \psi|^2}{\psi^2} \mathrm{d}x + \int_\Omega (m(x) - u^*) \mathrm{d}x = \sigma |\Omega|.$$

Since $u^* \to 0$ as $\alpha \to \infty$, it follows that for large α the semi-trivial equilibrium $(u^*,0)$ is unstable. For the stability of the semi-trivial equilibrium $(0,v^*)$ we would consider (13). By the maximum principle we have $\max v^* < \max m$. Using the new variable $\rho(x) = \mathrm{e}^{-(\alpha/\mu)m(x)}\phi(x)$ lets us convert (13) into a variational form. It turns out that using a test function that is concentrated near x_0 shows that for α large we have $\tau_0 > 0$ so $(0,v^*)$ is unstable. Since both semi-trivial states are unstable, the model (11) with $\beta = 0$ predicts coexistence for large α .

In the general case of (11), recent work by Chen, Hambrock, and Lou [10] shows that if β is small and $\alpha > \beta$ then the situation is similar to the case where $\beta = 0$. However, if β is large but $\alpha > \beta$ and α is large enough then the second competitor has the advantage. Also, for the case $\alpha = \beta$, if α and β are small the competitor with the smaller diffusion rate has the advantage, while if α and β are large the competitor with the faster diffusion rate has the advantage. The intuition is that if the competitors differ in their dispersal so that one competitor concentrates near resource peaks but the other is broadly spread there will be a type of spatial segregation that promotes coexistence but if one competitor can match the available resources more closely than the other it has an advantage. This last idea returns us to the old work by Hastings [14] and suggests connections with the ideal free distribution.

4 The Ideal Free Distribution

4.1 Definition and Connections to Dispersal Models

The ideal free distribution is a verbal ecological theory of how organisms would ideally locate themselves if they were free to choose their location. The key idea is that if an individual can increase its fitness by moving to another location, it will do so. This has certain consequences:

- At equilibrium organisms at all locations have equal fitness (otherwise some would move to improve theirs).
- At equilibrium there is no net movement of individuals.

In the context of a population model such as (2) the first of these means that at an equilibrium u^* we should have

$$f(x,u^*) = \text{constant};$$

the second means we should have

$$\nabla \cdot [\mu(x,u^*)\nabla u^* - \alpha u^* \nabla \mathrm{e}(x,u^*)] = 0.$$

These are not compatible in most models; however, they may be in certain models, or may asymptotically become compatible in certain limits. Recall that Hastings showed that if an equilibrium u^* of (6) has $f(x,u^*) \neq 0$ then a small population with a lower dispersal rate can invade the system. However, positive equilibria for the case $D = 0$ must have $f(x,u^*) = 0$, and at least in the case of constant diffusion ($\mu(x) =$ constant) and logistic growth ($f(x,u) = m(x) - u$), we have $u^* \to m(x)$ as $D \to 0$, so that asymptotically solutions approach an ideal free distribution as $D \to 0$ (see [4, Sects. 3.5.3]). Thus, in that case there is selective pressure toward dispersal strategies that conform to the ideal free distribution. However, if a strategy leads to an equilibrium that conforms to the ideal free condition $f(x,u^*) = 0$ then the analysis behind Hastings' result no longer applies. It is natural to ask if models can approximate the ideal free distribution without having the dispersal rate go to zero. One possibility would be to have the dispersal strategy involve moving up the gradient of fitness. Suppose $f(x,u) = m(x) - u$. Such a model is developed in [11]; it has the form

$$\begin{aligned} u_t &= \nabla \cdot [-\alpha u \nabla (m(x) - u)] + (m(x) - u)u \\ &= \alpha \nabla \cdot u \nabla u - \alpha \nabla \cdot u \nabla m + (m(x) - u)u \\ &\qquad \text{in } \Omega \times (0,\infty), \\ & u\left(\frac{\partial u}{\partial n} - \frac{\partial m}{\partial n}\right) = 0 \text{ on } \partial\Omega \times (0,\infty). \end{aligned} \tag{16}$$

This resembles a porous medium equation with an extra advective term. It supports $u^* = m(x)$ as an equilibrium that satisfies the ideal free properties, but it assumes that there is no ordinary diffusion term and it is degenerate as a parabolic equation. Adding some diffusion yields

$$\begin{aligned} u_t &= \nabla \cdot [\mu \nabla u - \alpha u \nabla (m(x) - u)] + (m(x) - u)u \\ &\qquad \text{in } \Omega \times (0,\infty), \end{aligned} \tag{17}$$

$$\mu \frac{\partial u}{\partial n} + \alpha u \left(\frac{\partial u}{\partial n} - \frac{\partial m}{\partial n}\right) = 0 \quad \text{on } \partial\Omega \times (0,\infty).$$

This model can be seen to be well posed by results of Amann [1, 2] .

4.2 *Results on an Approximately Ideal Free Model*

The model (17) is analyzed in [8]. Making the change of variables

$$w = u\mathrm{e}^{-(\alpha/\mu)(m-u)}$$

with inverse denoted as $u = h(x,w)$ converts (17) into

$$\begin{aligned} w_t &= (h_w(x,w))^{-1}[\mu e^{(\alpha/\mu)(m-h)}\Delta w \\ &\quad +\alpha e^{(\alpha/\mu)(m-h)}\nabla(m-h)\cdot\nabla w + h(m-h)] \\ &\quad \text{in } \Omega\times(0,\infty), \\ \frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial\Omega\times(0,\infty). \end{aligned} \tag{18}$$

Equation (18) has a Neumann boundary condition and satisfies a comparison principle. which leads to *a priori* bounds. If 0 is unstable and ϕ the principal eigenfunction of the linearized problem for (17) at $u = 0$ then $\varepsilon e^{-(\alpha/\mu)}m)\phi$ is a subsolution of the equilibrium problem for (18) for small ε so there exist minimal and maximal positive equilibria. (In general we do not know if they are the same.)

Theorem 4.1 ([8]). *For any positive equilibrium u of (17), $u \to m_+$ weakly in H^1 and strongly in L^2 as $\alpha/\mu \to \infty$.*

Sketch of Proof. We obtain *a priori* estimates on the equilibria of (17) partly by using another change of variables and using Nash-DiGiorgi type estimates. By writing the equilibrium equation in terms of w, dividing by w, and integrating by parts we obtain after some calculations

$$\int_\Omega (m-u)_+ \mathrm{d}x \le \frac{\mu}{\alpha}|\Omega|. \tag{19}$$

By integrating the equilibrium equation for u we obtain

$$\int_\Omega u(m-u)\mathrm{d}x = 0. \tag{20}$$

The *a priori* estimates imply that for any sequence of $\alpha/\mu \to \infty$ there is a subsequence so that the corresponding equilibria satisfy $u \to u^*$ in L^2 for some u^*. It follows from (19) that

$$\int_\Omega (m-u^*)_+ \mathrm{d}x = 0$$

so $u^* \ge m_+$ a.e. It then follows from (20) with u replaced by u^* that $u^* = m_+$ a.e. Hence, any sequence of values of α/μ there is a subsequence of corresponding equilibria that converges to m_+. Since all such subsequences converge to the same limit it follows that $u \to m_+$ as $\alpha/\mu \to \infty$. (If α is bounded below then the convergence is in C^γ for some $\gamma \in (0,1)$; if in addition $m > 0$ the convergence is in C^2.)

Theorem 4.1 has various consequences, including the uniqueness and stability of the positive equilibrium when the rate α of advection up the fitness gradient is large. Suppose $m(x) > 0$. If u_1 and u_2 are positive equilibria with u_2 minimal we have (by integrating over Ω)

$$\int_\Omega u_1(m-u_1)\mathrm{d}x = \int_\Omega u_2(m-u_2)\mathrm{d}x = 0$$

so that

$$\int_\Omega (u_1-u_2)(m-u_1-u_2) = 0. \tag{21}$$

Since $u_1 \to m$ and $u_2 \to m$ uniformly as $\alpha/\mu \to \infty$, we see that $m - u_1 - u_2 \to -m$ uniformly. Since m is strictly positive , for large α/μ we have $m - u_1 - u_2 < 0$. This along with $u_1 \geq u_2$ and $u_1 \not\equiv u_2$ contradicts (21), so we must have $u_1 = u_2$.

The linearization of the ideal free type model (17) around an equilibrium is

$$\nabla \cdot [\mu \nabla \phi - \alpha \phi \nabla (m-u) + \alpha u \nabla \phi] + (m-2u)\phi = \sigma\phi \qquad \text{in } \Omega,$$

$$[\mu \nabla \phi - \alpha \phi \nabla (m-u) + \alpha u \nabla \phi] \cdot n = 0 \quad \text{on } \partial\Omega.$$

This can be converted to a form that allows use of the maximum principle by a change of variables, which implies the existence of a principal eigenvalue σ_0 with positive eigenfunction ϕ_0. Integrating over Ω gives

$$\sigma_0 \int \phi_0 dx = \int_\Omega (m-2u)\phi_0 aq \mathrm{d}x.$$

If $m > 0$ it follows that for α/μ large we have $\sigma_0 < 0$ so that the equilibrium is stable.

4.3 Remarks

Approximately ideal free dispersal seems to allow organisms to track their resources (as described by the growth rate $m(x)$) accurately as $\alpha/\mu \to \infty$ so we expect that there will be selection for it in some cases. To examine that topic we would need to consider models analogous to (3), which would have the general form

$$\begin{aligned}
&u_t = \nabla \cdot [\mu \nabla u - \alpha u \nabla f(x,u+v)] + u f(x,u+v)\\
&v_t = \nabla \cdot [\nu \nabla v - \beta v \nabla g(x,u+v)] + v f(x,u+v)\\
&\text{in } \Omega \times (0,\infty),\\
&\mu \frac{\partial u}{\partial n} - \alpha u \frac{\partial f(x,u+v)}{\partial n} = \nu \frac{\partial v}{\partial n} - \beta v \frac{\partial g(x,u+v)}{\partial n} = 0\\
&\text{on } \partial\Omega \times (0,\infty).
\end{aligned}$$

Such models are challenging because they are strongly coupled and quasilinear, somewhat along the lines of cross-diffusion models. They present many issues which are the subject of ongoing research. Even in the case of a single equation there are many open questions, for example the uniqueness or multiplicity of equilibria when α/μ is not large.

An alternative approach to ideal free dispersal would be to look for specific forms of linear diffusion and advection with spatially varying coefficients that would admit ideal free solutions. It turns out that this is possible in the case of discrete diffusion models, and in that setting the ideal free property is usually necessary and sometimes sufficient for evolutionary stability [5]. Extending that approach to the infinite dimensional case is another direction of current research.

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Global Attractor of a Coupled Two-Cell Brusselator Model

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Dedicated to George R. Sell on the Occasion of His 70th Birthday

Abstract In this work the existence of a global attractor for the solution semi-flow of the coupled two-cell Brusselator model equations is proved. A grouping estimation method and a new decomposition approach are introduced to deal with the challenges in proving the absorbing property and the asymptotic compactness of this type of four-variable reaction-diffusion systems with cubic autocatalytic nonlinearity and with linear coupling. It is also proved that the Hausdorff dimension and the fractal dimension of the global attractor are finite.

Mathematics Subject Classification (2010): Primary 37L30, 35B40, 35K55, 35Q80; Secondary 80A32, 92B05

1 Introduction

Consider a coupled two-cell model of cubic autocatalytic reaction-diffusion systems with Brusselator kinetics [18, 20, 31],

$$\frac{\partial u}{\partial t} = d_1 \Delta u + a - (b+1)u + u^2 v + D_1(w - u), \tag{1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + bu - u^2 v + D_2(z - v), \tag{2}$$

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$$\frac{\partial w}{\partial t} = d_1 \Delta w + a - (b+1)w + w^2 z + D_1(u - w), \tag{3}$$

$$\frac{\partial z}{\partial t} = d_2 \Delta z + bw - w^2 z + D_2(v - z), \tag{4}$$

for $t > 0$, on a bounded domain $\Omega \subset \Re^n, n \leq 3$, that has a locally Lipschitz continuous boundary, with the homogeneous Dirichlet boundary condition

$$u(t,x) = v(t,x) = w(t,x) = z(t,x) = 0, \qquad t > 0, \;\; x \in \partial\Omega, \tag{5}$$

and an initial condition

$$u(0,x) = u_0(x),\; v(0,x) = v_0(x),\; w(0,x) = w_0(x),\; z(0,x) = z_0(x), \qquad x \in \Omega, \tag{6}$$

where d_1, d_2, a, b, D_1, and D_2 are positive constants. We do not assume that the initial data u_0, v_0, w_0, z_0, nor the solutions $u(t,x), v(t,x), w(t,x), z(t,x)$ to be nonnegative. In this work, we shall study the asymptotic dynamics of the solution semiflow generated by this problem.

The Brusselator is originally a system of two ordinary differential equations as a model for cubic autocatalytic chemical or biochemical reactions, cf. [2, 25, 35]. The name is after the home town of scientists who proposed it. Brusselator kinetics describes the following scheme of chemical reactions

$$\mathrm{A} \longrightarrow \mathrm{U}, \qquad \mathrm{B} + \mathrm{U} \longrightarrow \mathrm{V} + \mathrm{D},$$

$$2\mathrm{U} + \mathrm{V} \longrightarrow 3\mathrm{U}, \qquad \mathrm{U} \longrightarrow \mathrm{E},$$

where A and B are reactants, D and E are products, U and V are intermediate substances. Let $u(t,x)$ and $v(t,x)$ be the concentrations of U and V, and assume that the concentrations of the input compounds A and B are held constant during the reaction process, denoted by a and b respectively. Then one obtains a system of two nonlinear reaction-diffusion equations called (diffusive) *Brusselator equations*,

$$\frac{\partial u}{\partial t} = d_1 \Delta u + u^2 v - (b+1)u + a, \tag{7}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - u^2 v + bu. \tag{8}$$

There are several known examples of autocatalysis which can be modeled by the Brusselator equations, such as ferrocyanide-iodate-sulphite reaction, chlorite-iodide-malonic acid reaction, arsenite-iodate reaction, some enzyme catalytic reactions, and fungal mycelia growth, cf. [1, 2, 4, 9].

Numerous studies by numerical simulations or by mathematical analysis, especially after the publications [22, 23] in 1993, have shown that the autocatalytic reaction-diffusion systems such as the Brusselator equations and the Gray-Scott equations [13, 14] exhibit rich spatial patterns (including but not restricted to Turing patterns) and complex bifurcations [1, 3, 4, 7, 8, 11, 12, 16, 24, 26, 32, 38] as well as interesting dynamics [5, 9, 10, 17, 21, 27–29, 39, 40] on 1D or 2D domains.

For Brusselator equations and the other cubic autocatalytic model equations of space dimension $n \leq 3$, however, we have not seen substantial research results in the front of global dynamics until recently this author proved the existence of a global attractor for Brusselator equations [41], Gray-Scott equations [42], Selkov equations [33, 43], and the reversible Schnackenberg equations [30, 44].

In this paper, we shall show the existence of a global attractor in the product L^2 phase space for the solution semiflow of the coupled two-cell Brusselator equations (1)–(4) with homogeneous Dirichlet boundary conditions (5).

This study of global dynamics of the two-cell model of four coupled components is a substantial advance from the one-cell model of two-component reaction-diffusion systems toward the biological network dynamics [12, 19]. Multi-cell models generically mean the coupled ODEs or PDEs with large number of unknowns (components), which appear widely in the literature of systems biology as well as cell biology. Here understandably "cell" is a generic term that may not be narrowly or directly interpreted as a biological cell. Coupled cells with diffusive reaction and mutual mass exchange are often adopted as model systems for description of processes in living cells and tissues, or in distributed chemical reactions and transport for compartmental reactors [31, 37]. The mathematical analysis combined with semi-analytical simulations seems to become a common approach to understanding the complicated molecular interactions and signaling pathways in many cases.

In this regard, unfortunately, the problem with high dimensionality can arise and puzzle the research when the number of molecular species in the system turns out to be very large, which makes the behavior simulation extremely difficult or computationally too expensive. Thus theoretical results on multi-cell model dynamics can give insights to deeper exploration of various signal transductions and tempro-spatial pattern formations.

For most reaction-diffusion systems consisting of two or more equations arising from the scenarios of autocatalytic chemical reactions or biochemical activator-inhibitor reactions, such as the Brusselator equations and the coupled two-cell Brusselator equations here, the asymptotically dissipative sign condition in vector version

$$\lim_{|s|\to\infty} F(s)\cdot s \leq C,$$

where $C \geq 0$ is a constant, is inherently not satisfied by the opposite-signed and coupled nonlinear terms, see (11) later. Besides a serious challenge in dealing with this coupled two-cell model is that, due to the coupling of the two groups of variables u, v and w, z, one can no longer make a dissipative *a priori* estimate on the v-component by using the v-equation separately and then use the sum $y(t,x) = u(t,x) + v(t,x)$ separately to estimate the u-component in proving absorbing property and in proving asymptotical compactness of the solution semiflow as we did in [41–43]. The novel mathematical feature in this paper is to overcome this coupling obstacle and make the *a priori* estimates by a method of *grouping estimation* combined with a new decomposition approach.

We start with the formulation of an evolutionary equation associated with the two-cell Brusselator equations. Define the product Hilbert spaces as follows,

$$H = [L^2(\Omega)]^4, \quad E = [H_0^1(\Omega)]^4, \quad \Pi = [H_0^1(\Omega) \cap H^2(\Omega)]^4.$$

The norm and inner-product of H or the component space $L^2(\Omega)$ will be denoted by $\|\cdot\|$ and $\langle \cdot,\cdot \rangle$, respectively. The norm of $L^p(\Omega)$ will be denoted by $\|\cdot\|_{L^p}$ if $p \neq 2$. By the Poincar*é* inequality and the homogeneous Dirichlet boundary condition (5), there is a constant $\gamma > 0$ such that

$$\|\nabla \varphi\|^2 \geq \gamma \|\varphi\|^2, \quad \text{for } \varphi \in H_0^1(\Omega) \text{ or } E, \tag{9}$$

and we shall take $\|\nabla \varphi\|$ for the equivalent norm of the space E and of the component space $H_0^1(\Omega)$. We use $|\cdot|$ to denote an absolute value or a vector norm in a Euclidean space.

It is easy to check that, by the Lumer–Phillips theorem and the analytic semigroup generation theorem [34], the linear operator

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 & 0 \\ 0 & d_2\Delta & 0 & 0 \\ 0 & 0 & d_1\Delta & 0 \\ 0 & 0 & 0 & d_2\Delta \end{pmatrix} : D(A)(=\Pi) \longrightarrow H \tag{10}$$

is the generator of an analytic C_0-semigroup on the Hilbert space H, which will be denoted by $e^{At}, t \geq 0$. By the fact that $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous embedding for $n \leq 3$ and using the generalized Hölder inequality,

$$\|u^2 v\| \leq \|u\|_{L^6}^2 \|v\|_{L^6}, \quad \|w^2 z\| \leq \|w\|_{L^6}^2 \|z\|_{L^6}, \quad \text{for } u,v,w,z \in L^6(\Omega),$$

one can verify that the nonlinear mapping

$$F(g) = \begin{pmatrix} a-(b+1)u+u^2v+D_1(w-u) \\ bu-u^2v+D_2(z-v) \\ a-(b+1)w+w^2z+D_1(u-w) \\ bw-w^2z+D_2(v-z) \end{pmatrix} : E \longrightarrow H, \tag{11}$$

where $g = (u,v,w,z)$, is well defined on E and locally Lipschitz continuous. Then the initial-boundary value problem (1)–(6) is formulated into the following initial value problem,

$$\frac{\mathrm{d}g}{\mathrm{d}t} = Ag + F(g), \quad t > 0, \tag{12}$$

$$g(0) = g_0 = \mathrm{col}\,(u_0, v_0, w_0, z_0),$$

where $g(t) = \text{col}\,(u(t,\cdot), v(t,\cdot), w(t,\cdot), z(t,\cdot))$, simply written as $(u(t,\cdot), v(t,\cdot), w(t,\cdot), z(t,\cdot))$. Accordingly we shall write $g_0 = \text{col}\,(u_0, v_0, w_0, z_0)$.

By conducting *a priori* estimates on the Galerkin approximate solutions of the initial value problem (12) and the weak/weak* convergence argument, we can prove the local existence and uniqueness of the weak solution $g(t)$ of (12) in the sense given in [6, Chapter II and Chapter XV], which turns out to be a local strong solution for $t > 0$, cf. [34, Theorem 46.2]. Moreover, one can prove the continuous dependence of the solutions on the initial data and the following property,

$$g \in C([0, T_{max}); H) \cap C^1((0, T_{max}); H) \cap L^2(0, T_{max}; E), \tag{13}$$

where $I_{max} = [0, T_{max})$ is the maximal interval of existence.

We refer to [15, 34, 36] and many references therein for the concepts and basic facts in the theory of infinite dimensional dynamical systems, including few given below for clarity.

Definition 1.1. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space $\mathscr{X}$. A bounded subset B_0 of $\mathscr{X}$ is called an *absorbing set* in $\mathscr{X}$ if, for any bounded subset $B \subset \mathscr{X}$, there is some finite time $t_0 \geq 0$ depending on B such that $S(t)B \subset B_0$ for all $t > t_0$.

Definition 1.2. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space $\mathscr{X}$. A subset $\mathscr{A}$ of $\mathscr{X}$ is called a *global attractor* for this semiflow, if the following conditions are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in the sense that

$$S(t)\mathscr{A} = \mathscr{A} \quad \text{for any } \ t \geq 0.$$

(ii) $\mathscr{A}$ attracts any bounded set B of $\mathscr{X}$ in terms of the Hausdorff distance, i.e.

$$\text{dist}(S(t)B, \mathscr{A}) = \sup_{x\in B} \inf_{y\in\mathscr{A}} \|S(t)x - y\|_{\mathscr{X}} \to 0, \quad \text{as } t \to \infty.$$

Definition 1.3. A semiflow $\{S(t)\}_{t\geq 0}$ on a Banach space $\mathscr{X}$ is called *asymptotically compact* if for any bounded sequences $\{x_n\}$ in $\mathscr{X}$ and $\{t_n\} \subset (0, \infty)$ with $t_n \to \infty$, there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$, such that $\lim_{k\to\infty} S(t_{n_k})x_{n_k}$ exists in $\mathscr{X}$.

Here is the main result of this paper. We emphasize that this result is established unconditionally, neither assuming initial data or solutions are nonnegative, nor imposing any restriction on any positive parameters involved in (1)–(4).

Theorem 1.4 (Main Theorem). *For any positive parameters d_1, d_2, a, b, D_1, and D_2, there exists a global attractor $\mathscr{A}$ in the phase space H for the solution semiflow $\{S(t)\}_{t\geq 0}$ generated by the coupled two-cell Brusselator evolutionary equation* (12).

Kuratowski measure of noncompactness for bounded sets in a Banach space $\mathscr{X}$ is defined by

$$\kappa(B) \stackrel{\text{def}}{=} \inf\{\delta : B \text{ has a finite cover by open sets in } \mathcal{X} \text{ of diameters} < \delta\}.$$

If B is an unbounded set, then define $\kappa(B) = \infty$. The basic properties of the Kuratowski measure are listed here, cf. [34, Lemma 22.2],

(i) $\kappa(B) = 0$ if and only if B is precompact in $\mathcal{X}$, i.e. $Cl_{\mathcal{X}} B$ is a compact set in $\mathcal{X}$.
(ii) $\kappa(B_1) \leq \kappa(B_2)$ whenever $B_1 \subset B_2$.
(iii) $\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2)$, for any linear sum $B_1 + B_2$.

The following lemma states concisely the basic result on the existence of a global attractor for a semiflow and provides the connection of the κ-contracting concept to the asymptotical compactness, cf. [34, Chapter 2].

Lemma 1.5. *Let $\{S(t)\}_{t \geq 0}$ be a semiflow on a Banach space $\mathcal{X}$. If the following conditions are satisfied*:

(i) $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set in $\mathcal{X}$, and
(ii) $\{S(t)\}_{t \geq 0}$ is κ-contracting, i.e. $\lim_{t \to \infty} \kappa(S(t)B) = 0$ for any bounded set $B \subset \mathcal{X}$,

then $\{S(t)\}_{t \geq 0}$ is asymptotically compact and there exists a global attractor $\mathscr{A}$ in $\mathcal{X}$ for this semiflow. The global attractor is given by

$$\mathscr{A} = \omega(B_0) \stackrel{\text{def}}{=} \bigcap_{\tau \geq 0} Cl_{\mathcal{X}} \bigcup_{t \geq \tau} (S(t)B_0).$$

In Sect. 2, we shall prove the global existence of the weak solutions of the two-cell Brusselator evolutionary equation (12) and the absorbing property of this coupled Brusselator semiflow. In Sect. 3, a new decomposition technique is presented to deal with the asymptotic compactness issue of this problem and the κ-contracting property for the (v,z) components is proved. In Sect. 4, we treat the asymptotic compactness for the (u,w) components. Then we assemble these results to prove the existence of a global attractor in the phase space H for the coupled Brusselator semiflow. In Sect. 5, we show that the global attractor has a finite Hausdorff dimension and a finite fractal dimensions.

2 Absorbing Property

In this paper, we shall write $u(t,x), v(t,x), w(t,x)$, and $z(t,x)$ simply as $u(t)$, $v(t), w(t)$, and $z(t)$, or even as u, v, w, and z, and similarly for other functions of (t,x).

Lemma 2.1. *For any initial data $g_0 = (u_0, v_0, w_0, z_0) \in H$, there exists a unique, global, weak solution $g(t) = (u(t), v(t), w(t), z(t))$, $t \in [0, \infty)$, of the coupled Brusselator evolutionary equation* (12).

Proof. Taking the inner products $\langle(2), v(t)\rangle$ and $\langle(4), z(t)\rangle$ and summing them up, we get

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2\right) + d_2\left(\|\nabla v\|^2 + \|\nabla v\|^2\right) \\
&= \int_\Omega \left(-u^2v^2 + buv - w^2z^2 + bwz - D_2[v^2 - 2vz + z^2]\right)\,\mathrm{d}x \\
&= \int_\Omega - \left[\left(uv - \frac{b}{2}\right)^2 + \left(wz - \frac{b}{2}\right)^2 + D_2(v-z)^2\right]\mathrm{d}x + \frac{1}{2}b^2|\Omega| \le \frac{1}{2}b^2|\Omega|.
\end{aligned} \tag{14}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 + \|z\|^2\right) + 2\gamma d_2\left(\|v\|^2 + \|z\|^2\right) \le b^2|\Omega|,$$

which yields

$$\|v(t)\|^2 + \|z(t)\|^2 \le \mathrm{e}^{-2\gamma d_2 t}\left(\|v_0\|^2 + \|z_0\|^2\right) + \frac{b^2|\Omega|}{2\gamma d_2}, \quad \text{for } t \in [0, T_{max}). \tag{15}$$

Let $y(t,x) = u(t,x) + v(t,x) + w(t,x) + z(t,x)$. In order to treat the u-component and the w-component, first we add up (1), (2), (3) and (4) altogether to get the following equation satisfied by $y(t) = y(t,x)$,

$$\frac{\partial y}{\partial t} = d_1\Delta y - y + [(d_2 - d_1)\Delta(v+z) + (v+z) + 2a]. \tag{16}$$

Taking the inner-product $\langle(16), y(t)\rangle$ we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2 + d_1\|\nabla y\|^2 + \|y\|^2 &= \int_\Omega [(d_2 - d_1)\Delta(v+z) + (v+z) + 2a]y\,\mathrm{d}x \\
&\le |d_1 - d_2|\|\nabla(v+z)\|\|\nabla y\| + \|v+z\|\|y\| \\
&\quad + 2a|\Omega|^{1/2}\|y\| \\
&\le \frac{d_1}{2}\|\nabla y\|^2 + \frac{|d_1 - d_2|^2}{2d_1}\|\nabla(v+z)\|^2 + \frac{1}{2}\|y\|^2 \\
&\quad + \|v+z\|^2 + 4a^2|\Omega|,
\end{aligned}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2 + d_1\|\nabla y\|^2 + \|y\|^2 \le \frac{|d_1 - d_2|^2}{d_1}\|\nabla(v+z)\|^2 + 4\left(\|v\|^2 + \|z\|^2\right) + 8a^2|\Omega|.$$

Then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2 + d_1\|\nabla y\|^2 + \|y\|^2 \le \frac{|d_1 - d_2|^2}{d_1}\|\nabla(v+z)\|^2 + C_1(v_0, z_0, t), \tag{17}$$

where

$$C_1(v_0, z_0, t) = 4\mathrm{e}^{-2\gamma d_2 t}\left(\|v_0\|^2 + \|z_0\|^2\right) + \left(\frac{4b^2}{\gamma d_2} + 8a^2\right)|\Omega|. \tag{18}$$

Integrate the inequality (17) to see that the solution $y(t)$ of (16) satisfies the following estimate,

$$\begin{aligned} \|y(t)\|^2 \le\ & \|u_0 + v_0 + w_0 + z_0\|^2 + \frac{|d_1 - d_2|^2}{d_1}\int_0^t \|\nabla(v(s) + z(s))\|^2\,\mathrm{d}s \\ & + \frac{2}{\gamma d_2}\left(\|v_0\|^2 + \|v_0\|^2\right) + \left(\frac{4b^2}{\gamma d_2} + 8a^2\right)|\Omega|\,t, \quad t \in [0, T_{max}). \end{aligned} \tag{19}$$

From (14) we have

$$\begin{aligned} d_2\int_0^t \|\nabla(v(s) + z(s))\|^2\,\mathrm{d}s &\le 2d_2\int_0^t \left(\|\nabla v(s)\|^2 + \|\nabla z(s)\|^2\right)\mathrm{d}s \\ &\le \left(\|v_0\|^2 + \|z_0\|^2\right) + b^2|\Omega|t. \end{aligned}$$

Substitute this into (19) to obtain

$$\begin{aligned} \|y(t)\|^2 \le \|u_0 + v_0 + w_0 + z_0\|^2 &+ \left(\tfrac{|d_1 - d_2|^2}{d_1 d_2} + \tfrac{2}{\gamma d_2}\right)\left(\|v_0\|^2 + \|z_0\|^2\right) \\ &+ \left[\left(\tfrac{|d_1 - d_2|^2}{d_1 d_2} + \tfrac{4}{\gamma d_2}\right)b^2 + 8a^2\right]|\Omega|\,t, \quad t \in [0, T_{max}). \end{aligned} \tag{20}$$

Let $p(t) = u(t) + w(t)$. Then by (15) and (20) we have shown that

$$\begin{aligned} \|p(t)\|^2 &= \|u(t) + w(t)\|^2 = \|y(t) - (v(t) + z(t))\|^2 \\ &\le 2\left(\|u_0 + v_0 + w_0 + z_0\|^2 + \|v_0\|^2 + \|z_0\|^2\right) + C_2(g_0)\,t, \quad \text{for } t \in [0, T_{max}), \end{aligned} \tag{21}$$

where $C_2(g_0)$ is a constant depending on the initial data g_0.

On the other hand, let $\psi(t,x) = u(t,x) + v(t,x) - w(t,x) - z(t,x)$, which satisfies the equation

$$\frac{\partial \psi}{\partial t} = d_1\Delta\psi - (1 + 2D_1)\psi + \left[(d_2 - d_1)\Delta(v - z) + (1 + 2(D_1 - D_2))(v - z)\right]. \tag{22}$$

Taking the inner-product $\langle(22), \psi(t)\rangle$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2+d_1\|\nabla\psi\|^2+\|\psi\|^2 \le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2+d_1\|\nabla\psi\|^2+(1+2D_1)\|\psi\|^2$$
$$\le (d_1-d_2)\|\nabla(v-z)\|\|\nabla\psi\|+|1+2(D_1-D_2)|\|v-z\|\|\psi\|$$
$$\le \frac{d_1}{2}\|\nabla\psi\|^2+\frac{|d_1-d_2|^2}{2d_1}\|\nabla(v-z)\|^2+\frac{1}{2}\|\psi\|^2+\frac{1}{2}|1+2(D_1-D_2)|^2\|v-z\|^2,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2+d_1\|\nabla\psi\|^2+\|\psi\|^2 \le \frac{|d_1-d_2|^2}{d_1}\|\nabla(v-z)\|^2+C_3(v_0,z_0,t), \tag{23}$$

where

$$C_3(v_0,z_0,t)=2|1+2(D_1-D_2)|^2\left(\mathrm{e}^{-2\gamma d_2 t}\left(\|v_0\|^2+\|z_0\|^2\right)+\frac{b^2}{2\gamma d_2}|\Omega|\right). \tag{24}$$

Integration of (23) yields

$$\|\psi\|^2 \le \|u_0+v_0-w_0-z_0\|^2+\frac{|d_1-d_2|^2}{d_1}\int_0^t\|\nabla(v(s)-z(s))\|^2\,\mathrm{d}s$$
$$+|1+2(D_1-D_2)|^2\left(\frac{1}{\gamma d_2}(\|v_0\|^2+\|z_0\|^2)+\frac{b^2|\Omega|}{\gamma d_2}t\right), \quad t\in[0,T_{max}). \tag{25}$$

Note that

$$d_2\int_0^t\|\nabla(v(s)-z(s))\|^2\,\mathrm{d}s \le 2d_2\int_0^t\left(\|\nabla v(s)\|^2+\|\nabla z(s)\|^2\right)\mathrm{d}s$$
$$\le \left(\|v_0\|^2+\|z_0\|^2\right)+b^2|\Omega|t.$$

From (25) it follows that

$$\|\psi\|^2 \le \|u_0+v_0-w_0-z_0\|^2+\frac{|d_1-d_2|^2}{d_1 d_2}\left(\|v_0\|^2+\|z_0\|^2+b^2|\Omega|t\right)$$
$$+|1+2(D_1-D_2)|^2\left(\frac{1}{\gamma d_2}(\|v_0\|^2+\|z_0\|^2)+\frac{b^2|\Omega|}{\gamma d_2}t\right), \quad t\in[0,T_{max}). \tag{26}$$

Let $q(t)=u(t)-w(t)$. Then by (15) and (26) we find that

$$\|q(t)\|^2 = \|u(t)-w(t)\|^2=\|\psi(t)-(v(t)-z(t))\|^2$$
$$\le 2\left(\|u_0+v_0-w_0-z_0\|^2+\|v_0\|^2+\|z_0\|^2\right)+C_4(g_0)t, \quad \text{for } t\in[0,T_{max}), \tag{27}$$

where $C_4(g_0)$ is a constant depending on the initial data g_0.

Finally combining (21) and (27) we can conclude that for each initial data $g_0 \in H$, both $u(t) = (1/2)(p(t)+q(t))$ and $w(t) = (1/2)(p(t)-q(t))$ components are bounded if T_{max} of the maximal interval of existence of the solution is finite. Together with (15), this shows that, for each $g_0 \in H$, the weak solution $g(t) = (u(t), v(t), w(t), z(t))$ of (12) will never blow up in H at any finite time and it exists globally . □

Due to Lemma 2.1, the family of all the global weak solutions $\{g(t;g_0), t \geq 0, g_0 \in H\}$ of the coupled Brusselator evolutionary equation (12) defines a semiflow on H,

$$S(t) : g_0 \mapsto g(t;w_0), \quad g_0 \in H, \; t \geq 0,$$

which is called the **coupled Brusselator semiflow**.

Lemma 2.2. *There exists a constant* $K_0 > 0$*, such that the set*

$$B_0 = \left\{ \|g\| \in H : \|g\|^2 \leq K_0 \right\} \tag{28}$$

is a bounded absorbing set B_0 *in* H *for the coupled Brusselator semiflow* $\{S(t)\}_{t \geq 0}$.

Proof. For this coupled Brusselator semiflow, from (15) we obtain

$$\limsup_{t \to \infty} (\|v(t)\|^2 + \|z(t)\|^2) < R_0 = \frac{b^2|\Omega|}{\gamma d_2}. \tag{29}$$

Moreover, for any $t \geq 0$, (14) also implies that

$$\begin{aligned} \int_t^{t+1} (\|\nabla v(s)\|^2 + \|\nabla z(s)\|^2)\,\mathrm{d}s &\leq \frac{1}{d_2}(\|v(t)\|^2 + \|z(t)\|^2 + b^2|\Omega|) \\ &\leq \frac{1}{d_2}\left(\mathrm{e}^{-2\gamma d_2 t}(\|v_0\|^2 + \|z_0\|^2) + \frac{b^2|\Omega|}{2\gamma d_2} \right) + \frac{b^2|\Omega|}{d_2}. \end{aligned} \tag{30}$$

which is for later use.

From (17) we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t \|y(t)\|^2\right) \leq \frac{|d_1 - d_2|^2}{d_1} \mathrm{e}^t \|\nabla(v(t) + z(t))\|^2 + \mathrm{e}^t C_1(v_0, z_0, t). \tag{31}$$

Integrate (31) to obtain

$$\begin{aligned} \|y(t)\|^2 &\leq \mathrm{e}^{-t}\|u_0 + v_0 + w_0 + z_0\|^2 \\ &\quad + \frac{|d_1 - d_2|^2}{d_1} \int_0^t \mathrm{e}^{-(t-\tau)} \|\nabla(v(\tau) + z(\tau))\|^2 \,\mathrm{d}\tau + C_5(v_0, z_0, t), \end{aligned} \tag{32}$$

where

$$C_5(v_0,z_0,t) = \mathrm{e}^{-t}\int_0^t 4\mathrm{e}^{(1-2\gamma d_2)\tau}\,\mathrm{d}\tau\,(\|v_0\|^2+\|z_0\|^2) + \left(\frac{4b^2}{\gamma d_2}+8a^2\right)|\Omega|$$
$$\le 4\alpha(t)(\|v_0\|^2+\|z_0\|^2) + \left(\frac{4b^2}{\gamma d_2}+8a^2\right)|\Omega|,$$

in which

$$\alpha(t) = \mathrm{e}^{-t}\int_0^t \mathrm{e}^{(1-2\gamma d_2)\tau}\,d\tau = \begin{cases} \frac{1}{|1-2\gamma d_2|}\mathrm{e}^{-2\gamma d_2 t}, & \text{if } 1-2\gamma d_2 > 0; \\ t\mathrm{e}^{-t} \le 2\mathrm{e}^{-1}\mathrm{e}^{-t/2}, & \text{if } 1-2\gamma d_2 = 0; \\ \frac{1}{|1-2\gamma d_2|}\mathrm{e}^{-t}, & \text{if } 1-2\gamma d_2 < 0. \end{cases} \tag{33}$$

On the other hand, multiplying (14) by e^t and then integrating each term of the resulting inequality, we get

$$\frac{1}{2}\int_0^t \mathrm{e}^{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\|v(\tau)\|^2+\|z(\tau)\|^2\right)\mathrm{d}\tau + d_2\int_0^t \mathrm{e}^{\tau}(\|\nabla v(\tau)\|^2+\|z(\tau)\|^2)\,\mathrm{d}\tau \le \frac{1}{2}b^2|\Omega|\mathrm{e}^t,$$

so that, by integration by parts and using (15), we obtain

$$\begin{aligned} & d_2\int_0^t \mathrm{e}^{\tau}(\|\nabla v(\tau)\|^2+\|\nabla v(\tau)\|^2)\,\mathrm{d}\tau \\ & \le \frac{1}{2}b^2|\Omega|\mathrm{e}^t - \frac{1}{2}\int_0^t \mathrm{e}^{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\|v(\tau)\|^2+\|\nabla z(\tau)\|^2\right)\mathrm{d}\tau \\ & = \frac{1}{2}b^2|\Omega|\mathrm{e}^t - \frac{1}{2}\Big[\mathrm{e}^t(\|v(t)\|^2+\|z(t)\|^2) - (\|v_0\|^2+\|z_0\|^2) \\ & \quad - \int_0^t \mathrm{e}^{\tau}(\|v(\tau)\|^2+\|z(\tau)\|^2)\,d\tau\Big] \\ & \le b^2|\Omega|\mathrm{e}^t + (\|v_0\|^2+\|z_0\|^2) + \int_0^t \mathrm{e}^{(1-2\gamma d_2)\tau}(\|v_0\|^2+\|z_0\|^2)\,\mathrm{d}\tau + \frac{b^2|\Omega|}{2\gamma d_2}\mathrm{e}^t \\ & \le \left(1+\frac{1}{2\gamma d_2}\right)b^2|\Omega|\mathrm{e}^t + \left(1+\alpha(t)\mathrm{e}^t\right)(\|v_0\|^2+\|z_0\|^2), \quad \text{for } t \ge 0. \end{aligned} \tag{34}$$

Substituting (34) into (32), we obtain that, for $t \ge 0$,

$$
\begin{aligned}
\|y(t)\|^2 \le\ & e^{-t}\|u_0+v_0+w_0+z_0\|^2 + C_5(v_0,z_0,t) \\
& + \frac{2|d_1-d_2|^2}{d_1\,d_2} e^{-t}\left[\left(1+\frac{1}{2\gamma d_2}\right) b^2|\Omega|e^t + \left(1+e^t\alpha(t)\right)\left(\|v_0\|^2+\|z_0\|^2\right)\right] \\
\le\ & e^{-t}\|u_0+v_0+w_0+z_0\|^2 + 4\alpha(t)(\|v_0\|^2+\|z_0\|^2) + \left(\frac{4b^2}{\gamma d_2}+8a^2\right)|\Omega| \\
& + \frac{2|d_1-d_2|^2}{d_1\,d_2} e^{-t}\left[\left(1+\frac{1}{2\gamma d_2}\right) b^2|\Omega|e^t + \left(1+e^t\alpha(t)\right)\left(\|v_0\|^2+\|z_0\|^2\right)\right].
\end{aligned}
\tag{35}
$$

Note that (33) shows $\alpha(t) \to 0$, as $t \to 0$. From (35) we find that

$$
\limsup_{t\to\infty}\|y(t)\|^2 < R_1 = 1 + \left(\frac{4b^2}{\gamma d_2}+8a^2\right)|\Omega| + \frac{2|d_1-d_2|^2}{d_1\,d_2}\left(1+\frac{1}{2\gamma d_2}\right) b^2|\Omega|.
\tag{36}
$$

The combination of (29) and (36) gives us

$$
\limsup_{t\to\infty}\|u(t)+w(t)\|^2 = \limsup_{t\to\infty}\|y(t)-(v(t)+z(t))\|^2 < 4R_0 + 2R_1.
\tag{37}
$$

Similarly, from the inequality (23) satisfied by $\psi(t) = u(t)+v(t)-w(t)-z(t)$, we get

$$
\frac{d}{dt}\left(e^t\|\psi(t)\|^2\right) \le \frac{|d_1-d_2|^2}{d_1} e^t\|\nabla(v(t)-z(t))\|^2 + e^t C_3(v_0,z_0,t).
\tag{38}
$$

Integrate (38) to obtain

$$
\begin{aligned}
\|\psi(t)\|^2 \le\ & e^{-t}\|u_0+v_0-w_0-z_0\|^2 \\
& + \frac{|d_1-d_2|^2}{d_1}\int_0^t e^{-(t-\tau)}\|\nabla(v(\tau)-z(\tau))\|^2\,d\tau + C_6(v_0,z_0,t),
\end{aligned}
\tag{39}
$$

where

$$
\begin{aligned}
C_6(v_0,z_0,t) &= 2|1+2(D_1-D_2)|^2\left(e^{-t}\int_0^t e^{(1-2\gamma d_2)\tau}\,d\tau\,(\|v_0\|^2+\|z_0\|^2) + \frac{b^2}{\gamma d_2}|\Omega|\right) \\
&\le 2|1+2(D_1-D_2)|^2\left(\alpha(t)(\|v_0\|^2+\|z_0\|^2) + \frac{b^2}{\gamma d_2}|\Omega|\right).
\end{aligned}
$$

Using (34) to treat the integral term in (39), we obtain that, for $t \geq 0$,

$$\begin{aligned}\|\psi(t)\|^2 &\leq \mathrm{e}^{-t}\|u_0+v_0-w_0-z_0\|^2 + C_6(v_0,z_0,t)\\ &\quad + \frac{2|d_1-d_2|^2}{d_1\,d_2}\mathrm{e}^{-t}\int_0^t \mathrm{e}^{\tau}(\|\nabla v(\tau)\|^2+\|\nabla z(\tau)\|^2)\,\mathrm{d}\tau\\ &\leq \mathrm{e}^{-t}\|u_0+v_0-w_0-z_0\|^2 + 2|1+2(D_1-D_2)|^2\\ &\quad \times\left(\alpha(t)(\|v_0\|^2+\|z_0\|^2)+\frac{b^2}{\gamma d_2}|\Omega|\right)\\ &\quad + \frac{2|d_1-d_2|^2}{d_1\,d_2}\mathrm{e}^{-t}\left[\left(1+\frac{1}{2\gamma d_2}\right)b^2|\Omega|\mathrm{e}^t + \left(1+\mathrm{e}^t\alpha(t)\right)(\|v_0\|^2+\|z_0\|^2)\right].\end{aligned} \tag{40}$$

Therefore, since $\alpha(t) \to 0$, as $t \to 0$, from (40) we get

$$\limsup_{t\to\infty}\|\psi(t)\|^2 < R_2 = 1 + 2b^2|\Omega|\left[\frac{|1+2(D_1-D_2)|^2}{\gamma d_2} + \frac{|d_1-d_2|^2}{d_1\,d_2}\left(1+\frac{1}{2\gamma d_2}\right)\right]. \tag{41}$$

The combination of (29) and (41) gives us

$$\limsup_{t\to\infty}\|u(t)-w(t)\|^2 = \limsup_{t\to\infty}\|\psi(t)-(v(t)-z(t))\|^2 < 4R_0 + 2R_2. \tag{42}$$

Finally, putting together (37) and (42), we assert that

$$\limsup_{t\to\infty}(\|u(t)\|^2 + \|w(t)\|^2) < 8R_0 + 2(R_1+R_2). \tag{43}$$

Then assembling (29) and (43), we end up with

$$\limsup_{t\to\infty}\|g(t)\|^2 = \limsup_{t\to\infty}(\|u(t)\|^2+\|v(t)\|^2+\|w(t)\|^2+\|z(t)\|^2) < 9R_0 + 2(R_1+R_2).$$

Thus this lemma is proved with $K_0 = 9R_0 + 2(R_1+R_2)$ in the absorbing ball B_0 in (28). And K_0 is a uniform positive constant independent of initial data. □

3 κ-Contracting Property for the *(v, z)* Components

The lack of inherent dissipativity and the cross-cell coupling make the attempt of showing the asymptotic compactness of the coupled Brusselator semiflow more challenging. A good idea in dealing with this issue is through a decomposition approach. Here we introduce a new decomposition technique in the next lemma, which provides sufficient conditions for the existence of a global attractor.

Lemma 3.1. *For the solution semiflow $\{S(t)\}_{t\geq 0}$ generated by the coupled Brusselator evolutionary equation* (12) *on H, there exists a global attractor $\mathscr{A}$ in H if the following four conditions are satisfied:*

(i) There exists a bounded absorbing set B_0 in H for this semiflow.

(ii) For any $\varepsilon > 0$, there are positive constants $M = M(\varepsilon)$ and $T_1 = T_1(\varepsilon)$ such that

$$\int_{\Omega(|v(t)|\geq M)} |v(t)|^2 \, dx + \int_{\Omega(|z(t)|\geq M)} |z(t)|^2 \, dx < L_1 \varepsilon, \quad \textit{for any } t > T_1, \ g_0 \in B_0, \tag{44}$$

where $L_1 > 0$ is a uniform positive constant.

(iii) For any given $M > 0$,

$$\kappa\left(P_v\left[(S(t)B_0)_{\Omega(|v(t)|<M)}\right]\right) \longrightarrow 0, \ \textit{as } t \to \infty, \tag{45}$$

and

$$\kappa\left(P_z\left[(S(t)B_0)_{\Omega(|z(t)|<M)}\right]\right) \longrightarrow 0, \ \textit{as } t \to \infty, \tag{46}$$

where P_v and P_z are respectively the orthogonal projections from H onto the component spaces $L^2(\Omega)_v$ and $L^2(\Omega)_z$. Here

$$(S(t)B_0)_{\Omega(|v(t)|<M)} \stackrel{\text{def}}{=} \{(S(t)g_0)(\cdot)\theta_M(\cdot;t,g_0) : \text{for } g_0 \in B_0\},$$
$$(S(t)B_0)_{\Omega(|z(t)|<M)} \stackrel{\text{def}}{=} \{(S(t)g_0)(\cdot)\xi_M(\cdot;t,g_0) : \text{for } g_0 \in B_0\},$$

in which $\theta_M(x;t,g_0)$ and $\xi_M(x;t,g_0)$ are respectively the characteristic functions of the subsets $\Omega(|v(t)| < M)$ and $\Omega(|z(t)| < M)$, and $v(t) = v(t,x;g_0)$ is the v-component, $z(t) = z(t,x;g_0)$ is the z-component of the solutions of the coupled Brusselator evolutionary equations (12)*.*

(iv) There exists a uniform constant $L_2 > 0$ and time $T_2 > 0$ depending only on the absorbing ball B_0, such that

$$\|\nabla(u(t), w(t))\|^2 \leq L_2, \quad \textit{for any } t > T_2, \ g_0 \in B_0. \tag{47}$$

Proof. In light of Lemma 1.5, it suffices to show that this solution semiflow $\{S(t)\}_{t\geq 0}$ is κ-contracting on the space H. Since the absorbing set B_0 in (28) attracts every bounded set $B \subset H$, we need only to show

$$\lim_{t\to\infty} \kappa(S(t)B_0) = 0. \tag{48}$$

By the linear sum property of the Kuratowski measure listed in Sect. 1, we have

$$\kappa(S(t)B_0) \leq \kappa(S_1(t)B_0) + \kappa(S_2(t)B_0) + \kappa(S_3(t)B_0), \quad t > 0, \tag{49}$$

where

$$S_1(t)B_0 = \begin{pmatrix} (S(t)B_0)_u \\ 0 \\ (S(t)B_0)_w \\ 0 \end{pmatrix}, \quad S_2(t)B_0 = \begin{pmatrix} 0 \\ (S(t)B_0)_v \\ 0 \\ 0 \end{pmatrix},$$

and

$$S_3(t)B_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (S(t)B_0)_z \end{pmatrix},$$

and it holds that $\kappa(S_1(t)B_0) = \kappa(P_{u,w}[S(t)B_0])$, $\kappa(S_2(t)B_0) = \kappa(P_v[S(t)B_0])$ and $\kappa(S_3(t)B_0) = \kappa(P_z[S(t)B_0])$. Here $P_{u,w}$ is the orthogonal projections from H onto $L^2(\Omega)_u \times L^2(\Omega)_w$.

Note that for any given constant $M > 0$, we have

$$S_2(t)B_0 \subset (S_2(t)B_0)\theta_M + (S_2(t)B_0)(1-\theta_M), \tag{50}$$

where

$$(S_2(t)B_0)\theta_M = \{v(t,\cdot;g_0)\theta_M(\cdot;t,g_0) : g_0 \in B_0\},$$
$$(S_2(t)B_0)(1-\theta_M) = \{v(t,\cdot,;g_0)(1-\theta_M(\cdot;t,g_0)) : g_0 \in B_0\}.$$

By (44), for an arbitrarily given $\varepsilon > 0$, there exist constants $M > 0$ and $T_1 > 0$ such that

$$\int_\Omega |v(t,x;g_0)(1-\theta_M(x;t,g_0))|^2 \, \mathrm{d}x = \int_{\Omega(|v(t)| \geq M)} |v(t,x;g_0)|^2 \, \mathrm{d}x < L_1\varepsilon, \quad t > T_1,$$

which implies that

$$\kappa((S_2(t)B_0)(1-\theta_M)) < 2\sqrt{L_1\varepsilon}, \quad t > T_1. \tag{51}$$

Similarly we can deduce that

$$\kappa((S_3(t)B_0)(1-\theta_M)) < 2\sqrt{L_1\varepsilon}, \quad t > T_1. \tag{52}$$

On the other hand, by (45) and (46), for the same ε and M, there exists a sufficiently large $T^1 > 0$, such that

$$\kappa((S_2(t)B_0)\theta_M) + \kappa((S_3(t)B_0)\theta_M) < \varepsilon, \quad t > T^1. \tag{53}$$

Then by (50) and the monotone property of the κ-measure, (51), (52) and (53) show that

$$\kappa(S_2(t)B_0)) + \kappa(S_3(t)B_0)) < \varepsilon + 4\sqrt{L_1\varepsilon}, \quad \text{for } t > \max\{T_1, T^1\}.$$

Moreover, the condition (iv) implies that $\kappa(S_1(t)B_0) = 0$ for $t > T_2$, since a bounded subset in E must be precompact in H. Finally we obtain

$$\kappa(S(t)B_0) \le \sum_{i=1}^{3} \kappa(S_i(t)B_0) < \varepsilon + 4\sqrt{L_1\varepsilon}, \quad \text{for } t > \max\{T_1, T^1, T_2\}.$$

Therefore (48) is valid. □

Below we shall check that the conditions specified in the items (ii) and (iii) of Lemma 3.1 for the (v,z) components of the coupled Brusselator equations. We shall use the set notation

$$\begin{aligned} \Omega_M^{\phi} &= \Omega(\phi(t) \ge M) = \{x \in \Omega : \phi(t,x) \ge M\} \\ \Omega_{|\phi|,M} &= \Omega(|\phi(t)| < M) = \{x \in \Omega : |\phi(t,x)| < M\} \end{aligned} \tag{54}$$

where $\phi(t,x)$ is any measurable function on Ω for each given $t \ge 0$ and the norm notation

$$\|\rho\|^2_{\Omega_M^{\phi}} = \int_{\Omega(\phi(t)\ge M)} |\rho(x)|^2 \, dx \quad \text{and} \quad \|\rho\|^2_{\Omega_{|\phi|,M}} = \int_{\Omega(|\phi(t)|<M)} |\rho(x)|^2 \, dx.$$

We can use $m(S)$ or $|S|$ to denote the Lebesgue measure of a measurable subset S in Ω. For any measurable ϕ defined on Ω, let

$$(\varphi - M)_+ = \begin{cases} \varphi(x) - M, & \text{if } \varphi(x) \ge M, \\ 0, & \text{if } \varphi(x) < M; \end{cases}$$

and

$$(\varphi + M)_- = \begin{cases} \varphi(x) + M, & \text{if } \varphi(x) \le -M, \\ 0, & \text{if } \varphi(x) > -M. \end{cases}$$

As a preliminary remark, since B_0 in Lemma 2.2 is a bounded absorbing set in H for the coupled Brusselator semiflow $\{S(t)\}_{t\ge0}$, there exists a constants $T_0 > 0$, such that

$$\|S(t)g_0\|^2 \le K_0, \quad \text{for any } t > T_0, \ g_0 = (u_0, v_0, w_0, z_0) \in B_0, \tag{55}$$

where K_0 is the constant given in (28). Let this T_0 be fixed.

Lemma 3.2. *For any $\varepsilon > 0$, there exist positive constants $M_1 = M_1(\varepsilon)$ and $T_1 = T_1(\varepsilon)$, such that the v-component $v(t) = v(t,x;g_0)$ and the z-component $z(t) =$*

$z(t,x;g_0)$ of the solutions of the coupled Brusselator equations (1)–(4) satisfy the following estimate,

$$\int_{\Omega(|v(t)|\geq M_1)} |v(t)|^2 \,\mathrm{d}x + \int_{\Omega(|z(t)|\geq M_1)} |z(t)|^2 \,\mathrm{d}x < \frac{4b^2}{\gamma d_2}\varepsilon, \quad \textit{for } t > T_1,\ g_0 \in B_0, \tag{56}$$

where $L_1 \stackrel{def}{=} (4b^2)/(\gamma d_2)$ is a uniform constant.

Proof. By (55), for any $g_0 \in B_0$ and any $t > T_0$, we have $\|v(t)\|^2 + \|z(t)\|^2 \leq K_0$. Hence we have

$$\begin{aligned} &M^2\left[m(\Omega(|v(t)|\geq M)) + m(\Omega(|z(t)|\geq M))\right] \\ &\quad \leq \int_{\Omega(|v(t)|\geq M)} |v(t)|^2 \,\mathrm{d}x + \int_{\Omega(|z(t)|\geq M)} |v(t)|^2 \,\mathrm{d}x \leq K_0, \end{aligned}$$

so that there exists an $M = M(\varepsilon) > 0$ such that for any $t > T_0, g_0 \in B_0$,

$$m(\Omega(|v(t)|\geq M)) \leq \frac{K_0}{M^2} < \frac{\varepsilon}{2}, \quad \text{and} \quad m(\Omega(|z(t)|\geq M)) \leq \frac{K_0}{M^2} < \frac{\varepsilon}{2}. \tag{57}$$

Taking the inner-product $\langle(2),(v(t)-M)_+\rangle$, where M is given in (57), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{d\mathrm{d}t}\|(v-M)_+\|^2 + d_2\int_{\Omega_M^v} |\nabla(v-M)_+|^2 \,\mathrm{d}x \\ &= -\int_{\Omega_M^v} u^2 v(v-M)_+ \mathrm{d}x + \int_{\Omega_M^v} bu(v-M)_+ \mathrm{d}x + D_2\int_{\Omega_M^v} (z-v)(v-M)_+ \,\mathrm{d}x \\ &\leq -\int_{\Omega_M^v} \left[\left(u(v-M)_+ - \frac{b}{2}\right)^2 + u^2 M(v-M)_+\right] \mathrm{d}x + \frac{b^2}{4} m(\Omega(v(t)\geq M)) \\ &\quad + D_2\int_{\Omega_M^v} (z-M)(v-M)_+ \,\mathrm{d}x + D_2 M\int_{\Omega_M^v} (v-M)_+ \,\mathrm{d}x \\ &\quad - D_2\int_{\Omega_M^v} (v-M)_+^2 \,\mathrm{d}x - D_2 M\int_{\Omega_M^v} (v-M)_+ \,\mathrm{d}x \\ &\leq \frac{b^2}{4} m(\Omega(v(t)\geq M)) - D_2\|(v-M)_+\|^2 \\ &\quad + D_2\int_{\Omega(v(t)\geq M, z(t)\geq M)} (z-M)_+(v-M)_+ \,\mathrm{d}x, \end{aligned} \tag{58}$$

where we noticed that

$$D_2\int_{\Omega(v(t)\geq M, z(t)< M)} (z-M)(v-M)_+ \,\mathrm{d}x \leq 0.$$

Similarly, by taking the inner-product $\langle (4), (z(t)-M)_+ \rangle$, where M is given in (57), and through parallel steps we can get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(z-M)_+\|^2 + d_2\int_{\Omega_M^z}|\nabla(z-M)_+|^2\,\mathrm{d}x \\ \le \frac{b^2}{4}m(\Omega(z(t)\ge M)) - D_2\|(z-M)_+\|^2 \\ + D_2\int_{\Omega(v(t)\ge M, z(t)\ge M)}(z-M)_+(v-M)_+\,\mathrm{d}x.\end{aligned} \tag{59}$$

Sum up (58) and (59) and then use (57) to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|(v-M)_+\|^2 + \|(z-M)_+\|^2\right) + 2d_2\left(\|\nabla(v-M)_+\|^2 + \|\nabla(v-M)_+\|^2\right) \\ &\le \frac{b^2}{2}\varepsilon - 2D_2\Bigg(\|(v-M)_+\|^2 - 2\int_{\Omega(v(t)\ge M, z(t)\ge M)}(z-M)_+(v-M)_+\,\mathrm{d}x\|^2 \\ &\qquad + \|(z-M)_+\|^2\Bigg) \\ &\le \frac{b^2}{2}\varepsilon.\end{aligned}$$

By Poincaré inequality and Gronwall inequality, it follows that, for $t \ge 0, g_0 \in B_0$,

$$\|(v(t)-M)_+\|^2+\|(z(t)-M)_+\|^2 \le \mathrm{e}^{-2\gamma d_2 t}\left(\|(v_0-M)_+\|^2+\|(z_0-M)_+\|^2\right) + \frac{b^2\varepsilon}{4\gamma d_2}. \tag{60}$$

Thus there exists a time $T_+(\varepsilon) \ge T_0$ such that for any $t > T_+$ and any $g_0 \in B_0$, one has

$$\|(v(t)-M)_+\|^2 + \|(z(t)-M)_+\|^2 < \frac{b^2\varepsilon}{2\gamma d_2}. \tag{61}$$

Symmetrically we can prove that there exists a time $T_-(\varepsilon) \ge T_0$ such that for any $t > T_-$ and any $g_0 \in B_0$, one has

$$\|(v(t)+M)_-\|^2 + \|(z(t)+M)_-\|^2 < \frac{b^2\varepsilon}{2\gamma d_2}, \tag{62}$$

Adding up (61) and (62), we find that

$$\int_{\Omega(|v(t)|\ge M)}(|v(t)|-M)^2\,\mathrm{d}x + \int_{\Omega(|z(t)|\ge M)}(|z(t)|-M)^2\,d\mathrm{d}x < \frac{b^2\varepsilon}{\gamma d_2}, \tag{63}$$

for any $t > T_1 = \max\{T_+, T_-\}$ and for any $g_0 \in B_0$.

Moreover, since for any $g_0 \in B_0$ and any $T > T_0$, we have

$$m(\Omega(|v(t)| \geq kM)) + m(\Omega(|z(t)| \geq kM)) \leq \frac{K_0}{k^2 M^2},$$

there exists a sufficiently large integer $k > 0$ such that for any $t > T_1$ and $g_0 \in B_0$, it holds that

$$\begin{aligned}
&\int_{\Omega(|v(t)| \geq kM)} |v(t)|^2 \, \mathrm{d}x + \int_{\Omega(|z(t)| \geq kM)} |z(t)|^2 \, \mathrm{d}x \\
&\leq 2 \int_{\Omega(|v(t)| \geq M)} (|v(t)| - M)^2 \, \mathrm{d}x + 2M^2 m(\Omega(|v(t)| \geq kM)) \\
&\quad + 2 \int_{\Omega(|z(t)| \geq M)} (|z(t)| - M)^2 \, \mathrm{d}x + 2M^2 m(\Omega(|z(t)| \geq kM)) \\
&\leq \frac{2b^2 \varepsilon}{\gamma d_2} + \frac{2M^2 K_0}{k^2 M^2} = \frac{2b^2 \varepsilon}{\gamma d_2} + \frac{2K_0}{k^2} < \frac{4b^2 \varepsilon}{\gamma d_2}. \qquad (64)
\end{aligned}$$

Therefore (56) is proved with $M_1 = M_1(\varepsilon) = kM$, where M is given in (57) and k is the integer that validates (64), and $T_1 = T_1(\varepsilon) = \max\{T_+, T_-\}$. □

This lemma shows that the condition (ii) of Lemma 3.1 is satisfied by the coupled Brusselator semiflow. The next lemma is to check the condition (45) and (46) for the (v, z) components in item (iii) of Lemma 3.1.

Lemma 3.3. *For any given $M > 0$, it holds that*

$$\kappa \left(P_v(S(t)B_0)_{\Omega(|v(t)| < M)} \right) \longrightarrow 0, \quad \text{as } t \to \infty, \qquad (65)$$

$$\kappa \left(P_z(S(t)B_0)_{\Omega(|z(t)| < M)} \right) \longrightarrow 0, \quad \text{as } t \to \infty, \qquad (66)$$

in the respective component space $L^2(\Omega)$.

Proof. Taking the inner-product $\langle (2), -\Delta v(t) \rangle$, we have

$$-\langle v_t, \Delta v \rangle + d_2 \|\Delta v\|^2 = \langle u^2 v, \Delta v \rangle - b\langle u, \Delta v \rangle - D_2 \langle z - v, \Delta v \rangle.$$

By Green's formula and the homogeneous Dirichlet boundary condition, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla v\|^2 + d_2 \|\Delta v\|^2 \leq \langle u^2 v, \Delta v \rangle + \frac{b^2}{2d_2} \|u\|^2 + \frac{d_2}{2} \|\Delta v\|^2 - D_2 \langle z, \Delta v \rangle - D_2 \|\nabla v\|^2,$$

where

$$\begin{aligned}
\langle u^2 v, \Delta v \rangle &= -\int_\Omega u^2 |\nabla v|^2 \, \mathrm{d}x - 2 \int_\Omega uv (\nabla u \cdot \nabla v) \, \mathrm{d}x \\
&= -\int_\Omega |u \nabla v + v \nabla u|^2 \, \mathrm{d}x + \int_\Omega v^2 |\nabla u|^2 \, \mathrm{d}x \leq \int_\Omega v^2 |\nabla u|^2 \, \mathrm{d}x.
\end{aligned}$$

Consequently we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + d_2\|\Delta v\|^2 \le 2\int_\Omega v^2|\nabla u|^2\,\mathrm{d}x + \frac{b^2}{d_2}\|u\|^2 - 2D_2(\|\nabla v\|^2 + \langle z, \Delta v\rangle). \quad (67)$$

Similarly, we can get the following inequality for the z-component,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla z\|^2 + d_2\|\Delta z\|^2 \le 2\int_\Omega z^2|\nabla w|^2\,\mathrm{d}x + \frac{b^2}{d_2}\|w\|^2 - 2D_2(\|\nabla z\|^2 + \langle v, \Delta z\rangle). \quad (68)$$

We can also establish the inequality similar to (67) but with integrals over the set $\Omega_{|v|,M} = \Omega(|v(t)| < M)$ and the inequality similar to (68) but with integrals over the set $\Omega_{|z|,M} = \Omega(|z(t)| < M)$. Then sum up the two to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2_{\Omega_{|v|,M}} + \|\nabla z\|^2_{\Omega_{|z|,M}}\right) + d_2\left(\|\Delta v\|^2_{\Omega_{|v|,M}} + \|\Delta z\|^2_{\Omega_{|z|,M}}\right) \\
&\le \frac{b^2}{d_2}\left(\|u\|^2_{\Omega_{|v|,M}} + \|w\|^2_{\Omega_{|z|,M}}\right) + 2\int_{\Omega_{|v|,M}} v^2|\nabla u|^2\,\mathrm{d}s + 2\int_{\Omega_{|z|,M}} z^2|\nabla w|^2\,\mathrm{d}x \\
&\quad - 2D_2\left(\|\nabla v\|^2_{\Omega_{|v|,M}} + \|\nabla z\|^2_{\Omega_{|z|,M}} + \langle z, \Delta v\rangle_{\Omega_{|v|,M}} + \langle v, \Delta z\rangle_{\Omega_{|z|,M}}\right) \\
&\le \frac{b^2}{d_2}K_0 + 2M^2\left(\|\nabla u\|^2_{\Omega_{|v|,M}} + \|\nabla w\|^2_{\Omega_{|z|,M}}\right) \\
&\quad + \frac{2D_2^2}{d_2}\|z\|^2_{\Omega_{|v|,M}} + \frac{d_2}{2}\|\Delta v\|^2_{\Omega_{|v|,M}} + \frac{2D_2^2}{d_2}\|v\|^2_{\Omega_{|z|,M}} + \frac{d_2}{2}\|\Delta z\|^2_{\Omega_{|z|,M}}. \quad (69)
\end{aligned}$$

Since $\|z\|^2_{\Omega_{|v|,M}} + \|v\|^2_{\Omega_{|z|,M}} \le \|S(t)g_0\|^2 \le K_0$ due to (55), and by Poincaré inequality, it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2_{\Omega_{|v|,M}} + \|\nabla z\|^2_{\Omega_{|z|,M}}\right) + \frac{\gamma d_2}{2}\left(\|\nabla v\|^2_{\Omega_{|v|,M}} + \|\nabla z\|^2_{\Omega_{|z|,M}}\right) \\
&\le \frac{K_0}{d_2}(b^2 + 2D_2^2) + 2M^2\left(\|\nabla u\|^2_{\Omega_{|v|,M}} + \|\nabla w\|^2_{\Omega_{|z|,M}}\right), \quad t > T_0,\ g_0 \in B_0. \quad (70)
\end{aligned}$$

This inequality (70) implies that

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} \le r\beta + h, \quad t > T_0, \quad (71)$$

where

$$\beta(t) = \|\nabla v\|^2_{\Omega_{|v|,M}} + \|\nabla z\|^2_{\Omega_{|z|,M}}, \quad r(t) = \frac{1}{2}\gamma d_2, \quad \text{and}$$

$$h(t) = \frac{K_0}{d_2}(b^2 + 2D_2^2) + 2M^2\left(\|\nabla u\|^2_{\Omega_{|v|,M}} + \|\nabla w\|^2_{\Omega_{|z|,M}}\right).$$

By (30), there exists a constant $T_3 = T_3(K_0) > 0$ such that $T_3 \geq T_0$ and

$$\int_t^{t+1} \left(\|\nabla v(s)\|^2_{\Omega_{|v|,M}} + \|\nabla z(s)\|^2_{\Omega_{|z|,M}} \right) \mathrm{d}s \leq \int_t^{t+1} \left(\|\nabla v(s)\|^2 + \|\nabla z(s)\|^2 \right) \mathrm{d}s$$
$$\leq C_7 = \frac{b^2|\Omega|}{d_2}\left(1 + \frac{1}{\gamma d_2}\right), \quad \text{for } t > T_3, \; g_0 \in B_0. \tag{72}$$

By integrating the inequality (17) on the time interval $[t, t+1]$ and using (72) and (36), we can deduce that there exists $T_4 = T_4(K_0) > 0$ such that $T_4 \geq T_3$ and

$$d_1 \int_t^{t+1} \|\nabla y(s)\|^2 \, \mathrm{d}s \leq \|y(t)\|^2 + \frac{2|d_1 - d_2|^2}{d_1} C_7 + 4K_0 + \left(\frac{4b^2}{\gamma d_2} + 8a^2\right)|\Omega|$$
$$\leq R_1 + \frac{2|d_1 - d_2|^2}{d_1} C_7 + 4K_0 + \left(\frac{4b^2}{\gamma d_2} + 8a^2\right)|\Omega|, \quad t > T_4, \tag{73}$$

where R_1 is the constant given in (36).

Similarly, doing the same to (23) and using (72) and (41), we find that there exists $T_5 = T_5(K_0) > 0$ such that $T_5 \geq T_3$ and

$$d_1 \int_t^{t+1} \|\nabla \psi(s)\|^2 \, \mathrm{d}s$$
$$\leq \|\psi(t)\|^2 + \frac{2|d_1 - d_2|^2}{d_1} C_7 + 2|1 + 2(D_1 - D_2)|^2 \left(K_0 + \frac{b^2}{2\gamma d_2}|\Omega|\right)$$
$$\leq R_2 + \frac{2|d_1 - d_2|^2}{d_1} C_7 + 2|1 + 2(D_1 - D_2)|^2 \left(K_0 + \frac{b^2}{2\gamma d_2}|\Omega|\right), \quad t > T_5, \tag{74}$$

where R_2 is the constant given in (41).

From (72), (73) and (74) it follows that, for $t > \max\{T_4, T_5\}$ and any $g_0 \in B_0$,

$$\int_t^{t+1} \|\nabla u(s)\|^2_{\Omega_{|v|,M}} \, \mathrm{d}s \leq \int_t^{t+1} \|\nabla u(s)\|^2 \, \mathrm{d}s = \int_t^{t+1} \left\| \frac{1}{2}(y(s) + \psi(s)) - \nabla v(s) \right\|^2 \mathrm{d}s$$
$$\leq \int_t^{t+1} \|\nabla y(s)\|^2 \, \mathrm{d}s + \int_t^{t+1} \|\nabla \psi(s)\|^2 \, \mathrm{d}s + 2\int_t^{t+1} \|\nabla v(s)\|^2 \, \mathrm{d}ds \leq C_8, \tag{75}$$

where

$$C_8 = 2C_7\left(1 + \frac{2|d_1 - d_2|^2}{d_1^2}\right) + \frac{1}{d_1}(R_1 + R_2) + \frac{2K_0}{d_1}\left(2 + |1 + 2(D_1 - D_2)|^2\right)$$
$$+ \frac{|\Omega|}{d_1}\left[\left(\frac{4b^2}{\gamma d_2} + 8a^2\right) + |1 + 2(D_1 - D_2)^2| \frac{b^2}{\gamma d_2}\right].$$

We can also assert that

$$\int_t^{t+1} \|\nabla w(s)\|^2_{\Omega_{|z|,M}} \, \mathrm{d}s \le \int_t^{t+1} \|\nabla w(s)\|^2 \mathrm{d}s$$
$$= \int_t^{t+1} \left\| \frac{1}{2}(y(s) - \psi(s)) - \nabla z(s) \right\|^2 \mathrm{d}s \le C_8. \tag{76}$$

According to (75) and (76), we have

$$\int_t^{t+1} h(s)\, \mathrm{d}s \le 4M^2 C_8 + \frac{K_0}{d_2}(b^2 + 2D_2^2), \quad \text{for } t \ge \max\{T_3, T_4\},\ g_0 \in B_0. \tag{77}$$

Besides we have $\int_t^{t+1} r(s)\, \mathrm{d}s \le \gamma d_2$.

Finally by (72) and (77) and applying the uniform Gronwall inequality [34, 36] to (71), we obtain

$$\|\nabla v(t)\|^2_{\Omega_{|v|,M}} + \|\nabla z(t)\|^2_{\Omega_{|z|,M}} \le \left(C_7 + 4M^2 C_8 + \frac{K_0}{d_2}(b^2 + 2D_2^2) \right) \mathrm{e}^{\gamma d_2},$$
$$t > T_6,\ g_0 \in B_0, \tag{78}$$

where $T_6 = \max\{T_4, T_5\} + 1$. Note that the right-hand side of (78) is a uniform constant depending on the constant K_0 in (28) and the arbitrarily fixed constant M only. The inequality (78) shows that for any given $t > T_6$,

$$P_v(S(t)B_0)_{\Omega(|v(t)|<M)} \text{ and } P_z(S(t)B_0)_{\Omega(|z(t)|<M)} \text{ are bounded sets in } H_0^1(\Omega).$$

Due to the compact Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ for space dimension $n \le 3$, it shows that for any given $t > T_6$,

$$P_v(S(t)B_0)_{\Omega(|v(t)|<M)} \text{ and } P_z(S(t)B_0)_{\Omega(|z(t)|<M)} \text{ are precompact sets in } L^2(\Omega).$$

Thus (65) and (66) are proved. □

This lemma shows that the conditions (iii) of Lemma 3.1 is verified for the coupled Brusselator semiflow.

4 κ-Contracting Property for the *(u, w)* Components

In this section, we shall check that the condition (iv) of Lemma 3.1 for the (u, w) components of the coupled two-cell Brusselator equations.

Lemma 4.1. *There exists a uniform constant* $K_2 > 0$ *such that*

$$\limsup_{t\to\infty}\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) < K_2, \tag{79}$$

for the (v,z) *components of the solutions of the coupled Brusselator evolutionary equation* (12) *with any initial data* $g_0 \in H$.

Proof. According to the solution property (13) satisfied by all the global weak solutions on $[0,\infty)$, we know that for any given initial status $g_0 \in H$ there exists a time $t_0 \in (0,1)$ such that

$$S(t_0)g_0 \in E = [H_0^1(\Omega)]^6 \hookrightarrow \mathbb{L}^6(\Omega). \tag{80}$$

Then the weak solution $g(t) = S(t)g_0$ becomes a strong solution on $[t_0,\infty)$ and satisfies

$$S(\cdot)g_0 \in C([t_0,\infty);E)\cap L^2(t_0,\infty;\Pi) \subset C([t_0,\infty);\mathbb{L}^6(\Omega)) \subset C([t_0,\infty);\mathbb{L}^4(\Omega)), \tag{81}$$

for $n \le 3$. Based on this observation, without loss of generality, we can simply *assume* that $g_0 \in \mathbb{L}^6(\Omega)$ for the purpose of studying the long-time dynamics. Thus parabolic regularity (81) of strong solutions implies the $S(t)g_0 \in E \subset \mathbb{L}^6(\Omega), t \ge 0$. Then by the bootstrap argument, again without loss of generality, one can *assume* that $g_0 \in \Pi \subset \mathbb{L}^8(\Omega)$ so that $S(t)g_0 \in \Pi \subset \mathbb{L}^8(\Omega), t \ge 0$.

Take the L^2 inner-product $\langle (2), v^5\rangle$ and $\langle (4), z^5\rangle$ and sum up to obtain

$$\begin{aligned}
&\frac{1}{6}\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) + 5d_2\left(\|v(t)^2\nabla v(t)\|^2 + \|z(t)^2\nabla z(t)\|^2\right) \\
&\quad = \int_\Omega \Big(bu(t,x)v^5(t,x) - u^2(t,x)v^6(t,x) + bw(t,x)z^5(t,x) - w^2(t,x)z^6(t,x)\Big)\,\mathrm{d}x \\
&\qquad + D_2\int_\Omega \Big[(z(t,x)-v(t,x))v^5(t,x) + (v(t,x)-z(t,x))z^5(t,x)\Big]\,\mathrm{d}x.
\end{aligned} \tag{82}$$

By Young's inequality, we have

$$\begin{aligned}
&\int_\Omega \Big[\Big(buv^5 - u^2v^6\Big) + \Big(bwz^5 - w^2z^6\Big)\Big]\,\mathrm{d}x \\
&\qquad \le \frac{1}{2}\left(\int_\Omega b^2(v^4+z^4)\,\mathrm{d}x - \int_\Omega (u^2v^6 + w^2z^6)\,\mathrm{d}x\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \Big[(z-v)v^5 + (v-z)z^5\Big]\,\mathrm{d}x \\
&\qquad \le \int_\Omega \left[-v^6 + \left(\frac{1}{6}z^6 + \frac{5}{6}v^6\mathrm{d}x\right) + \left(\frac{1}{6}v^6 + \frac{5}{6}z^6\right) - z^6\right]\mathrm{d}x = 0.
\end{aligned}$$

Substitute the above two inequalities into (82) and use Poincaré inequality, we get the following inequality relating $\|(v,z)\|_{L^6}^6$ to $\|(v,z)\|_{L^4}^4$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) + 10\gamma d_2\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) \\
&\quad \le \frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) + 10 d_2\left(\|\nabla v^3(t)\|^2 + \|\nabla z^3(t)\|^2\right) \\
&\quad \le 3b^2(\|v(t)\|_{L^4}^4 + \|z(t)\|_{L^4}^4).
\end{aligned}$$

Similarly we can get the corresponding inequality relating $\|(v,z)\|_{L^4}^4$ to $\|(v,z)\|^2$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|_{L^4}^4 + \|z(t)\|_{L^4}^4\right) + 6\gamma d_2\left(\|v(t)\|_{L^4}^4 + \|z(t)\|_{L^4}^4\right) \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|_{L^4}^4 + \|z(t)\|_{L^4}^4\right) + 6d_2\left(\|\nabla v^2(t)\|^2 + \|\nabla z^2(t)\|^2\right) \le 2b^2(\|v(t)\|^2 + \|z(t)\|^2).
\end{aligned}$$

Applying Gronwall inequality to the above two inequalities and using (15), we get

$$\begin{aligned}
&\|v(t)\|_{L^4}^4 + \|z(t)\|_{L^4}^4 \\
&\quad \le \mathrm{e}^{-6\gamma d_2 t}\left(\|v_0\|_{L^4}^4 + \|z_0\|_{L^4}^4\right) + \int_0^t \mathrm{e}^{-6\gamma d_2(t-\tau)} 2b^2(\|v(\tau)\|^2 + \|z(\tau)\|^2)\mathrm{d}\tau \\
&\quad \le \mathrm{e}^{-6\gamma d_2 t}\left(\|v_0\|_{L^4}^4 + \|z_0\|_{L^4}^4\right) + \int_\Omega \mathrm{e}^{-6\gamma d_2(t-\tau)-2\gamma d_2\tau} 2b^2(\|v_0\|^2 + \|z_0\|^2)\,\mathrm{d}\tau \\
&\qquad + \frac{b^4|\Omega|}{6\gamma^2 d_2^2} \\
&\quad \le \mathrm{e}^{-2\gamma d_2 t} C_9\left(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6\right) + \frac{b^4|\Omega|}{6\gamma^2 d_2^2},
\end{aligned}$$

for $t > 0$, where C_9 is a uniform positive constant, and then

$$\begin{aligned}
&\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6 \\
&\quad \le \mathrm{e}^{-10\gamma d_2 t}\left(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6\right) + \int_0^t \mathrm{e}^{-10\gamma d_2(t-\tau)} 3b^2(\|v(\tau)\|_{L^4}^4 + \|z(\tau)\|_{L^4}^4)\mathrm{d}\tau \\
&\quad \le \mathrm{e}^{-10\gamma d_2 t}\left(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6\right) + \int_0^t \mathrm{e}^{-10\gamma d_2(t-\tau)-2\gamma d_2\tau} 3b^2 C_9(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6)\,\mathrm{d}\tau \\
&\qquad + \frac{b^6|\Omega|}{20\gamma^3 d_2^3} \\
&\quad \le \mathrm{e}^{-2\gamma d_2 t}\left(1 + \frac{3b^2 C_9}{8\gamma d_2}\right)\left(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6\right) + \frac{b^6|\Omega|}{20\gamma^3 d_2^3},
\end{aligned}$$

for $t > 0$. It follows that

$$\limsup_{t\to\infty} \left(\|v(t)\|_{L^4}^4 + \|z(t)\|_{L^4}^4\right) < K_1 = 1 + \frac{b^4|\Omega|}{6\gamma^2 d_2^2}, \tag{83}$$

$$\limsup_{t\to\infty} \left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) < K_2 = 1 + \frac{b^6|\Omega|}{20\gamma^3 d_2^3}. \tag{84}$$

Thus (79) is proved. □

Lemma 4.2. *There exists a finite time $T^* > 0$ depending only on the absorbing set B_0 such that, for any initial data $g_0 \in B_0$, where B_0 is the absorbing set shown in* (28)*, the following inequalities hold for the respective components of the solution trajectory $S(t)g_0$ of* (12)*,*

$$\int_t^{t+1} (\|\nabla v(s)\|^2 + \|\nabla z(s)\|^2)\, \mathrm{d}s \le C_{10} \quad \text{and} \quad \int_t^{t+1} (\|\nabla y(s)\|^2 + \|\nabla \psi(s)\|^2)\, \mathrm{d}s \le C_{11}, \tag{85}$$

for any $t > T^*$, where C_{10} and C_{11} are uniform positive constants depending only on K_0 and $|\Omega|$.

Proof. The first inequality follows directly from (30) and it holds for any $t \ge 0$, with

$$C_{10} = \frac{1}{d_2}\left(K_0 + b^2|\Omega|\left(1 + \frac{1}{2\gamma d_2}\right)\right).$$

Now we prove the second inequality in (85). From (17) and (23) we obtain

$$\begin{aligned}
&\int_t^{t+1} (\|\nabla y(s)\|^2 + \|\nabla \psi(s)\|^2)\, \mathrm{d}s \\
&\le \frac{1}{d_1}(\|y(t)\|^2 + \|\psi(t)\|^2) + \frac{4|d_1 - d_2|^2}{d_1^2}\int_t^{t+1} (\|\nabla v(s)\|^2 + \|\nabla z(s)\|^2)\, \mathrm{d}s \\
&\quad + 4K_0 + \left(\frac{4b^2}{\gamma d_2} + 8a^2\right)|\Omega| + 2|1 + 2(D_1 - D_2)|^2\left(K_0 + \frac{b^2}{2\gamma d_2}|\Omega|\right).
\end{aligned} \tag{86}$$

By (36), (41), and using the first inequality in (85), we can assert that there is a finite time $T^* > 0$ such that

$$\int_t^{t+1} (\|\nabla y(s)\|^2 + \|\nabla \psi(s)\|^2)\, \mathrm{d}s \le C_{11}, \quad \text{for any } t > T^*,$$

where

$$\begin{aligned}
C_{11} = \frac{R_1 + R_2}{d_1} + \frac{4|d_1 - d_2|^2 C_{10}}{d_1^2} + 4K_0 + \left(\frac{4b^2}{\gamma d_2} + 8a^2\right)|\Omega| \\
+ 2|1 + 2(D_1 - D_2)|^2\left(K_0 + \frac{b^2}{2\gamma d_2}|\Omega|\right).
\end{aligned}$$

Thus (85) is proved. □

Lemma 4.3. *There exist a uniform constant $L_2 > 0$ depending on K_0, K_1 and $|\Omega|$ but independent of initial data and a finite time $T_2 > 0$ only depending on the absorbing set B_0, such that the (u,w) components of any solution trajectory $S(t)g_0, t \geq 0$, of the Brusselator evolutionary equation (12) satisfies*

$$\|\nabla(u(t), w(t))\|^2 \leq L_2, \quad \text{for any } t > T_2,\ g_0 \in B_0. \tag{87}$$

Proof. The continuous imbedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ implies that there is a uniform constant $\eta > 0$ such that for any $\phi \in L^6(\Omega)$,

$$\|\phi\|_{L^6} \leq \eta\, \|\nabla \phi\|. \tag{88}$$

We shall use the notation $\|(\phi_1, \phi_2)\|^2 = \|\phi\|^2 + \|\phi_2\|^2$.

Take the inner-products $\langle (1), -\Delta u(t,\cdot)\rangle$ and $\langle (3), -\Delta w(t,\cdot)\rangle$ and add up the two resulting equalities. We get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(u,w)\|^2 + d_1\|\Delta(u,w)\|^2 + (b+1)\|\nabla(u,w)\|^2 \\
&= -\int_\Omega a(\Delta u + \Delta w)\,\mathrm{d}x - \int_\Omega (u^2 v \Delta u + w^2 z \Delta w)\,\mathrm{d}x \\
&\quad - D_1 \int_\Omega (|\nabla u|^2 - 2\nabla u \cdot \nabla w + |\nabla w|^2)\,\mathrm{d}x \\
&\leq \left(\frac{d_1}{2} + \frac{d_1}{2}\right)\|\Delta(u,w)\|^2 + \frac{a^2}{d_1}|\Omega| + \frac{1}{2d_1}\int_\Omega \left(u^4 v^2 + w^4 z^2\right)\mathrm{d}x.
\end{aligned}$$

By using the generalized Hölder inequality and the imbedding property (88), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(u,w)\|^2 + 2(b+1)\|\nabla(u,w)\|^2 &\leq \frac{2a^2}{d_1}|\Omega| + \frac{1}{d_1}\left(\|u\|_{L^6}^4\|v\|_{L^6}^2 + \|w^2\|_{L^6}^4\|z\|_{L^6}^2\right) \\
&\leq \frac{2a^2}{d_1}|\Omega| + \frac{\eta^4}{d_1}\left(\|v\|_{L^6}^2\|\nabla u\|^4 + \|z\|_{L^6}^2\|\nabla w\|^4\right).
\end{aligned} \tag{89}$$

By Lemma 4.1, there is a finite time $T^{**} > 0$ such that, for any $g_0 \in B_0$, we have

$$\|(v(t,\cdot))\|_{L^6}^6 + \|z(t,\cdot)\|_{L^6}^6 < K_2, \quad \text{for } t > T^{**}. \tag{90}$$

From the proof of Lemma 4.1 it is seen that T^{**} only depends on the absorbing set B_0. Substitute (90) into (89) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(u,w)\|^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(u,w)\|^2 + 2(b+1)\|\nabla(u,w)\|^2 \leq \frac{\eta^4 K_2^{1/3}}{d_1}\|\nabla(u,w)\|^4 + \frac{2a^2}{d_1}|\Omega|, \tag{91}$$

for any $t > T^{**}$, which can be written as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} \leq \zeta \Gamma + \frac{2a^2}{d_1}|\Omega|, \quad \text{for } t > T^{**}, \tag{92}$$

where

$$\Gamma(t) = \|\nabla(u(t), w(t))\|^2 \quad \text{and} \quad \zeta(t) = \frac{\eta^4 K_2^{1/3}}{d_1}\Gamma(t).$$

By Lemma 4.2, we have

$$\begin{aligned}
&\int_t^{t+1} \Gamma(\tau)\,\mathrm{d}\tau \\
&= \int_t^{t+1} (\|\nabla u(\tau)\|^2 + \|\nabla w(\tau)\|^2)\,\mathrm{d}\tau \\
&\leq \int_t^{t+1} (\|\nabla(u(\tau)+w(\tau))\|^2 + \|\nabla(u(\tau)-w(\tau))\|^2)\,\mathrm{d}\tau \\
&\leq \int_t^{t+1} \left(\|\nabla(y(\tau)-(v(\tau)+z(\tau)))\|^2 + \|\nabla(\psi(\tau)-(v(\tau)-z(\tau)))\|^2\right)\mathrm{d}\tau \\
&\leq \int_t^{t+1} 2\left(\|\nabla y(\tau)\|^2 + \|\nabla\psi(\tau)\|^2 + \|\nabla(v+z)\|^2 + \|\nabla(v-z)\|^2\right)\mathrm{d}\tau \\
&\leq 8C_{10} + 2C_{11}, \quad \text{for } t > T^*.
\end{aligned} \tag{93}$$

Apply the uniform Gronwall inequality [34,36] to (92) and use (93) to conclude that (87) holds with

$$L_2 = \left(8C_{10} + 2C_{11} + \frac{2a^2}{d_1}|\Omega|\right)\exp\left(\frac{1}{d_1}\eta^4 K_2^{1/3}(8C_{10}+2C_{11})\right), \tag{94}$$

and $T_2 = \max\{T^*, T^{**}\} + 1$. The proof is completed. □

Finally we prove Theorem 1.4 (Main Theorem) on the existence of a global attractor, denoted by $\mathscr{A}$, for the coupled Brusselator semiflow $\{S(t)\}_{t\geq 0}$.

Proof of Theorem 1.4. In Lemma 2.2, Lemma 3.2, Lemma 3.3, and Lemma 4.3 we have shown respectively that the conditions (i), (ii), (iii), and (iv) of Lemma 3.1 are all satisfied by this coupled Brusselator semiflow $\{S(t)\}_{t\geq 0}$. Then we apply Lemma 3.1 to reach the conclusion that there exists a global attractor $\mathscr{A}$ in H for this coupled Brusselator semiflow $\{S(t)\}_{t\geq 0}$. □

5 Finite Dimensionality of the Global Attractor

In this section we show that the global attractor $\mathscr{A}$ of the coupled Brusselator semiflow $\{S(t)\}_{t\ge 0}$ has a finite Hausdorff dimension and a finite fractal dimension. Let $q_m = \limsup_{t\to\infty} q_m(t)$, where

$$q_m(t) = \sup_{g_0\in\mathscr{A}} \sup_{\substack{g_i\in H, \|g_i\|=1 \\ i=1,\cdots,m}} \left(\frac{1}{t}\int_0^t \operatorname{Tr}\left(A + F'(S(\tau)g_0)\right) \circ Q_m(\tau)\,\mathrm{d}\tau \right), \tag{95}$$

where $\mathscr{A}$ is the global attractor of the semiflow $\{S(t)\}_{t\ge 0}$ in H, $\operatorname{Tr}(A+F'(S(\tau)g_0))$ is the trace of the linear operator $A + F'(S(\tau)g_0)$, with $F(g)$ being the nonlinear map in (12), and $Q_m(t)$ stands for the orthogonal projection of the space H on the subspace spanned by $G_1(t),\dots,G_m(t)$, with

$$G_i(t) = L(S(t),g_0)g_i, \quad i = 1,\dots,m. \tag{96}$$

Here $F'(S(\tau)g_0)$ is the Fréchet derivative of the map F at $S(\tau)g_0$, and $L(S(t),g_0)$ is the Fréchet derivative of the map $S(t)$ at g_0, with t being fixed.

Lemma 5.1 ([36]). *If there is an integer m such that $q_m < 0$, then the Hausdorff dimension $d_H(\mathscr{A})$ and the fractal dimension $d_F(\mathscr{A})$ of $\mathscr{A}$ satisfy*

$$d_H(\mathscr{A}) \le m, \quad \text{and} \quad d_F(\mathscr{A}) \le m \max_{1\le j\le m-1}\left(1 + \frac{(q_j)_+}{|q_m|}\right) \le 2m. \tag{97}$$

It can be shown that for any given $t > 0$, $S(t)$ is Fréchet differentiable in H and its Fréchet derivative at g_0 is the bounded linear operator $L(S(t),g_0)$ given by

$$L(S(t),g_0)G_0 \stackrel{\text{def}}{=} G(t) = (U(t),V(t),W(t),Z(t)), \text{ for any } G_0 = (U_0,V_0,W_0,Z_0) \in H,$$

where $(U(t),V(t),W(t),Z(t))$ is the strong solution of the following initial-boundary value problem of the coupled Brusselator variational equations

$$\begin{aligned}
\frac{\partial U}{\partial t} &= d_1\Delta U + 2u(t)v(t)U + u^2(t)V - (b+1)U + D_1(W-U),\\
\frac{\partial V}{\partial t} &= d_2\Delta V - 2u(t)v(t)U - u^2(t)V + bU + D_2(Z-V),\\
\frac{\partial W}{\partial t} &= d_1\Delta W + 2w(t)z(t)W + w^2(t)Z - (b+1)W + D_1(U-W),\\
\frac{\partial Z}{\partial t} &= d_2\Delta Z - 2w(t)z(t)W - w^2(t)Z + bW + D_2(V-Z),\\
U &= V = W = Z = 0, \quad t>0,\ x\in\partial\Omega,\\
U(0) &= U_0, \quad V(0) = V_0, \quad W(0) = W_0, \quad Z(0) = Z_0.
\end{aligned} \tag{98}$$

Here $g(t) = (u(t), v(t), w(t), z(t)) = S(t)g_0$ is the solution of (12) with the initial condition $g(0) = g_0$. The initial value problem (98) can be written as

$$\frac{\mathrm{d}G}{\mathrm{d}t} = (A + F'(S(t)g_0))G, \quad G(0) = G_0. \tag{99}$$

Note that the invariance of $\mathscr{A}$ implies $\mathscr{A} \subset B_0$, where B_0 is the bounded absorbing set given in Lemma 2.2. Hence we have

$$\sup_{g_0 \in \mathscr{A}} \|S(t)g_0\|^2 \le K_0,$$

Theorem 5.2. *The global attractors $\mathscr{A}$ for the coupled Brusselator semiflow $\{S(t)\}_{t\ge 0}$ has a finite Hausdorff dimension and a finite fractal dimension.*

Proof. By Lemma 5.1, we shall estimate $\mathrm{Tr}\,(A + F'(S(\tau)g_0)) \circ Q_m(\tau)$. At any given time $\tau > 0$, let $\{\varphi_j(\tau) : j = 1, \ldots, m\}$ be an H-orthonormal basis for the subspace

$$Q_m(\tau)H = \mathrm{Span}\,\{G_1(\tau), \ldots, G_,(\tau)\},$$

where $G_1(t), \ldots, G_m(t)$ satisfy (99) with the respective initial values $G_{1,0}, \ldots, G_{m,0}$ and, without loss of generality, assuming that $G_{1,0}, \ldots, G_{m,0}$ are linearly independent in H. By Gram-Schmidt orthogonalization scheme,

$$\varphi_j(\tau) = (\varphi_j^1(\tau), \varphi_j^2(\tau), \varphi_j^3(\tau), \varphi_j^4(\tau)) \in E,$$

$j = 1, \ldots, m$, and $\varphi_j(\tau)$ are strongly measurable in τ. Let $d_0 = \min\{d_1, d_2\}$. Then we have

$$\begin{aligned} \mathrm{Tr}\,(A + F'(S(\tau)g_0) \circ Q_m(\tau) &= \sum_{j=1}^m \big(\langle A\varphi_j(\tau), \varphi_j(\tau)\rangle + \langle F'(S(\tau)g_0)\varphi_j(\tau), \varphi_j(\tau)\rangle\big) \\ &\le -d_0 \sum_{j=1}^m \|\nabla \varphi_j(\tau)\|^2 + J_1 + J_2 + J_3, \end{aligned} \tag{100}$$

where

$$\begin{aligned} J_1 &= \sum_{j=1}^m \int_\Omega 2u(\tau)v(\tau)\left(|\varphi_j^1(\tau)|^2 - \varphi_j^1(\tau)\varphi_j^2(\tau)\right)\mathrm{d}x \\ &\quad + \sum_{j=1}^m \int_\Omega 2w(\tau)z(\tau)\left(|\varphi_j^3(\tau)|^2 - \varphi_j^3(\tau)\varphi_j^4(\tau)\right)\mathrm{d}x, \\ J_2 &= \sum_{j=1}^m \int_\Omega \left(u^2(\tau)\left(\varphi_j^1(\tau)\varphi_j^2(\tau) - |\varphi_j^2(\tau)|^2\right) + w^2(\tau)\left(\varphi_j^3(\tau)\varphi_j^4(\tau) - |\varphi_j^4(\tau)|^2\right)\right)\mathrm{d}x \\ &\le \sum_{j=1}^m \int_\Omega \left(u^2(\tau)|\varphi_j^1(\tau)||\varphi_j^2(\tau)| + w^2(\tau)|\varphi_j^3(\tau)||\varphi_j^4(\tau)|\right)\mathrm{d}x, \end{aligned}$$

and

$$J_3 = \sum_{j=1}^{m} \int_\Omega \left(-(b+1)(|\varphi_j^1(\tau)|^2 + |\varphi_j^3(\tau)|^2) + b(\varphi_j^1(\tau)\varphi_j^2(\tau) + \varphi_j^3(\tau)\varphi_j^4(\tau))\right) \mathrm{d}x$$
$$- \sum_{j=1}^{m} \int_\Omega \left(D_1\left(\varphi_j^1(\tau) - \varphi_j^3(\tau)\right)^2 + D_2\left(\varphi_j^3(\tau) - \varphi_j^4(\tau)\right)^2\right) \mathrm{d}x$$
$$\le \sum_{j=1}^{m} \int_\Omega b\left(\varphi_j^1(\tau)\varphi_j^2(\tau) + \varphi_j^3(\tau)\varphi_j^4(\tau)\right) \mathrm{d}x.$$

We can estimate each of the three terms as follows. First, by the generalized Hölder inequality, Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ for $n \le 3$, which implies that there is a uniform constant $\delta > 0$ such that

$$\|\phi\|_{L^4} \le \delta \|\nabla\phi\|, \tag{101}$$

and using (83), (87), as well as the invariance of the global attractor $\mathscr{A}$, we get

$$\begin{aligned} J_1 \; &\le 2\sum_{j=1}^{m} \|u(\tau)\|_{L^4}\|v(\tau)\|_{L^4}\left(\|\varphi_j^1(\tau)\|_{L^4}^2 + \|\varphi_j^1(\tau)\|_{L^4}\|\varphi_j^2(\tau)\|_{L^4}\right) \\ &\quad + 2\sum_{j=1}^{m} \|w(\tau)\|_{L^4}\|z(\tau)\|_{L^4}\left(\|\varphi_j^3(\tau)\|_{L^4}^2 + \|\varphi_j^3(\tau)\|_{L^4}\|\varphi_j^4(\tau)\|_{L^4}\right) \\ &\le 4\delta L_2^{1/2}K_1^{1/4}\sum_{j=1}^{m}\|\varphi_j(\tau)\|_{L^4}^2. \end{aligned} \tag{102}$$

Now we apply the Garliardo–Nirenberg interpolation inequality, cf. [34, Theorem B.3],

$$\|\varphi\|_{W^{k,p}} \le C\|\varphi\|_{W^{m,q}}^{\theta}\|\varphi\|_{L^r}^{1-\theta}, \quad \text{for } \varphi \in W^{m,q}(\Omega), \tag{103}$$

provided that $p,q,r \ge 1, 0 < \theta \le 1$, and

$$k - \frac{n}{p} \le \theta\left(m - \frac{n}{q}\right) - (1-\theta)\frac{n}{r}, \quad \text{where } n = \dim\Omega.$$

Here with $W^{k,p}(\Omega) = L^4(\Omega), W^{m,q}(\Omega) = H_0^1(\Omega), L^r(\Omega) = L^2(\Omega)$, and $\theta = n/4 \le 3/4$, it follows from (103) that

$$\|\varphi_j(\tau)\|_{L^4} \le C\|\nabla\varphi_j(\tau)\|^{\frac{n}{4}}\|\varphi_j(\tau)\|^{1-\frac{n}{4}} = C\|\nabla\varphi_j(\tau)\|^{\frac{n}{4}}, \quad j = 1,\dots,m, \tag{104}$$

since $\|\varphi_j(\tau)\| = 1$, where C is a uniform constant. Substitute (104) into (102) to obtain

$$J_1 \le 4\delta L_2^{1/2}K_1^{1/4}C^2\sum_{j=1}^{m}\|\nabla\varphi_j(\tau)\|^{\frac{n}{2}}. \tag{105}$$

By a similar argument, we can get

$$J_2 \le \delta^2 L_2 \sum_{j=1}^{m} \|\varphi_j(\tau)\|_{L^4}^2 \le \delta^2 L_2 C^2 \sum_{j=1}^{m} \|\nabla\varphi_j(\tau)\|^{\frac{n}{2}}. \tag{106}$$

Moreover, we have

$$J_3 \le \sum_{j=1}^{m} b\|\varphi_j(\tau)\|^2 = bm. \tag{107}$$

Substituting (105), (106) and (107) into (100), we obtain

$$\begin{aligned}&\mathrm{Tr}\,(A+F'(S(\tau)g_0)\circ Q_m(\tau)\\ &\quad\le -d_0\sum_{j=1}^{m}\|\nabla\varphi_j(\tau)\|^2 + (4\delta L_2^{1/2}K_1^{1/4}+\delta^2L_2)C^2\sum_{j=1}^{m}\|\nabla\varphi_j(\tau)\|^{\frac{n}{2}} + bm.\end{aligned} \tag{108}$$

By Young's inequality, for $n \le 3$, we have

$$(4\delta L_2^{1/2}K_1^{1/4}+\delta^2L_2)C^2\sum_{j=1}^{m}\|\nabla\varphi_j(\tau)\|^{\frac{n}{2}} \le \frac{d_0}{2}\sum_{j=1}^{m}\|\nabla\varphi_j(\tau)\|^2 + \Phi(n)m,$$

where $\Phi(n)$ is a uniform constant depending only on $n = \dim(\Omega)$. Hence,

$$\mathrm{Tr}\,(A+F'(S(\tau)g_0)\circ Q_m(\tau) \le -\frac{d_0}{2}\sum_{j=1}^{m}\|\nabla\varphi_j(\tau)\|^2 + (\Phi(n)+b)\,m, \quad \tau>0,\, g_0\in\mathscr{A}.$$

According to the generalized Sobolev–Lieb–Thirring inequality [36, Appendix, Corollary 4.1], since $\{\varphi_1(\tau),\dots,\varphi_m(\tau)\}$ is an orthonormal set in H, so there exists a uniform constant $\Psi > 0$ only depending on the shape and dimension of Ω, such that

$$\sum_{j=1}^{m}\|\nabla\varphi_j(\tau)\|^2 \ge \Psi\frac{m^{1+\frac{2}{n}}}{|\Omega|^{\frac{2}{n}}}. \tag{109}$$

Therefore, we end up with

$$\mathrm{Tr}\,(A+F'(S(\tau)g_0)\circ Q_m(\tau) \le -\frac{d_0\Psi}{2|\Omega|^{\frac{2}{n}}}m^{1+\frac{2}{n}} + (\Phi(n)+b)\,m, \quad \tau>0,\, g_0\in\mathscr{A}. \tag{110}$$

Then we can conclude that

$$\begin{aligned}q_m(t) &= \sup_{g_0\in\mathscr{A}}\ \sup_{\substack{g_i\in H,\|g_i\|=1\\ i=1,\cdots,m}}\left(\frac{1}{t}\int_0^t \mathrm{Tr}\left(A+F'(S(\tau)g_0)\right)\circ Q_m(\tau)\,\mathrm{d}\tau\right)\\ &\le -\frac{d_0\Psi}{2|\Omega|^{\frac{2}{n}}}m^{1+\frac{2}{n}} + (\Phi(n)+b)\,m, \quad \text{for any } t>0,\end{aligned} \tag{111}$$

so that

$$q_m = \limsup_{t\to\infty} q_m(t) \le -\frac{d_0\Psi}{2|\Omega|^{\frac{2}{n}}} m^{1+\frac{2}{n}} + (\Phi(n)+b)m < 0, \tag{112}$$

if the integer m satisfies the following condition,

$$m-1 \le \left(\frac{2(\Phi(n)+b)}{d_0\Psi}\right)^{n/2} |\Omega| < m. \tag{113}$$

According to Lemma 5.1, we have shown that the Hausdorff dimension and the fractal dimension of the global attractor $\mathscr{A}$ are finite and their upper bounds are given by

$$d_H(\mathscr{A}) \le m \quad \text{and} \quad d_F(\mathscr{A}) \le 2m, \tag{114}$$

where m satisfies (113). □

As a remark, with some adjustment in proof, the corresponding results are valid for the coupled two-cell Gray-Scott equations, Selkov equations, and Schnackenberg equations.

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Projectors on the Generalized Eigenspaces for Partial Differential Equations with Time Delay

Arnaut Ducrot, Pierre Magal, and Shigui Ruan

Abstract To study the nonlinear dynamics, such as Hopf bifurcation, of partial differential equations with delay, one needs to consider the characteristic equation associated to the linearized equation and to determine the distribution of the eigenvalues; that is, to study the spectrum of the linear operator. In this paper we study the projectors on the generalized eigenspaces associated to some eigenvalues for linear partial differential equations with delay. We first rewrite partial differential equations with delay as non-densely defined semilinear Cauchy problems, then obtain formulas for the integrated solutions of the semilinear Cauchy problems with non-dense domain by using integrated semigroup theory, from which we finally derive explicit formulas for the projectors on the generalized eigenspaces associated to some eigenvalues. As examples, we apply the obtained results to study a reaction-diffusion equation with delay and an age-structured model with delay.

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1 Introduction

Taking the interactions of spatial diffusion and time delay into account, a single species population model can be described by a partial differential equation with time delay as follows:

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$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \mathrm{d}\dfrac{\partial^2 u(t,x)}{\partial x^2} - au(t-r,x)[1+u(t,x)], \ t>0, \ x \in [0,\pi], \\ \dfrac{\partial u(t,x)}{\partial x} = 0, \ x = 0,\pi, \\ u(0,.) = u_0 \in C([0,\pi],\mathbb{R}), \end{cases} \tag{1}$$

where $u(t,x)$ denotes the density of the species at time t and location x, $d>0$ is the diffusion rate of the species, $r>0$ is the time delay constant, and $a>0$ is a constant. Equation (1) has been studied by many researchers, for example, Yoshida [55], Memory [36], and Busenberg and Huang [12] investigated Hopf bifurcation of the equation.

We consider the Banach space $Y = C([0,\pi],\mathbb{R})$ endowed with the usual supremum norm. Define $B : D(B) \subset Y \to Y$ by

$$B\varphi = \varphi''$$

with

$$D(B) = \left\{\varphi \in C^2([0,\pi],\mathbb{R}) : \varphi'(0) = \varphi'(\pi) = 0\right\}.$$

Denote

$$\hat{L}(y) = -ay(-r), \ f(y) = -ay(0)y(-r).$$

Equation (1) can be written as an abstract partial functional differential equations (PFDE) (see, e.g., Travis and Webb [48, 49], Wu [54] and Faria [18]):

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = By(t) + \widehat{L}(y_t) + f(t,y_t), \forall t \geq 0, \\ y_0 = \varphi \in C_B, \end{cases} \tag{2}$$

where

$$C_B := \{\varphi \in C([-r,0];Y) : \ \varphi(0) \in \overline{D(B)}\},$$

$y_t \in C_B$ satisfies $y_t(\theta) = y(t+\theta), \theta \in [-r,0]$, $\widehat{L} : C_B \to Y$ is a bounded linear operator, and $f : \mathbb{R} \times C_B \to Y$ is a continuous map. In fact, many other partial differential equations with time delay can also be written in the form of system (2) (see Wu [54]).

In the last 30 years, PFDE have been studied extensively by many researchers. For example, Travis and Webb [48, 49], Fitzgibbon [20], Martin and Smith [30, 31], Arino and Sanchez [8] investigated the fundamental theory; Parrot [38] considered the linearized stability; Memory [37] studied the stable and unstable manifolds; Lin et al. [27], Faria et al. [19] and Adimy et al. [4] established the existence and smoothness of center manifolds; Faria [18] developed the normal form theory; Ruan et al. [41] and Ruan and Zhang [42] discussed the homoclinic bifurcation. For more detailed theories and results, we refer to the monograph of Wu [54].

In order to study the dynamics of system (2), such as Hopf bifurcation, we need to consider the characteristic equation associated to the linearized equation and to determine the distribution of the eigenvalues; that is, to carry out the spectrum analysis of the linear operator. The aim of this article is to obtain explicit formulas for the projectors on the generalized eigenspaces associated to some eigenvalues for the linear PFDE

$$\begin{cases} \frac{\mathrm{d}y(t)}{\mathrm{d}t} = By(t) + \widehat{L}(y_t), \forall t \geq 0 \\ y_0 = \varphi \in C_B. \end{cases} \tag{3}$$

In the context of ordinary functional differential equations with $Y = \mathbb{R}^n$ (and B is bounded), this problem has been studied since the 1970s (see Hale and Verduyn Lunel [22], Hassard et al. [23] and Arino et al. [7]); the usual approach is based on the formal adjoint system. The method was recently further studied in the monograph of Diekmann et al. [14] using the so called sun star adjoint spaces, see also Kaashoek and Verduyn Lunel [24], Frasson and Verduyn Lunel [21], Verduyn Lunel [50], Diekmann et al. [13] and the references cited therein. We refer to Liu et al. [28] for a more recent study on this topic. Let us also mention that the explicit formula for the projectors on the generalized eigenspaces turns to be a crucial tool to study the bifurcations by using normal form arguments (see Liu et al [29] in the context of abstract non-densely defined Cauchy problems).

There are a few approaches to treat problem (2). Webb [51] and Travis and Webb [48, 49] viewed the problem as a nonlinear Cauchy problem and focused on many aspects using this approach. Arino and Sanchez [9] and Kappel [25] used the variation of constant formula and worked directly with the system. See also Ruess [43, 44], Rhandi [40] and the references cited therein. We would like to point out that the results and techniques in the above mentioned papers do not apply directly to our problem, as we are not discussing the existence and local stability of solutions for linear partial differential equations with delay. Instead, we study the projectors on the generalized eigenspaces associated to some eigenvalues for linear partial differential equations with delay so that we can study bifurcations in such equations. Recently, Adimy [1,2], Adimy and Arino [3], and Thieme [45] employed the integrated semigroup theory (see Ezzinbi and Adimy [17] for a survey). Here we use a formulation that is between the formulations of Adimy [1,2] and Thieme [45] and more closely related to the one of Travis and Webb [48, 49]. See also Adimy et al. [4].

The rest of the paper is organized as follows. In Sect. 2 we will show how to formulate the PFDE as a semilinear Cauchy problem with non-dense domain. In Sect. 3 we recall some results on integrated semigroup theory and spectrum analysis. Section 4 presents main results on projectors on the eigenspaces. Section 5 deals with a special case for a simple eigenvalue. Some examples and discussions are given in Sect. 6.

2 Preliminaries

Let $B : D(B) \subset Y \to Y$ be a linear operator on a Banach space $(Y, \| \|_Y)$. Assume that it is a Hille–Yosida operator; that is, there exist two constants, $\omega_B \in \mathbb{R}$ and $M_B > 0$, such that $(\omega_B, +\infty) \subset \rho(B)$ and

$$\left\|(\lambda I - B)^{-n}\right\| \leq \frac{M_B}{(\lambda - \omega_B)^n}, \ \forall \lambda > \omega_B, \ \forall n \geq 1.$$

Set

$$Y_0 := \overline{D(B)}.$$

Consider B_0, the part of B in Y_0, which is defined by

$$B_0 y = By \text{ for each } y \in D(B_0)$$

with

$$D(B_0) := \{y \in D(B) : By \in Y_0\}.$$

For $r \geq 0$, set

$$C := C([-r, 0]; Y)$$

which is endowed with the supremum norm

$$\|\varphi\|_\infty = \sup_{\theta \in [-r,0]} \|\varphi(\theta)\|_Y.$$

Consider the PFDE:

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = By(t) + \widehat{L}(y_t) + f(t, y_t), \forall t \geq 0, \\ y_0 = \varphi \in C_B, \end{cases} \tag{4}$$

where $y_t \in C_B$ satisfies $y_t(\theta) = y(t+\theta), \theta \in [-r, 0]$, $\widehat{L} : C_B \to Y$ is a bounded linear operator, and $f : \mathbb{R} \times C_B \to Y$ is a continuous map. Since B is a Hille–Yosida operator, it is well known that B_0, the part of B in Y_0, generates a C_0-semigroup of bounded linear operators $\left\{T_{B_0}(t)\right\}_{t\geq 0}$ on Y_0, and B generates an integrated semigroup $\{S_B(t)\}_{t\geq 0}$ on Y. The solution of the Cauchy problem (4) must be understood as a fixed point of

$$y(t) = T_{B_0}(t)\varphi(0) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S_B(t-s)\left[\widehat{L}(y_s) + f(s, y_s)\right]\mathrm{d}s.$$

Since $\left\{T_{B_0}(t)\right\}_{t\geq 0}$ acts on Y_0, we observe that it is necessary to assume that

$$\varphi(0) \in Y_0 \Rightarrow \varphi \in C_B.$$

In order to study the PFDE (4) by using the integrated semigroup theory, we consider the PFDE (4) as an abstract non-densely defined Cauchy problem. Firstly, we regard the PFDE (4) as a PDE. Define $u \in C([0,+\infty) \times [-r,0], Y)$ by

$$u(t,\theta) = y(t+\theta), \ \forall t \geq 0, \ \forall \theta \in [-r,0].$$

Note that if $y \in C^1([-r,+\infty), Y)$, then

$$\frac{\partial u(t,\theta)}{\partial t} = y'(t+\theta) = \frac{\partial u(t,\theta)}{\partial \theta}.$$

Hence, we must have

$$\frac{\partial u(t,\theta)}{\partial t} - \frac{\partial u(t,\theta)}{\partial \theta} = 0, \ \forall t \geq 0, \ \forall \theta \in [-r,0].$$

Moreover, for $\theta = 0$, we obtain

$$\frac{\partial u(t,0)}{\partial \theta} = y'(t) = By(t) + \widehat{L}(y_t) + f(t,y_t) = Bu(t,0) + \widehat{L}(u(t,.)) + f(t,u(t,.)), \forall t \geq 0.$$

Therefore, we deduce formally that u must satisfy a PDE

$$\begin{cases} \dfrac{\partial u(t,\theta)}{\partial t} - \dfrac{\partial u(t,\theta)}{\partial \theta} = 0, \\ \dfrac{\partial u(t,0)}{\partial \theta} = Bu(t,0) + \widehat{L}(u(t,.)) + f(t,u(t,.)), \forall t \geq 0, \\ u(0,.) = \varphi \in C_B. \end{cases} \tag{5}$$

In order to rewrite the PDE (5) as an abstract non-densely defined Cauchy problem, we extend the state space to take into account the boundary conditions. This can be accomplished by adopting the following state space

$$X = Y \times C$$

taken with the usual product norm

$$\left\| \begin{pmatrix} y \\ \varphi \end{pmatrix} \right\| = \|y\|_Y + \|\varphi\|_\infty.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}, \ \forall \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in D(A) \tag{6}$$

with

$$D(A) = \{0_Y\} \times \{\varphi \in C^1([-r,0], Y), \ \varphi(0) \in D(B)\}.$$

Note that A is non-densely defined because

$$X_0 := \overline{D(A)} = \{0_Y\} \times C_B \neq X.$$

We also define $L : X_0 \to X$ by

$$L\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} := \begin{pmatrix} \widehat{L}(\varphi) \\ 0_C \end{pmatrix}$$

and $F : \mathbb{R} \times X_0 \to X$ by

$$F\left(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}\right) := \begin{pmatrix} f(t,\varphi) \\ 0_C \end{pmatrix}.$$

Set

$$v(t) := \begin{pmatrix} 0_Y \\ u(t) \end{pmatrix}.$$

Now we can consider the PDE (5) as the following non-densely defined Cauchy problem

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + L(v(t)) + F(t, v(t)),\ t \geq 0;\ \ v(0) = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in X_0. \tag{7}$$

3 Some Results on Integrated Solutions and Spectra

In this section we will first study the integrated solutions of the Cauchy problem (7) in the special case

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix} h(t) \\ 0 \end{pmatrix}, t \geq 0,\ v(0) = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in X_0, \tag{8}$$

where $h \in L^1((0,\tau), Y)$. Recall that $v \in C([0,\tau], X)$ is an integrated solution of (8) if and only if

$$\int_0^t v(s)\mathrm{d}s \in D(A), \forall t \in [0,\tau] \tag{9}$$

and

$$v(t) = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} + A\int_0^t v(s)\mathrm{d}s + \int_0^t \begin{pmatrix} h(s) \\ 0 \end{pmatrix} \mathrm{d}s. \tag{10}$$

In the sequel, we will use the integrated semigroup theory to define such an integrated solution. We refer to Arendt [5], Thieme [46], Kellermann and Hieber [26], and the book of Arendt et al. [6] for further details on this subject. We also refer to Magal and Ruan [33–35] for more results and update references.

From (9) we note that if v is an integrated solution we must have

$$v(t) = \lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} v(s)\mathrm{d}s \in \overline{D(A)}.$$

Hence

$$v(t) = \begin{pmatrix} 0_Y \\ u(t) \end{pmatrix}$$

with

$$u \in C([0,\tau], C_B).$$

We introduce some notations. Let $L : D(L) \subset X \to X$ be a linear operator on a complex Banach space X. Denote by $\rho(L)$ the resolvent set of $L, N(L)$ the null space of L, and $R(L)$ the range of L, respectively. The spectrum of L is $\sigma(L) = \mathbb{C}\backslash\rho(L)$. The *point spectrum* of L is the set

$$\sigma_P(L) := \{\lambda \in \mathbb{C} : N(\lambda I - L) \neq \{0\}\}.$$

The *essential spectrum* (in the sense of Browder [11]) of L is denoted by $\sigma_{ess}(L)$. That is, the set of $\lambda \in \sigma(L)$ such that at least one of the following holds: (1) $R(\lambda I - L)$ is not closed; (2) λ is a limit point of $\sigma(L)$; (3) $N_\lambda(L) := \bigcup_{k=1}^{\infty} N\left((\lambda I - L)^k\right)$ is infinite dimensional. Define

$$X_{\lambda_0} = \bigcup_{n\geq 0} N((\lambda_0 - L)^n).$$

Let Y be a subspace of X. Then we denote by $L_Y : D(L_Y) \subset Y \to Y$ the part of L on Y, which is defined by

$$L_Y y = Ly, \forall y \in D(L_Y) := \{y \in D(L) \cap Y : Ly \in Y\}.$$

Definition 3.1. Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}_{t\geq 0}$ on a Banach space X. We define the *growth bound* $\omega_0(L) \in [-\infty, +\infty)$ of L by

$$\omega_0(L) := \lim_{t\to+\infty} \frac{\ln\left(\|T_L(t)\|_{\mathcal{L}(X)}\right)}{t}.$$

The *essential growth bound* $\omega_{0,\text{ess}}(L) \in [-\infty, +\infty)$ of L is defined by

$$\omega_{0,\text{ess}}(L) := \lim_{t\to+\infty} \frac{\ln(\|T_L(t)\|_{\text{ess}})}{t},$$

where $\|T_L(t)\|_{\text{ess}}$ is the essential norm of $T_L(t)$ defined by

$$\|T_L(t)\|_{\text{ess}} = \kappa(T_L(t)B_X(0,1)),$$

here $B_X(0,1) = \{x \in X : \|x\|_X \leq 1\}$, and for each bounded set $B \subset X$,

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon\}$$

is the Kuratovsky measure of non-compactness.

We have the following result. The existence of the projector was first proved by Webb [52, 53] and the fact that there is a finite number of points of the spectrum is proved by Engel and Nagel [16, Corollary 2.1, p. 258].

Theorem 3.2. *Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}$ on a Banach space X. Then*

$$\omega_0(L) = \max\left(\omega_{0,\mathrm{ess}}(L), \max_{\lambda \in \sigma(L) \setminus \sigma_{\mathrm{ess}}(L)} \mathrm{Re}(\lambda)\right).$$

Assume in addition that $\omega_{0,\mathrm{ess}}(L) < \omega_0(L)$. Then for each $\gamma \in (\omega_{0,\mathrm{ess}}(L), \omega_0(L)]$, $\{\lambda \in \sigma(L) : \mathrm{Re}(\lambda) \geq \gamma\} \subset \sigma_p(L)$ is nonempty, finite and contains only poles of the resolvent of L. Moreover, there exists a finite rank bounded linear operator of projection $\Pi : X \to X$ satisfying the following properties:

(a) $\Pi(\lambda - L)^{-1} = (\lambda - L)^{-1}\Pi, \forall \lambda \in \rho(L)$;
(b) $\sigma\left(L_{\Pi(X)}\right) = \{\lambda \in \sigma(L) : \mathrm{Re}(\lambda) \geq \gamma\}$;
(c) $\sigma\left(L_{(I-\Pi)(X)}\right) = \sigma(L) \setminus \sigma\left(L_{\Pi(X)}\right)$.

In Theorem 3.2 the projector Π is the projection on the direct sum of the generalized eigenspaces of L associated to all points $\lambda \in \sigma(L)$ with $\mathrm{Re}(\lambda) \geq \gamma$. As a consequence of Theorem 3.2 we have following corollary.

Corollary 3.3. *Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}$ on a Banach space X, and assume that $\omega_{0,\mathrm{ess}}(L) < \omega_0(L)$. Then*

$$\{\lambda \in \sigma(L) : \mathrm{Re}(\lambda) > \omega_{0,\mathrm{ess}}(L)\} \subset \sigma_P(L)$$

and each $\widehat{\lambda} \in \{\lambda \in \sigma(L) : \mathrm{Re}(\lambda) > \omega_{0,\mathrm{ess}}(L)\}$ is a pole of the resolvent of L. That is, $\widehat{\lambda}$ is isolated in $\sigma(L)$, and there exists an integer $k_0 \geq 1$ (the order of the pole) such that the Laurent's expansion of the resolvent takes the following form

$$(\lambda I - L)^{-1} = \sum_{n=-k_0}^{\infty} (\lambda - \lambda_0)^n B_n^{\lambda_0},$$

where $\{B_n^{\lambda_0}\}$ are bounded linear operators on X, and the above series converges in the norm of operators whenever $|\lambda - \lambda_0|$ is small enough.

The following result is due to Magal and Ruan [35, see Lemma 2.1 and Proposition 3.6].

Theorem 3.4. *Let* $(X,\|.\|)$ *be a Banach space and* $L : D(L) \subset X \to X$ *be a linear operator. Assume that* $\rho(L) \neq \emptyset$ *and* L_0*, the part of* L *in* $\overline{D(L)}$*, is the infinitesimal generator of a linear* C_0*-semigroup* $\left\{T_{L_0}(t)\right\}_{t\geq 0}$ *on the Banach space* $\overline{D(L)}$*. Then*

$$\sigma(L) = \sigma(L_0).$$

Let $X_0 := \overline{D(L)}$, $\Pi_0 : \overline{D(L)} \to \overline{D(L)}$ *be a bounded linear operator of projection. Assume that*

$$\Pi_0 (\lambda I - L_0)^{-1} = (\lambda I - L_0)^{-1} \Pi_0, \ \forall \lambda > \omega$$

and

$$\Pi_0\left(\overline{D(L)}\right) \subset D(L_0) \text{ and } L_0\mid_{\Pi_0\left(\overline{D(L)}\right)} \text{ is bounded.}$$

Then there exists a unique bounded linear operator of projection Π *on* X *satisfying the following properties:*

(i) $\Pi\mid_{\overline{D(L)}} = \Pi_0$.
(ii) $\Pi(X) \subset \overline{D(L)}$.
(iii) $\Pi(\lambda I - L)^{-1} = (\lambda I - L)^{-1}\Pi, \forall \lambda > \omega$.

Moreover, for each $x \in X$ *we have the following approximation formula*

$$\Pi x = \lim_{\lambda\to+\infty} \Pi_0 \lambda (\lambda I - L)^{-1} x.$$

We return to the Cauchy problem (8) and investigate some properties of the linear operator A.

Theorem 3.5. *For the operator* A *defined in (6), the resolvent set of* A *satisfies*

$$\rho(A) = \rho(B).$$

Moreover, for each $\lambda \in \rho(A)$, *we have the following explicit formula for the resolvent of* A :

$$\begin{aligned} &(\lambda I - A)^{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix}0_Y\\ \psi\end{pmatrix} \\ &\Leftrightarrow \psi(\theta) = e^{\lambda\theta}(\lambda I - B)^{-1}[\varphi(0) + \alpha] + \int_\theta^0 e^{\lambda(\theta - s)}\varphi(s)\,ds. \end{aligned} \tag{11}$$

Proof. Let us first prove that $\rho(B) \subset \rho(A)$. If $\lambda \in \rho(B)$, for $\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} \in X$ we must find $\begin{pmatrix}0_{\mathbb{R}^n}\\ \psi\end{pmatrix} \in D(A)$ such that

$$
\begin{aligned}
&(\lambda I-A)\begin{pmatrix}0_Y\\ \psi\end{pmatrix}=\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\\
&\quad\Leftrightarrow\begin{cases}\psi'(0)-B\psi(0)=\alpha\\ \lambda\psi-\psi'=\varphi\end{cases}\\
&\quad\Leftrightarrow\begin{cases}(\lambda I-B)\,\psi(0)=\alpha+\varphi(0)\\ \lambda\psi-\psi'=\varphi\end{cases}\\
&\quad\Leftrightarrow\begin{cases}(\lambda I-B)\,\psi(0)=\alpha+\varphi(0)\\ \psi(\theta)=\mathrm{e}^{\lambda(\theta-\widehat{\theta})}\psi\left(\widehat{\theta}\right)+\int_{\widehat{\theta}}^{\theta}\mathrm{e}^{\lambda(\theta-l)}\varphi(l)\,\mathrm{d}l,\forall\theta\geq\widehat{\theta}\end{cases}\\
&\quad\Leftrightarrow\begin{cases}(\lambda I-B)\,\psi(0)=\alpha+\varphi(0)\\ \psi\left(\widehat{\theta}\right)=\mathrm{e}^{\lambda\widehat{\theta}}\psi(0)-\int_0^{\widehat{\theta}}\mathrm{e}^{\lambda(\widehat{\theta}-l)}\varphi(l)\,\mathrm{d}l,\forall\widehat{\theta}\in[-r,0],\end{cases}\\
&\quad\Leftrightarrow\psi\left(\widehat{\theta}\right)=\mathrm{e}^{\lambda\widehat{\theta}}(\lambda I-B)^{-1}[\alpha+\varphi(0)]-\int_0^{\widehat{\theta}}\mathrm{e}^{\lambda(\widehat{\theta}-l)}\varphi(l)\,\mathrm{d}l,\forall\widehat{\theta}\in[-r,0].
\end{aligned}
$$

Therefore, we obtain that $\lambda\in\rho(A)$ and the formula in (11) holds.

It remains to prove that $\rho(A)\subset\rho(B)$, that is $\sigma(B)\subset\sigma(A)$. First, from the above computations we have the following result

$$
R(\lambda I-A)=\left\{\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\in X:\ \ \alpha+\varphi(0)\in R(\lambda I-B)\right\}. \tag{12}
$$

Moreover, we have

$$
\begin{pmatrix}0_Y\\ \psi\end{pmatrix}\in N(\lambda I-A)\ \Leftrightarrow\ \exists y\in D(B)\ \begin{cases}y\in N(\lambda I-B)\\ \psi(\theta)=\mathrm{e}^{\lambda\theta}y.\end{cases}
$$

Thus if $\lambda\in\sigma_P(B)$ (the point spectrum of B) then there exists $y\in N(\lambda I-B)\setminus\{0_Y\}$, and a vector $\begin{pmatrix}0_Y\\ \psi\end{pmatrix}\in N(\lambda I-A)\setminus\{0_C\}$ with $\psi(\theta)=\mathrm{e}^{\lambda\theta}y$. Thus $\lambda\in\sigma_P(A)$.

Assume that $\lambda\in\sigma(B)\setminus\sigma_P(B)$. Then $N(\lambda I-B)=\{0_Y\}$, and since $\rho(B)\neq\emptyset$, we deduce that B is closed. So if $R(\lambda I-B)=Y$, by using Theorem II.20 p.30 in [10], we deduce that $(\lambda I-B)$ is invertible, that is, $(\lambda I-B)$ is a bijection from $D(B)$ into Y, and there exists $C>0$ such that

$$
\left\|(\lambda I-B)^{-1}\right\|\leq C,
$$

so $\lambda\in\rho(B)$, a contradiction.

We deduce that $\lambda\in\sigma(B)\setminus\sigma_P(B)$, then $N(\lambda I-B)=\{0_Y\}$ and $R(\lambda I-B)\neq Y$. Thus $N(\lambda I-A)=\{0_C\}$. Therefore $(\lambda I-A)$ is one-to-one but not onto because of (12). Thus $\sigma(B)\subset\sigma(A)$ and this completes the proof of Theorem 13. □

Lemma 3.6. *The linear operator $A:D(A)\subset X\to X$ is a Hille–Yosida operator. More precisely, for each $\omega_A>\omega_B$, there exists $M_A\geq 1$ such that*

$$
\left\|(\lambda I-A)^{-n}\right\|_{\mathcal{L}(X)}\leq\frac{M_A}{(\lambda-\omega_A)^n},\forall n\geq 1,\forall\lambda>\omega_A. \tag{13}
$$

Proof. Let $\omega_A > \omega_B$. Since B is a Hille–Yosida operator on Y, following Lemma 5.1 in Pazy [39], we can find an equivalent norm $|.|_Y$ on Y such that

$$\left|(\lambda I - B)^{-1} x\right| \leq \frac{|x|}{\lambda - \omega_B} \ \ \forall \lambda > \omega_B, \ \ \forall x \in Y.$$

Then we define $|.|$ the equivalent norm on X by

$$\left|\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}\right| = |\alpha| + \|\varphi\|_{\omega_A},$$

where

$$\|\varphi\|_{\omega_A} := \sup_{\theta \in [-r,0]} \left| \mathrm{e}^{-\omega_A \theta} \varphi(\theta) \right|.$$

Using (11) and the above results, we obtain for each $\lambda > \omega_A$ that

$$\begin{aligned}
&\left|(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}\right| \\
&\leq \sup_{\theta \in [-r,0]} \left[\mathrm{e}^{-\omega_A \theta} \mathrm{e}^{\lambda \theta} \left| (\lambda I - B)^{-1} [\varphi(0) + \alpha] \right| + \mathrm{e}^{-\omega_A \theta} \int_\theta^0 \mathrm{e}^{\lambda(\theta - s)} |\varphi(s)| \, \mathrm{d}s \right] \\
&\leq \sup_{\theta \in [-r,0]} \left[\mathrm{e}^{-\omega_A \theta} \mathrm{e}^{\lambda \theta} \frac{1}{\lambda - \omega_B} [|\varphi(0)| + |\alpha|] + \mathrm{e}^{-\omega_A \theta} \mathrm{e}^{\lambda \theta} \int_\theta^0 \mathrm{e}^{-(\lambda - \omega_A)s} \mathrm{d}s \, \|\varphi\|_{\omega_A} \right] \\
&= \frac{1}{\lambda - \omega_A} |\alpha| + \sup_{\theta \in [-r,0]} [\frac{\mathrm{e}^{-\omega_A \theta} \mathrm{e}^{\lambda \theta}}{\lambda - \omega_A} |\varphi(0)| + \frac{\mathrm{e}^{-\omega_A \theta} \mathrm{e}^{\lambda \theta} \left[\mathrm{e}^{-(\lambda - \omega_A)\theta} - 1 \right]}{\lambda - \omega_A} \|\varphi\|_{\omega_A}] \\
&\leq \frac{1}{\lambda - \omega_A} \left[|\alpha| + \|\varphi\|_{\omega_A} \right] \\
&= \frac{1}{\lambda - \omega_A} \left| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right|.
\end{aligned}$$

Therefore, (13) holds and the proof is completed. □

Since A is a Hille–Yosida operator, A generates a non-degenerated integrated semigroup $\{S_A(t)\}_{t \geq 0}$ on X. It follows from Thieme [46] and Kellerman and Hieber [26] that the abstract Cauchy problem (8) has at most one integrated solution.

Lemma 3.7. *Let $h \in L^1((0,\tau), Y)$ and $\varphi \in C([-r,0], Y)$ with $\varphi(0) \in Y_0$. Then there exists an unique integrated solution $t \to v(t)$ of the Cauchy problem (8) which can be expressed explicitly by the following formula*

$$v(t) = \begin{pmatrix} 0_Y \\ u(t) \end{pmatrix}$$

with

$$u(t)(\theta) = y(t+\theta), \forall t \in [0,\tau], \forall \theta \in [-r,0], \tag{14}$$

where

$$y(t) = \begin{cases} \varphi(t), & t \in [-r,0], \\ T_{B_0}(t)\varphi(0) + (S_B \diamond h)(t), & t \in [0,\tau] \end{cases}$$

and

$$(S_B \diamond h)(t) := \frac{\mathrm{d}}{\mathrm{d}t}(S_B * h)(t),\ (S_B * h)(t) := \int_0^t S_B(t-s)h(s)\mathrm{d}s.$$

Proof. Since A is a Hille–Yosida operator, there is at most one integrated solution of the Cauchy problem (8). So it is sufficient to prove that u defined by (14) satisfies for each $t \in [0,\tau]$ the following

$$\begin{pmatrix} 0_Y \\ \int_0^t u(l)dl \end{pmatrix} \in D(A) \tag{15}$$

and

$$\begin{pmatrix} 0_Y \\ u(t) \end{pmatrix} = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} + A\begin{pmatrix} 0_Y \\ \int_0^t u(l)\mathrm{d}l \end{pmatrix} + \begin{pmatrix} \int_0^t h(l)\mathrm{d}l \\ 0 \end{pmatrix}. \tag{16}$$

Since

$$\int_0^t u(l)(\theta)\,\mathrm{d}l = \int_0^t y(l+\theta)\mathrm{d}l = \int_\theta^{t+\theta} y(s)\mathrm{d}s$$

and $y \in C([-r,\tau],Y)$, the map $\theta \to \int_0^t u(l)(\theta)\,\mathrm{d}l$ belongs to $C^1([-r,0],Y)$. We also observe that

$$\int_0^t u(l)(0)\,\mathrm{d}l = \int_0^t y(l)\,\mathrm{d}l = \int_0^t T_{B_0}(l)\varphi(0) + (S_B \diamond h)(l)\mathrm{d}l$$
$$= \int_0^t T_{B_0}(l)\varphi(0)\,\mathrm{d}l + (S_B * h)(t) \in D(B),$$

therefore, (15) follows. Moreover,

$$A\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}$$

whenever $\varphi \in C^1([-r,0],Y)$ with $\varphi(0) \in D(B)$. Hence

$$A\begin{pmatrix} 0 \\ \int_0^t u(l)dl \end{pmatrix} = \begin{pmatrix} -[y(t)-y(0)] + B\int_0^t y(s)\mathrm{d}s \\ [y(t+.)-y(.)] \end{pmatrix}$$
$$= -\begin{pmatrix} 0 \\ \varphi \end{pmatrix} + \begin{pmatrix} -[y(t)-\varphi(0)] + B\int_0^t y(s)\mathrm{d}s \\ y(t+.) \end{pmatrix}.$$

Therefore, (16) is satisfied if and only if

$$y(t) = \varphi(0) + B\int_0^t y(s)\mathrm{d}s + \int_0^t h(s)\mathrm{d}s. \tag{17}$$

Since B is a Hille–Yosida operator, we deduce that (17) is equivalent to

$$y(t) = T_{B_0}(t)\varphi(0) + (S_B \diamond h)(t).$$

The proof is completed. □

Recall that $A_0 : D(A_0) \subset \overline{D(A)} \to \overline{D(A)}$, the part of A in $\overline{D(A)}$, is defined by

$$A_0 \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = A \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}, \forall \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in D(A_0),$$

where

$$D(A_0) = \left\{ \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in D(A) : A \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in \overline{D(A)} \right\}.$$

From the definition of A in (6) and the fact that

$$\overline{D(A)} = \{0_Y\} \times \{\varphi \in C([-r,0],Y),\ \varphi(0) \in Y_0\},$$

we know that A_0 is a linear operator defined by

$$A_0 \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \varphi' \end{pmatrix}, \forall \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in D(A_0),$$

where

$$D(A_0)$$
$$= \left\{ \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in \{0_Y\} \times \{\varphi \in C^1([-r,0],Y) : \varphi(0) \in D(B),\ -\varphi'(0) + B\varphi(0) = 0 \right\}.$$

Now by using the fact that A is a Hille–Yosida operator, we deduce that A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ and

$$v(t) = T_{A_0}(t) \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}$$

is an integrated solution of

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t),\ t \geq 0,\ v(0) = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in X_0.$$

Using Lemma 3.7 with $h = 0$, we obtain the following result.

Lemma 3.8. *The linear operator A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of bounded linear operators on $\overline{D(A)}$ which is defined by*

$$T_{A_0}(t)\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \widehat{T}_{A_0}(t)(\varphi) \end{pmatrix}, \tag{18}$$

where

$$\widehat{T}_{A_0}(t)(\varphi)(\theta) = \begin{cases} T_{B_0}(t+\theta)\,\varphi(0), & t+\theta \geq 0, \\ \varphi(t+\theta), & t+\theta \leq 0. \end{cases}$$

Since A is a Hille–Yosida operator, we know that A generates an integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X, and $t \to S_A(t)\begin{pmatrix} y \\ \varphi \end{pmatrix}$ is an integrated solution of

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + \begin{pmatrix} y \\ \varphi \end{pmatrix},\ t \geq 0,\ v(0) = 0.$$

Since $S_A(t)$ is linear we have

$$S_A(t)\begin{pmatrix} y \\ \varphi \end{pmatrix} = S_A(t)\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} + S_A(t)\begin{pmatrix} y \\ 0_C \end{pmatrix},$$

where

$$S_A(t)\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \int_0^t T_{A_0}(l)\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}\mathrm{d}l$$

and $S_A(t)\begin{pmatrix} y \\ 0 \end{pmatrix}$ is an integrated solution of

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + \begin{pmatrix} y \\ 0 \end{pmatrix},\ t \geq 0,\ v(0) = 0.$$

Therefore, by using Lemma 3.7 with $h(t) = y$ and the above results, we obtain the following result.

Lemma 3.9. *The linear operator A generates an integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X. Moreover,*

$$S_A(t)\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \widehat{S}_A(t)(y,\varphi) \end{pmatrix},\ \begin{pmatrix} y \\ \varphi \end{pmatrix} \in X,$$

where $\widehat{S}_A(t)$ is the linear operator defined by

$$\widehat{S}_A(t)(y,\varphi) = \widehat{S}_A(t)(0,\varphi) + \widehat{S}_A(t)(y,0)$$

with

$$\widehat{S}_A(t)(0,\varphi)(\theta) = \int_0^t \widehat{T}_{A_0}(s)(\varphi)(\theta)\,\mathrm{d}s = \int_{-\theta}^t T_{B_0}(s+\theta)\varphi(0)\mathrm{d}s + \int_0^{-\theta} \varphi(s+\theta)\mathrm{d}s$$

and

$$\widehat{S}_A(t)(y,0)(\theta) = \begin{cases} S_B(t+\theta)y, & t+\theta \geq 0, \\ 0, & t+\theta \leq 0. \end{cases}$$

Now we focus on the spectra of A and $A+L$. Since A is a Hille–Yosida operator, so is $A+L$. Moreover, $(A+L)_0 : D((A+L)_0) \subset \overline{D(A)} \to \overline{D(A)}$, the part of $A+L$ in $\overline{D(A)}$, is a linear operator defined by

$$(A+L)_0 \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \varphi' \end{pmatrix}, \ \forall \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in D\left((A+L)_0\right),$$

where

$$D\left((A+L)_0\right) =$$
$$\left\{ \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in \{0_Y\} \times \{\varphi \in C^1\left([-r,0],Y\right) : \ \varphi(0) \in D(B), \ \ \varphi'(0) = B\varphi(0) + \widehat{L}(\varphi) \right\}.$$

From Theorems 3.4 and 3.5, we know that

$$\sigma(B) = \sigma(A) = \sigma(A_0) \text{ and } \sigma(A+L) = \sigma\left((A+L)_0\right).$$

From (18), we have

$$\widehat{T}_{A_0}(t)(\varphi)(\theta) = T_{B_0}(r+\theta)\, T_{B_0}(t-r)\, \varphi(0), t \geq r, \theta \in [-r,0].$$

Therefore we get

$$\omega_{0,\mathrm{ess}}(A_0) \leq \omega_{0,\mathrm{ess}}(B_0). \tag{19}$$

In the following lemma, we specify the essential growth rate of the C_0-semigroup generated by $(A+L)_0$ in some cases. Unfortunately, this problem is not fully understood.

Lemma 3.10. *Assume that one of the two following properties are satisfied:*

(a) For each $t > 0$, $\widehat{L}\widehat{T}_{A_0}(t)$ is compact from C into Y.
(b) For each $t > 0$, $T_{B_0}(t)$ is compact on Y.

Then we have

$$\omega_{0,\mathrm{ess}}((A+L)_0) \leq \omega_{0,\mathrm{ess}}(B_0).$$

Proof. For Assumption (a) the result is a direct consequence of Theorem 1.2 in Ducrot et al. [15] or the results in Thieme [47]. The case (b) has been treated by Adimy et al. [4, Theorem 2.7]. □

From now on we set

$$\Omega = \{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) > \omega_{0,\mathrm{ess}}((A+L)_0)\}.$$

From the discussion in this section, we obtain the following proposition.

Proposition 3.11. *The linear operator $A+L : D(A) \subset X \to X$ is a Hille–Yosida operator. $(A+L)_0$ is the infinitesimal generator of a strongly continuous semigroup $\left\{T_{(A+L)_0}(t)\right\}_{t\geq 0}$ of bounded linear operators on $\overline{D(A)}$. Moreover,*

$$T_{(A+L)_0}(t)\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \widehat{T}_{(A+L)_0}(t)(\varphi) \end{pmatrix} \tag{20}$$

with

$$\widehat{T}_{(A+L)_0}(t)(\varphi)(\theta) = y(t+\theta), \ \forall t \geq 0, \ \forall \theta \in [-r,0],$$

where

$$y(t) = \begin{cases} \varphi(t), & \forall t \in [-r,0], \\ T_{B_0}(t)\varphi(0) + \left(S_B \diamond \widehat{L}(y_\cdot)\right)(t), & \forall t \geq 0. \end{cases}$$

Furthermore, we have that

$$\sigma((A+L)_0) \cap \Omega = \sigma_P((A+L)_0) \cap \Omega = \{\lambda \in \Omega : \ N(\Delta(\lambda)) \neq 0\},$$

where $\Delta(\lambda) : D(B) \subset Y \to Y$ is the following linear operator

$$\Delta(\lambda) = \lambda I - B - \widehat{L}\left(e^{\lambda \cdot} I_Y\right). \tag{21}$$

Then each $\lambda_0 \in \sigma\left((A+L)_0\right) \cap \Omega$ is a pole in Ω of $(\lambda I - (A+L))^{-1}$. For each $\gamma > \omega_{0,\mathrm{ess}}((A+L)_0)$, the subset $\{\lambda \in \sigma\left((A+L)_0\right) : \mathrm{Re}(\lambda) \geq \gamma\}$ is either empty or finite.

Proof. The first part of the result follows immediately from Lemma 3.7 applied with $h(t) = \widehat{L}(y_t)$, and the last part of the proof follows from Theorem 3.2, Corollary 3.3, and Theorem 3.4. □

4 Projectors on the Eigenspaces

Let $\lambda_0 \in \sigma(A+L) \cap \Omega$. From the above discussion we already knew that λ_0 is a pole of $(\lambda I - (A+L))^{-1}$ of finite order $k_0 \geq 1$. This means that λ_0 is isolated in $\sigma(A+L)$ and the Laurent's expansion of the resolvent around λ_0 takes the following form

$$(\lambda I - (A+L))^{-1} = \sum_{n=-k_0}^{+\infty} (\lambda - \lambda_0)^n B_n^{\lambda_0}. \tag{22}$$

The bounded linear operator $B_{-1}^{\lambda_0}$ is the projector on the generalized eigenspace of $(A+L)$ associated to λ_0. The goal of this section is to provide a method to compute $B_{-1}^{\lambda_0}$. We remark that

$$(\lambda-\lambda_0)^{k_0}(\lambda I-(A+L))^{-1}=\sum_{m=0}^{+\infty}(\lambda-\lambda_0)^m B_{m-k_0}^{\lambda_0}.$$

So we have the following approximation formula

$$B_{-1}^{\lambda_0}=\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-1)!}\frac{\mathrm{d}^{k_0-1}}{\mathrm{d}\lambda^{k_0-1}}\left((\lambda-\lambda_0)^{k_0}(\lambda I-(A+L))^{-1}\right). \tag{23}$$

In order to compute an explicit formula for the resolvent of $A+L$ we will use the following lemma.

Lemma 4.1. *Let $A:D(A)\subset X\to X$ be a linear operator on a Banach space X with $\rho(A)\neq\emptyset$. Let $B:\overline{D(A)}\to X$ be a bounded linear operator. Then for each $\lambda\in\rho(A)$ we have*

$$\lambda\in\rho(A+B)\Leftrightarrow 1\in\rho\left(B(\lambda I-A)^{-1}\right).$$

Moreover, for each $\lambda\in\rho(A+B)$ we have

$$(\lambda I-A-B)^{-1}=(\lambda I-A)^{-1}\left[I-B(\lambda I-A)^{-1}\right]^{-1}$$

$$\left[I-B(\lambda I-A)^{-1}\right]^{-1}=I+B(\lambda I-A-B)^{-1}$$

Proof. Assume first that $1\in\rho\left(B(\lambda I-A)^{-1}\right)$. Then

$$\begin{aligned}&(\lambda I-A-B)(\lambda I-A)^{-1}\left[I-B(\lambda I-A)^{-1}\right]^{-1}\\&=\left[I-B(\lambda I-A)^{-1}\right]\left[I-B(\lambda I-A)^{-1}\right]^{-1}=I,\end{aligned}$$

and

$$\begin{aligned}&(\lambda I-A)^{-1}\left[I-B(\lambda I-A)^{-1}\right]^{-1}(\lambda I-A-B)\\&=(\lambda I-A)^{-1}\left[I-B(\lambda I-A)^{-1}\right]^{-1}\left(I-B(\lambda I-A)^{-1}\right)(\lambda I-A)\\&=I_{D(A)}.\end{aligned}$$

Thus $\lambda\in\rho(A+B)$, and

$$(\lambda I-A-B)^{-1}=(\lambda I-A)^{-1}\left[I-B(\lambda I-A)^{-1}\right]^{-1}.$$

Conversely, assume that $\lambda \in \rho(A+B)$, then

$$\begin{aligned}
&\left[I-B(\lambda I-A)^{-1}\right]\left[I+B(\lambda I-A-B)^{-1}\right]\\
&\quad=\left[I-B(\lambda I-A)^{-1}\right]\left[(\lambda I-A-B)(\lambda I-A-B)^{-1}+B(\lambda I-A-B)^{-1}\right]\\
&\quad=\left[I-B(\lambda I-A)^{-1}\right]\left[(\lambda I-A)(\lambda I-A-B)^{-1}\right]\\
&\quad=(\lambda I-A)(\lambda I-A-B)^{-1}-B(\lambda I-A-B)^{-1}=I
\end{aligned}$$

and

$$\begin{aligned}
&\left[I+B(\lambda I-A-B)^{-1}\right]\left[I-B(\lambda I-A)^{-1}\right]\\
&\quad=\left[(\lambda I-A)(\lambda I-A-B)^{-1}\right]\left[I-B(\lambda I-A)^{-1}\right]\\
&\quad=(\lambda I-A)(\lambda I-A-B)^{-1}[\lambda I-A-B](\lambda I-A)^{-1}\\
&\quad=I.
\end{aligned}$$

Thus, $1 \in \rho\left(B(\lambda I-A)^{-1}\right)$ and

$$\left[I-B(\lambda I-A)^{-1}\right]^{-1}=I+B(\lambda I-A-B)^{-1}.$$

This completes the proof. □

In order to give an explicit formula for $B_{-1}^{\lambda_0}$, we need the following results.

Lemma 4.2. *We have the following equivalence:*

$$\lambda \in \rho(A+L)\cap\Omega \Leftrightarrow \Delta(\lambda) \text{ is invertible.}$$

Moreover, we have the following explicit formula for the resolvent of $A+L$

$$\begin{aligned}
&(\lambda I-(A+L))^{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{pmatrix}0_Y\\ \psi\end{pmatrix}\\
&\Leftrightarrow\\
&\psi(\theta)=\int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\,ds+e^{\lambda\theta}\Delta(\lambda)^{-1}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_.^0 e^{\lambda(.-s)}\varphi(s)\,ds\right)\right].
\end{aligned} \tag{24}$$

Proof. We consider the linear operator $A_\gamma : D(A)\subset X \to X$ defined by

$$A_\gamma\begin{pmatrix}0_Y\\ \varphi\end{pmatrix}=\begin{pmatrix}-\varphi'(0)+(B-\gamma I)\varphi(0)\\ \varphi'\end{pmatrix},\ \forall\begin{pmatrix}0_Y\\ \varphi\end{pmatrix}\in D(A),$$

and

$$L_\gamma \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} \widehat{L}(\varphi) + \gamma\varphi(0) \\ 0_C \end{pmatrix}.$$

Then we have

$$A + L = A_\gamma + L_\gamma.$$

Moreover,

$$\omega_0(B_0 - \gamma I) = \omega_0(B_0) - \gamma.$$

Hence by Theorem 3.5, for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega_0(B_0) - \gamma$, we have $\lambda \in \rho(A_\gamma)$ and

$$\begin{aligned} &(\lambda I - A_\gamma)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \psi \end{pmatrix} \\ &\Leftrightarrow \psi(\theta) = \mathrm{e}^{\lambda\theta}(\lambda I - (B - \gamma I))^{-1}[\varphi(0) + \alpha] + \int_\theta^0 \mathrm{e}^{\lambda(\theta - s)}\varphi(s)\,\mathrm{d}s. \end{aligned} \tag{25}$$

Therefore, for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega_0(B_0) - \gamma$, by Lemma 4.1 we deduce that $[\lambda I - (A_\gamma + L_\gamma)]$ is invertible if and only if $I - L_\gamma(\lambda I - A_\gamma)^{-1}$ is invertible, and

$$(\lambda I - (A_\gamma + L_\gamma))^{-1} = (\lambda I - A_\gamma)^{-1}\left[I - L_\gamma(\lambda I - A_\gamma)^{-1}\right]^{-1}. \tag{26}$$

We also know that $\left[I - L_\gamma(\lambda I - A_\gamma)^{-1}\right]\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \widehat{\varphi} \end{pmatrix}$ is equivalent to $\varphi = \widehat{\varphi}$ and

$$\begin{aligned} &\alpha - \left[\widehat{L}\left(\mathrm{e}^{\lambda \cdot}(\lambda I - (B - \gamma I))^{-1}\alpha\right) + \gamma(\lambda I - (B - \gamma I))^{-1}\alpha\right] \\ &\quad = \widehat{\alpha} + \begin{bmatrix} \widehat{L}\left(\mathrm{e}^{\lambda \cdot}(\lambda I - (B - \gamma I))^{-1}\widehat{\varphi}(0) + \int_\cdot^0 \mathrm{e}^{\lambda(\cdot - s)}\widehat{\varphi}(s)\,\mathrm{d}s\right) \\ +\gamma(\lambda I - (B - \gamma I))^{-1}\widehat{\varphi}(0) \end{bmatrix}. \end{aligned}$$

Because

$$\begin{aligned} &\alpha - \widehat{L}\left(\mathrm{e}^{\lambda \cdot}(\lambda I - (B - \gamma I))^{-1}\alpha\right) - \gamma(\lambda I - (B - \gamma I))^{-1}\alpha \\ &= \left[\lambda I - (B - \gamma I) - \widehat{L}\left(\mathrm{e}^{\lambda \cdot} I\right) - \gamma I\right](\lambda I - (B - \gamma I))^{-1}\alpha \\ &= \left[\lambda I - B - \widehat{L}\left(\mathrm{e}^{\lambda \cdot} I\right)\right](\lambda I - (B - \gamma I))^{-1}\alpha \\ &= \Delta(\lambda)(\lambda I - (B - \gamma I))^{-1}\alpha \end{aligned}$$

and B is closed, we deduce that $\Delta(\lambda)$ is closed, and by using the same arguments as in the proof of Theorem 3.5 to

$$\Delta(\lambda)(\lambda I-(B-\gamma I))^{-1}\alpha$$
$$=\widehat{\alpha}+\left[\begin{array}{c}\widehat{L}\left(\mathrm{e}^{\lambda\cdot}(\lambda I-(B-\gamma I))^{-1}\widehat{\varphi}(0)+\int_{\cdot}^{0}\mathrm{e}^{\lambda(\cdot-s)}\widehat{\varphi}(s)\,\mathrm{d}s\right)\\+\gamma(\lambda I-(B-\gamma I))^{-1}\widehat{\varphi}(0)\end{array}\right]$$

we deduce that $\left[I-L_{\gamma}\left(\lambda I-A_{\gamma}\right)^{-1}\right]$ is invertible if and only if

$$\Delta(\lambda)=\left[\lambda I-B-\widehat{L}\left(\mathrm{e}^{\lambda\cdot}I\right)\right]$$

is invertible. So for $\lambda\in\Omega$, $[\lambda I-(A+L)]$ is invertible if and only if $\Delta(\lambda)$ is invertible.

Moreover,

$$\left[I-L_{\gamma}\left(\lambda I-A_{\gamma}\right)^{-1}\right]^{-1}\begin{pmatrix}\widehat{\alpha}\\\widehat{\varphi}\end{pmatrix}=\begin{pmatrix}\alpha\\\varphi\end{pmatrix}$$

is equivalent to $\varphi=\widehat{\varphi}$ and

$$\alpha=(\lambda I-(B-\gamma I))\Delta(\lambda)^{-1}$$
$$\times\left[\begin{array}{c}\widehat{\alpha}+\widehat{L}\left(\mathrm{e}^{\lambda\cdot}(\lambda I-(B-\gamma I))^{-1}\widehat{\varphi}(0)+\int_{\cdot}^{0}\mathrm{e}^{\lambda(\cdot-s)}\widehat{\varphi}(s)\,\mathrm{d}s\right)\\+\gamma(\lambda I-(B-\gamma I))^{-1}\widehat{\varphi}(0)\end{array}\right]. \tag{27}$$

Note that $A+L=A_{\gamma}+L_{\gamma}$. By using (25), (26) and (27), we obtain for each $\gamma>0$ large enough that

$$(\lambda I-(A+L))^{-1}\begin{pmatrix}\alpha\\\varphi\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}^n}\\\psi\end{pmatrix}$$
$$\Leftrightarrow$$
$$\psi(\theta)=\mathrm{e}^{\lambda\theta}(\lambda I-(B-\gamma I))^{-1}\varphi(0)+\int_{\theta}^{0}\mathrm{e}^{\lambda(\theta-s)}\varphi(s)\,\mathrm{d}s$$
$$+\mathrm{e}^{\lambda\theta}\Delta(\lambda)^{-1}\left[\begin{array}{r}\alpha+\widehat{L}\left(\mathrm{e}^{\lambda\cdot}(\lambda I-(B-\gamma I))^{-1}\varphi(0)+\int_{\cdot}^{0}\mathrm{e}^{\lambda(\cdot-s)}\varphi(s)\,\mathrm{d}s\right)\\+\gamma(\lambda I-(B-\gamma I))^{-1}\varphi(0)\end{array}\right],$$

but

$$(\lambda I-(B-\gamma I))^{-1}\varphi(0)+\Delta(\lambda)^{-1}\left[\widehat{L}\left(\mathrm{e}^{\lambda\cdot}I\right)+\gamma I\right](\lambda I-(B-\gamma I))^{-1}\varphi(0)$$
$$=\left(I+\left[\lambda I-B-\widehat{L}\left(\mathrm{e}^{\lambda\cdot}I\right)\right]^{-1}\left[\widehat{L}\left(\mathrm{e}^{\lambda\cdot}I\right)+B-\lambda I+\lambda I-(B-\gamma I)\right]\right)$$
$$\times(\lambda I-(B-\gamma I))^{-1}\varphi(0)$$
$$=\left(I-I+\left[\lambda I-B-\widehat{L}\left(\mathrm{e}^{\lambda\cdot}I\right)\right]^{-1}[\lambda I-(B-\gamma I)]\right)(\lambda I-(B-\gamma I))^{-1}\varphi(0)$$
$$=\left[\lambda I-B-\widehat{L}\left(\mathrm{e}^{\lambda\cdot}I\right)\right]^{-1}\varphi(0)$$

and the result follows. □

Next we introduce the following operators $\widetilde{\Pi} : X_0 \to C$ and $F : Y \to X$ such that

$$\widetilde{\Pi}\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \varphi(0), \;\; F\alpha = \begin{pmatrix} \alpha \\ 0_C \end{pmatrix}.$$

Then from Lemma 4.2 we have:

$$\widetilde{\Pi}(\lambda I - (A+L))^{-1} F\alpha = \Delta^{-1}(\lambda)\alpha, \;\; \forall \lambda \in \rho(A+L) \cap \Omega, \;\; \forall \alpha \in Y. \quad (28)$$

Since $\lambda \to (\lambda I - (A+L))^{-1}$ is holomorphic from Ω into $\mathcal{L}(X)$, we deduce from the above formula that the map $\lambda \to \Delta^{-1}(\lambda)$ is holomorphic in Ω. Moreover, by Proposition 3.11 we know that $\Delta^{-1}(.)$ has only finite order poles. Therefore, $\Delta^{-1}(\lambda)$ has a Laurent's expansion around λ_0 as follows

$$\Delta(\lambda)^{-1} = \sum_{n=-\widehat{k}_0}^{+\infty} (\lambda - \lambda_0)^n \Delta_n, \;\; \Delta_n \in \mathcal{L}(Y), \;\; \forall n \geq -\widehat{k}_0.$$

From the following lemma we know that $\widehat{k}_0 = k_0$.

Lemma 4.3. *Let $\lambda_0 \in \sigma(A+L) \cap \Omega$. Then the following statements are equivalent*

(a) λ_0 is a pole of order k_0 of $(\lambda I - (A+L))^{-1}$.
(b) λ_0 is a pole of order k_0 of $\Delta(\lambda)^{-1}$.
(c) $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{k_0} \Delta(\lambda)^{-1} \neq 0$ and $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{k_0+1} \Delta(\lambda)^{-1} = 0$.

Moreover, if one the above assertions is satisfied, then for each $n \geq -k_0$,

$$R(\Delta_n) \subset D(B)$$

and

$$B\Delta_n \in \mathcal{L}(Y).$$

Proof. The proof of the equivalence follows from the explicit formula of the resolvent of $A+L$ obtained in Lemma 4.2. It remains to prove the last part of the lemma. Let $\lambda_0 \in \sigma(A+L) \cap \Omega$ be a pole of order k_0 of the resolvent of $\lambda \to \Delta(\lambda)^{-1}$. Let $\gamma \in \rho(B)$. Then by Proposition 3.11, $\lambda \in \rho(A+L) \cap \Omega \Leftrightarrow \Delta(\lambda)$ is invertible. But

$$\Delta(\lambda) = \gamma I - B + C_\gamma(\lambda)$$

with

$$C_\gamma(\lambda) = \left(\lambda I - \gamma I - \widehat{L}\left(e^{\lambda \cdot} I\right)\right).$$

So by Lemma 4.1, if $\Delta(\lambda)$ is invertible then $1 \in \rho\left(I - C_\gamma(\lambda)(\gamma I - B)^{-1}\right)$ and

$$\Delta(\lambda)^{-1} = (\gamma I - B)^{-1}\left[I - C_\gamma(\lambda)(\gamma I - B)^{-1}\right]^{-1}.$$

Clearly $\lambda \to C_\gamma(\lambda)$ is holomorphic, λ_0 is pole of order k_0. It follows that

$$\left[I - C_\gamma(\lambda)(\gamma I - B)^{-1}\right]^{-1} = \sum_{n=-k_0}^{+\infty} (\lambda - \lambda_0)^n \widetilde{\Delta}_n^\gamma$$

and by the uniqueness of the Laurent's expansion we obtain

$$\Delta_n = (\gamma I - B)^{-1} \widetilde{\Delta}_n^\gamma, \ \forall n \geq -k_0.$$

This completes the proof. □

Lemma 4.4. *The operators* $\Delta_{-1}, \ldots, \Delta_{-k_0}$ *satisfy*

$$\Delta_{k_0}(\lambda_0) \begin{pmatrix} \Delta_{-1} \\ \Delta_{-2} \\ \vdots \\ \Delta_{-k_0+1} \\ \Delta_{-k_0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\left(\Delta_{-k_0}\ \Delta_{-k_0+1} \cdots \Delta_{-2}\ \Delta_{-1}\right) \Delta_{k_0}(\lambda_0) = \left(0 \cdots 0\right),$$

where $\Delta_{k_0}(\lambda_0)$ *is the following operator matrix (from* $D(B)^{k_0}$ *into* Y^{k_0}*)*

$$\Delta_{k_0}(\lambda_0) = \begin{pmatrix} \Delta^{(0)}(\lambda_0) & \Delta^{(1)}(\lambda_0) & \Delta^{(2)}(\lambda_0)/2! & \cdots & \Delta^{(k_0-1)}(\lambda_0)/(k_0-1)! \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \Delta^{(2)}(\lambda_0)/2! \\ \vdots & & \ddots & \ddots & \Delta^{(1)}(\lambda_0) \\ 0 & \cdots & \cdots & 0 & \Delta^{(0)}(\lambda_0) \end{pmatrix},$$

where

$$\Delta^{(0)}(\lambda) = \Delta(\lambda) = \lambda I - B - \widehat{L}\left(e^{\lambda \cdot} I_Y\right)$$

and

$$\Delta^{(n)}(\lambda) = \frac{d^n}{d\lambda^n}\left(\lambda I - \widehat{L}\left(e^{\lambda \cdot} I\right)\right), \ \forall n \geq 1.$$

Proof. We have

$$(\lambda - \lambda_0)^{k_0} I = \Delta(\lambda)\left(\sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n \Delta_{n-k_0}\right) = \left(\sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n \Delta_{n-k_0}\right)\Delta(\lambda)$$

and

$$\Delta(\lambda) = \Delta(\lambda_0) + \left[(\lambda - \lambda_0) I - \left(\widehat{L}\left(e^{\lambda \cdot} I\right) - \widehat{L}\left(e^{\lambda_0 \cdot} I\right)\right)\right].$$

So

$$\Delta(\lambda) = \Delta(\lambda_0) + \sum_{n=1}^{+\infty} (\lambda - \lambda_0)^n \frac{\Delta^{(n)}(\lambda_0)}{n!}.$$

Hence,

$$(\lambda - \lambda_0)^{k_0} I = \left(\sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n \frac{\Delta^{(n)}(\lambda_0)}{n!} \right) \left(\sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n \Delta_{n-k_0} \right).$$

By using the last part of Lemma 4.3, we know that for each $n \geq -k_0$, $\Delta^{(0)}(\lambda_0)\Delta_n = \Delta(\lambda_0)\Delta_n$ is bounded and linear for Y into itself, so we obtain

$$(\lambda - \lambda_0)^{k_0} I = \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n \sum_{k=0}^{n} \frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!} \Delta_{k-k_0}$$

and

$$(\lambda - \lambda_0)^{k_0} I = \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n \sum_{k=0}^{n} \Delta_{k-k_0} \frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}.$$

By the uniqueness of the Taylor's expansion for analytic maps, we obtain that for $n \in \{0, \ldots, k_0 - 1\}$,

$$0 = \sum_{k=0}^{n} \Delta_{k-k_0} \frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!} = \sum_{k=0}^{n} \frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!} \Delta_{k-k_0}.$$

Therefore, the result follows. □

Now we look for an explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated to λ_0. Set

$$\Psi_1(\lambda)(\varphi)(\theta) := \int_\theta^0 e^{\lambda(\theta - s)} \varphi(s)\, ds$$

and

$$\Psi_2(\lambda)\left(\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}\right)(\theta) := e^{\lambda\theta} \left[\alpha + \varphi(0) + \widehat{L}\left(\int_.^0 e^{\lambda(.-s)} \varphi(s)\, ds \right) \right].$$

Then both maps are analytic and

$$(\lambda I - (A+L))^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \Psi_1(\lambda)(\varphi)(\theta) + \Delta(\lambda)^{-1} \Psi_2(\lambda) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}(\theta) \end{pmatrix}.$$

We observe that the only singularity in the last expression is $\Delta(\lambda)^{-1}$. Since Ψ_1 and Ψ_2 are analytic, we have for $j = 1,2$ that

$$\Psi_j(\lambda) = \sum_{n=0}^{+\infty} \frac{(\lambda-\lambda_0)^n}{n!} L_n^j(\lambda_0),$$

where $|\lambda-\lambda_0|$ is small enough and $L_n^j(.) := \frac{\mathrm{d}^n \Psi_j(.)}{\mathrm{d}\lambda^n}, \forall n \geq 0, \forall j = 1,2$. Hence, we get

$$\begin{aligned}
&\lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{\mathrm{d}^{k_0-1}}{\mathrm{d}\lambda^{k_0-1}} \left((\lambda-\lambda_0)^{k_0} \Psi_1(\lambda)\right) \\
&= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \sum_{n=0}^{+\infty} \frac{(n+k_0)!}{(n+1)!} \frac{(\lambda-\lambda_0)^{n+1}}{n!} L_n^1(\lambda_0) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{\mathrm{d}^{k_0-1}}{\mathrm{d}\lambda^{k_0-1}} (\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1} \Psi_2(\lambda) \\
&= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{\mathrm{d}^{k_0-1}}{\mathrm{d}\lambda^{k_0-1}} \left[\left(\sum_{n=-k_0}^{+\infty} (\lambda-\lambda_0)^{n+k_0} \Delta_n \right) \left(\sum_{n=0}^{+\infty} \frac{(\lambda-\lambda_0)^n}{n!} L_n^2(\lambda_0) \right) \right] \\
&= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{\mathrm{d}^{k_0-1}}{\mathrm{d}\lambda^{k_0-1}} \left[\left(\sum_{n=0}^{+\infty} (\lambda-\lambda_0)^{n} \Delta_{n-k_0} \right) \left(\sum_{n=0}^{+\infty} \frac{(\lambda-\lambda_0)^n}{n!} L_n^2(\lambda_0) \right) \right] \\
&= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{\mathrm{d}^{k_0-1}}{\mathrm{d}\lambda^{k_0-1}} \left[\sum_{n=0}^{+\infty} \sum_{j=0}^{n} (\lambda-\lambda_0)^{n-j} \Delta_{n-j-k_0} \frac{(\lambda-\lambda_0)^j}{j!} L_j^2(\lambda_0) \right] \\
&= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{\mathrm{d}^{k_0-1}}{\mathrm{d}\lambda^{k_0-1}} \left[\sum_{n=0}^{+\infty} (\lambda-\lambda_0)^{n} \sum_{j=0}^{n} \Delta_{n-j-k_0} \frac{1}{j!} L_j^2(\lambda_0) \right] \\
&= \sum_{j=0}^{k_0-1} \frac{1}{j!} \Delta_{-1-j} L_j^2(\lambda_0).
\end{aligned}$$

From the above results we can obtain the explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated to λ_0, which is given in the following proposition.

Proposition 4.5. *Each* $\lambda_0 \in \sigma((A+L))$ *with* $\operatorname{Re}(\lambda_0) > \omega_{0,\mathrm{ess}}((A+L)_0)$ *is a pole of* $(\lambda I - (A+L))^{-1}$ *of order* $k_0 \geq 1$. *Moreover,* k_0 *is the only integer such that there exists* $\Delta_{-k_0} \in \mathcal{L}(Y)$ *with* $\Delta_{-k_0} \neq 0$, *such that*

$$\Delta_{-k_0} = \lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1}.$$

Furthermore, the projector $B_{-1}^{\lambda_0}$ *on the generalized eigenspace of* $(A+L)$ *associated to* λ_0 *is defined by the following formula*

$$B_{-1}^{\lambda_0}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{bmatrix}0_Y\\ \sum_{j=0}^{k_0-1}\frac{1}{j!}\Delta_{-1-j}L_j^2(\lambda_0)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\end{bmatrix}, \tag{29}$$

where

$$\Delta_{-j}=\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-j)!}\frac{\mathrm{d}^{k_0-j}}{\mathrm{d}\lambda^{k_0-j}}\left((\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1}\right),\ j=1,\ldots,k_0,$$

$$L_0^2(\lambda)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\mathrm{e}^{\lambda\theta}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_.^0\mathrm{e}^{\lambda(.-s)}\varphi(s)\,\mathrm{d}s\right)\right],$$

and

$$\begin{aligned}L_j^2(\lambda)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}&=\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\left[L_0^2(\lambda)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\right]\\ &=\sum_{k=0}^{j}C_j^k\theta^k\mathrm{e}^{\lambda\theta}\frac{\mathrm{d}^{j-k}}{\mathrm{d}\lambda^{j-k}}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_.^0\mathrm{e}^{\lambda(.-s)}\varphi(s)\,\mathrm{d}s\right)\right],\ \ j\geq 1,\end{aligned}$$

in which

$$\frac{\mathrm{d}^i}{\mathrm{d}\lambda^i}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_.^0\mathrm{e}^{\lambda(.-s)}\varphi(s)\,\mathrm{d}s\right)\right]=\widehat{L}\left(\int_.^0(.-s)^i\mathrm{e}^{\lambda(.-s)}\varphi(s)\,\mathrm{d}s\right),\ i\geq 1.$$

5 Projector for a Simple Eigenvalue

In studying Hopf bifurcation it usually requires to consider the projector for a simple eigenvalue. In this section we study the case when λ_0 is a simple eigenvalue of $(A+L)$. That is, λ_0 is a pole of order 1 of the resolvent of $(A+L)$ and the dimension of the eigenspace of $(A+L)$ associated to the eigenvalue λ_0 is 1.

We know that λ_0 is a pole of order 1 of the resolvent of $(A+L)$ if and only if there exists $\Delta_{-1}\neq 0$, such that

$$\Delta_{-1}=\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\Delta(\lambda)^{-1}.$$

From Lemma 4.4, we have

$$\Delta_{-1}\Delta(\lambda_0)=0\text{ and }\Delta(\lambda_0)\Delta_{-1}=0.$$

Hence

$$\Delta_{-1}\left[By+\widehat{L}\left(e^{\lambda_0 \cdot}y\right)\right]=\lambda_0\Delta_{-1}y, \forall y\in D(B),$$

$$\left[B+\widehat{L}\left(e^{\lambda_0 \cdot}I\right)\right]\Delta_{-1}y=\lambda_0\Delta_{-1}y, \forall y\in Y.$$

So if we assume that λ_0 is a pole of order 1 of $\Delta(\lambda_0)^{-1}$, then λ_0 is simple if and only if $\dim[N(\Delta(\lambda_0))]=1$. Hence,

$$\Delta_{-1}=\left\langle W_{\lambda_0},.\right\rangle_{Y^*,Y}V_{\lambda_0}.$$

Since

$$\left[B+\widehat{L}\left(e^{\lambda_0 \cdot}I\right)\right]\Delta_{-1}=\lambda_0\Delta_{-1},$$

we must have $V_{\lambda_0}\in D(B)$ and

$$\Delta(\lambda_0)V_{\lambda_0}=0\Leftrightarrow BV_{\lambda_0}+\widehat{L}\left(e^{\lambda_0 \cdot}V_{\lambda_0}\right)=\lambda_0V_{\lambda_0}, \tag{30}$$

so V_{λ_0} is an eigenvector of $\Delta(\lambda_0)_0$, the part of $\Delta(\lambda_0)$ in $\overline{D(B)}$ (which is the infinitesimal generator of a C_0-semigroup).

Since B is not densely defined, the characterization of $W_{\lambda_0}\in Y^*$ is more delicate. First, since

$$\Delta_{-1}\left[B+\widehat{L}\left(e^{\lambda_0 \cdot}I\right)\right]=\lambda_0\Delta_{-1},$$

it follows that

$$\left\langle W_{\lambda_0},By+\widehat{L}\left(e^{\lambda \cdot}y\right)\right\rangle_{Y^*,Y}=\lambda_0\left\langle W_{\lambda_0},y\right\rangle_{Y^*,Y}, \ \forall y\in D(B). \tag{31}$$

So $W_{\lambda_0}\mid_{\overline{D(B)}}$ (the restriction of W_{λ_0} to $\overline{D(B)}$) is an adjoint eigenvector of $\Delta(\lambda_0)_0$, the part of $\Delta(\lambda_0)$ in $\overline{D(B)}$. But $D(B)$ is not dense (in general) in Y, so $\Delta(\lambda_0)^*$ is not defined as a linear operator on Y^*. In order to characterize W_{λ_0}, we observe that $W^0_{\lambda_0}:=W_{\lambda_0}\mid_{\overline{D(B)}}\in N\left(\Delta(\lambda_0)^*_0\right)$, and by Theorem 3.4

$$\left\langle W_{\lambda_0},y\right\rangle_{Y^*,Y}=\lim_{\lambda\to+\infty}\left\langle W^0_{\lambda_0},\lambda\left(\lambda I-\left(B+\widehat{L}\left(e^{\lambda_0 \cdot}I\right)\right)\right)^{-1}y\right\rangle_{Y_0^*,Y_0}, \ \forall y\in Y.$$

Since $k_0=1$ and

$$B^{\lambda_0}_{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{bmatrix}0_Y\\ \Delta_{-1}L^2_0(\lambda_0)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\end{bmatrix}$$

with

$$L^2_0(\lambda_0)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=e^{\lambda_0\theta}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_.^0 e^{\lambda_0(.-s)}\varphi(s)\,ds\right)\right],$$

we can see that $B_{-1}^{\lambda_0}B_{-1}^{\lambda_0} = B_{-1}^{\lambda_0}$ if and only if

$$\Delta_{-1} = \Delta_{-1}\left[I+\widehat{L}\left(\int_{\cdot}^{0} e^{\lambda_0 \cdot} ds\right)\right]\Delta_{-1}. \tag{32}$$

Therefore, we obtain the following corollary.

Corollary 5.1. *$\lambda_0 \in \sigma((A+L))$ is a simple eigenvalue of $(A+L)$ if and only if*

$$\lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)^2\Delta(\lambda)^{-1} = 0$$

and

$$\dim[N(\Delta(\lambda_0))] = 1.$$

Moreover, the projector on the eigenspace associated to λ_0 is

$$B_{-1}^{\lambda_0}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{bmatrix} 0_Y \\ e^{\lambda_0\theta}\Delta_{-1}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_{\cdot}^{0} e^{\lambda_0(\cdot-s)}\varphi(s)\,ds\right)\right]\end{bmatrix}, \tag{33}$$

where

$$\Delta_{-1} = \left\langle W_{\lambda_0}^{*},.\right\rangle V_{\lambda_0}$$

in which

$$V_{\lambda_0} \in D(B)\setminus\{0\},\ BV_{\lambda_0}+\widehat{L}\left(e^{\lambda_0\cdot}V_{\lambda_0}\right) = \lambda_0 V_{\lambda_0},\ W_{\lambda_0}\in Y^*\setminus\{0\},$$

$$\left\langle W_{\lambda_0}, By+\widehat{L}\left(e^{\lambda\cdot}y\right)\right\rangle_{Y^*,Y} = \lambda_0\langle W_{\lambda_0},y\rangle_{Y^*,Y},\ \forall y\in D(B),$$

and

$$\Delta_{-1} = \Delta_{-1}\left[I+\widehat{L}\left(\int_{\cdot}^{0} e^{\lambda_0\cdot} ds\right)\right]\Delta_{-1}.$$

6 Comments on Semilinear Problems and Examples

In this section we give a few comments and remarks concerning the results obtained in this paper. In order to study the semilinear PFDE

$$\begin{cases} \dfrac{dy(t)}{dt} = By(t)+\widehat{L}(y_t)+f(y_t), \forall t\geq 0, \\ y_0^{\varphi} = \varphi \in C_B = \{\varphi\in C([-r,0];Y):\ \ \varphi(0)\in\overline{D(B)}\}, \end{cases} \tag{34}$$

we considered the associated abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t)+L(v(t))+F(v(t)), t\geq 0,\ v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi\end{pmatrix}\in\overline{D(A)}, \tag{35}$$

where

$$F\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}f(\varphi)\\ 0\end{pmatrix}.$$

By using Lemma 3.7 we can check that the integrated solutions of (35) are the usual solutions of the PFDE (34).

Now we are in the position to investigate the properties of the semiflows generated by the PFDE by using the known results on non-densely defined semilinear Cauchy problems. In particular when f is Lipschitz continuous, from the results of Thieme [45], for each $\varphi \in C_B$ we obtain a unique solution $t \to y^{\varphi}(t)$ on $[-r,+\infty)$ of (34), and we can define a nonlinear C^0-semigroup $\{U(t)\}_{t\geq 0}$ on C_B by

$$U(t)\varphi = y_t^{\varphi}.$$

From the results in Magal [32], one may also consider the case where f is Lipschitz on bounded sets of C_B. The non-autonomous case has also been considered in Thieme [45] and Magal [32]. We refer to Ezzinbi and Adimy [17] for more results about the existence of solutions using integrated semigroups.

In order to describe the local asymptotic behavior around some equilibrium, we assume that $\overline{y} \in D(B)$ is an equilibrium of the PFDE (34), that is,

$$0 = B\overline{y} + L\left(\overline{y}1_{[-r,0]}\right) + f\left(\overline{y}1_{[-r,0]}\right).$$

Then by the stability result of Thieme [45], we obtain the following stability results for PFDE.

Theorem 6.1 (Exponential Stability). *Assume that $f : C_B \to \mathbb{R}^n$ is continuously differentiable in some neighborhood of $\overline{y}1_{[-r,0]}$ and $Df\left(\overline{y}1_{[-r,0]}\right) = 0$. Assume in addition that*

$$\omega_{0,\text{ess}}((A+L)_0) < 0$$

and each $\lambda \in \mathbb{C}$ such that

$$N(\Delta(\lambda)) \neq 0$$

has strictly negative real part. Then there exist $\eta, M, \gamma \in [0,+\infty)$ such that for each $\varphi \in C$ with $\left\|\varphi - \overline{y}1_{[-r,0]}\right\|_{\infty} \leq \eta$, the PFDE (34) has a unique solution $t \to y^{\varphi}(t)$ on $[-r,+\infty)$ satisfying

$$\left\|y_t^{\varphi} - \overline{y}1_{[-r,0]}\right\|_{\infty} \leq Me^{-\gamma t}\left\|\varphi - \overline{y}1_{[-r,0]}\right\|_{\infty}, \forall t \geq 0.$$

The above theorem is well known in the context of FDE and PFDE (see, e.g., Arino et al. [7], Hale and Verduyn Lunel [22, Corollary 6.1, p. 215] and Wu [54, Corollary 1.11, p. 71]).

The existence and smoothness of center manifolds was also investigated for abstract non-densely defined Cauchy problems by Magal and Ruan [35]. More precisely, if we denote $\Pi_c : X \to X$ the bounded linear operator of projection

$$\Pi_c = B_{-1}^{\lambda_1} + \cdots + B_{-1}^{\lambda_m}$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} = \sigma_C(A+L) := \{\lambda \in \sigma(A+L) : \mathrm{Re}(\lambda) = 0\}$. Then

$$X_c = \Pi_c(X)$$

is the direct sum of the generalized eigenspaces associated to the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. Moreover,

$$\Pi_c(X) \subset X_0$$

and Π_c commutes with the resolvent of $(A+L)$. Set

$$X_h = R(I - \Pi_c) \ (\nsubseteq X_0).$$

Then we have the following state space decomposition

$$X = X_c \oplus X_h \text{ and } X_0 = X_{0c} \oplus X_{0h},$$

where

$$X_{0c} = X_c \cap X_0 = X_c \text{ and } X_{0h} = X_h \cap X_0 \neq X_h.$$

Then we can split the original abstract Cauchy problem (35) into the following system

$$\begin{cases} \dfrac{\mathrm{d}u_c(t)}{\mathrm{d}t} = (A+L)_c u_c(t) + \Pi_c F(u_c(t) + u_h(t)), \\ \dfrac{\mathrm{d}u_h(t)}{\mathrm{d}t} = (A+L)_h u_h(t) + \Pi_h F(u_c(t) + u_h(t)), \end{cases} \tag{36}$$

where $(A+L)_c$, the part of $A+L$ in X_c, is a bounded linear operator (since $\dim(X_c) < +\infty$), and $(A+L)_h$, the part of $A+L$ in X_h, is a non-densely defined Hille–Yosida operator. So the first equation of (36) is an ordinary differential equation and the second equation of (36) is a new non-densely defined Cauchy problem with

$$\sigma((A+L)_h) = \sigma((A+L)) \setminus \sigma_C(A+L).$$

Assume for simplicity that f is C^k in some neighborhood of the equilibrium 0_{C_B} and that

$$f(0) = 0 \text{ and } Df(0) = 0.$$

Then we can find (see [35, Theorem 4.21]) a manifold

$$M = \{x_c + \psi(x_c) : x_c \in X_c\},$$

where the map $\psi : X_c \to X_h \cap \overline{D(A)}$ is C^k with

$$\psi(0) = 0, \ D\psi(0) = 0,$$

and M is locally invariant by the semiflow generated by (35).

More precisely, we can find a neighborhood Ω of 0 in C_B such that if $I \subset \mathbb{R}$ is an interval and $u_c : I \to X_c$ is a solution of the ordinary differential equation

$$\frac{\mathrm{d}u_c(t)}{\mathrm{d}t} = (A+L)_c u_c(t) + \Pi_c F(u_c(t) + \psi(u_c(t))) \tag{37}$$

satisfying

$$u(t) := u_c(t) + \psi(u_c(t)) \in \Omega, \ \forall t \in I,$$

then $u(t)$ is an integrated solution of (35), that is,

$$u(t) = u(s) + A\int_s^t u(l)\mathrm{d}l + \int_s^t F(u(l))\mathrm{d}l, \ \forall t,s \in I \text{ with } t \geq s.$$

Conversely, if $u : \mathbb{R} \to X_0$ is an integrated solution of (35) satisfying

$$u(t) \in \Omega, \ \forall t \in \mathbb{R},$$

then $u_c(t) = \Pi_c u(t)$ is a solution of the ordinary differential equation (37). This result leads to the Hopf bifurcation results for PFDE and we refer to Hassard et al. [23] and Wu [54] for more results on this subject.

6.1 A Reaction-Diffusion Model with Delay (B is Densely Defined)

Reconsider an example from Wu [54]

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t-r,x), \ x \in (0,1), \\ \dfrac{\partial u(t,x)}{\partial x} = 0, \ x = 0,1, \\ u(0,.) = u_0 \in L^1(0,1), \end{cases} \tag{38}$$

where $\alpha \in \mathbb{R}$ and

$$B\varphi = \varphi''$$

with

$$D(B) = \left\{\varphi \in W^{2,1}(0,1) : \varphi'(0) = \varphi'(1) = 0\right\}.$$

We compute the projectors. Firstly, we have the following lemma.

Lemma 6.2. *The linear operator $B : D(B) \subset L^1(0,1) \to L^1(0,1)$ is the infinitesimal generator of an analytic semigroup $\{T_B(t)\}_{t\geq 0}$ on $L^1(0,1)$. Moreover, we have the following properties*

(a) $T_B(t)$ is compact for each $t > 0$.
(b) $T_B(t)L^1_+(0,1) \subset L^1_+(0,1)$ for each $t \geq 0$.

(c) $\sigma(B) = \sigma_P(B) = \left\{ \lambda_n = -(n\pi\varepsilon)^2 : n \geq 0 \right\}$.

(d) *For each* $n \geq 0$, $\lambda_n = -(n\pi\varepsilon)^2$ *is a simple eigenvalue of* B, *the projector on the eigenspace associated to* λ_n *is given by*

$$\Pi_n(\varphi)(x) = \begin{cases} \int_0^1 \varphi(y)\mathrm{d}y, & \text{if } n = 0, \\ 2\left(\int_0^1 \cos(n\pi y)\varphi(y)\mathrm{d}y \right)\cos(n\pi x), & \text{if } n \geq 1. \end{cases}$$

Here

$$C_B = C = C\left([-r,0], L^1(0,1)\right)$$

and the linear operator $\widehat{L} : C \to L^1(0,1)$ is defined by

$$\widehat{L}(\varphi) = \delta\varphi(-r).$$

Moreover, by applying Lemma 3.10(b), we obtain

$$\omega_{0,\mathrm{ess}}((A+L)_0) = -\infty.$$

The characteristic function is given by

$$\Delta(\lambda)x = \lambda x - Bx - \delta \mathrm{e}^{-\lambda r} x,$$

so $\Delta(\lambda)$ is invertible if and only if $\lambda - \delta\mathrm{e}^{-\lambda r} \notin \sigma(B)$. Thus,

$$\sigma((A+L)_0) = \left\{ \lambda \in \mathbb{C} : \lambda - \delta\mathrm{e}^{-\lambda r} \in \sigma(B) \right\}.$$

Let $\widehat{\lambda} \in \sigma((A+L)_0)$ be given and let $n_0 \geq 0$ such that

$$\widehat{\lambda} - \delta\mathrm{e}^{-\widehat{\lambda} r} = \lambda_{n_0},$$

where $\lambda_{n_0} \in \sigma(B)$.

Now we prove the following result.

Proposition 6.3. *Suppose that*

$$\gamma = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\lambda - \delta\mathrm{e}^{-\lambda r} - \lambda_{n_0} \right)|_{\lambda=\widehat{\lambda}} \neq 0.$$

Then the eigenvalue $\widehat{\lambda}$ *is a simple eigenvalue of* $(A+L)$. *Moreover, we have*

$$\Delta_{-1} = \gamma^{-1}\Pi_{n_0}$$

and

$$B_{-1}^{\widehat{\lambda}}\begin{pmatrix}\alpha \\ \varphi\end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{e}^{\widehat{\lambda}\theta}\gamma^{-1}\Pi_{n_0}\left(\alpha + \varphi(0) + \delta\int_{-r}^0 \mathrm{e}^{-\widehat{\lambda}(r+s)}\varphi(s)\mathrm{d}s \right) \end{pmatrix}, \ \forall \begin{pmatrix}\delta \\ \varphi\end{pmatrix} \in Y \times C.$$

Proof. Let us first notice that

$$N(\Delta(\widehat{\lambda})) = N(\lambda_{n_0} I - B).$$

Thus, due to Lemma 6.2, it is a one-dimensional space. Now we compute Δ_{-1}. For $\lambda \neq \widehat{\lambda}$ with $\left|\lambda - \widehat{\lambda}\right|$ small enough,

$$\begin{aligned}
\Delta(\lambda)^{-1} &= \left(\lambda I - \delta \mathrm{e}^{-\lambda r} I - B\right)^{-1} \\
&= \left(\lambda I - \delta \mathrm{e}^{-\lambda r} I - B\right)^{-1} \Pi_{n_0} + \left(\lambda I - \delta \mathrm{e}^{-\lambda r} I - B\right)^{-1} \left(I - \Pi_{n_0}\right) \\
&= \frac{1}{\lambda - \delta \mathrm{e}^{-\lambda r} - \lambda_{n_0}} \Pi_{n_0} + \left(\lambda I - \delta \mathrm{e}^{-\lambda r} I - B\right)^{-1} \left(I - \Pi_{n_0}\right)
\end{aligned}$$

and $\left(\lambda I - \delta e^{-\lambda r} I - B\right)^{-1} \left(I - \Pi_{n_0}\right)$ is bounded in the norm of operators for λ close enough to $\widehat{\lambda}$. Since $\gamma \neq 0$, it follows that $\widehat{\lambda}$ is a pole of order 1 of $\Delta(\lambda)^{-1}$ and we have

$$\lim_{\lambda \to \widehat{\lambda}} \left(\lambda - \widehat{\lambda}\right) \Delta(\lambda)^{-1} = \gamma^{-1} \Pi_{n_0}.$$

Moreover, we can easily get that

$$\lim_{\lambda \to \widehat{\lambda}} \left(\lambda - \widehat{\lambda}\right)^2 \Delta(\lambda)^{-1} = 0.$$

Thus, Corollary 5.1 applies and also provides the formula for the generalized projector. This completes the proof of the proposition. □

Remark 6.4. If $\gamma = 0$, then $\alpha \neq 0$ and we obtain that $\widehat{\lambda}$ is a pole of order two of $\Delta(\lambda)^{-1}$. Indeed, we can easily obtain that

$$\lim_{\lambda \to \widehat{\lambda}} \left(\lambda - \widehat{\lambda}\right) \Delta(\lambda)^{-1} = \frac{2}{\widehat{\gamma}} \Pi_{n_0},$$

where $\widehat{\gamma} = -\delta \tau^2 e^{-\widehat{\lambda} \tau} \neq 0$, while

$$\lim_{\lambda \to \widehat{\lambda}} \left(\lambda - \widehat{\lambda}\right)^2 \Delta(\lambda)^{-1} = 0.$$

Then we can derive the expression for the corresponding eigenprojector according to Proposition 4.5.

6.2 An Age-Structured Model with Delay (B is Non-Densely Defined)

By taking into account the time $r > 0$ between the reproduction and the birth of individuals, one may introduce the following (simplified) age-structured model with delay

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = -\mu u(t,a), \; a \geq 0, \\ u(t,0) = \int_0^{+\infty} b(a) u(t-r,a) \mathrm{d}a \\ u(0,.) = u_0 \in L^1(0,+\infty), \end{cases} \tag{39}$$

where $r > 0$, $\mu > 0$, and $b \in L^\infty_+(0,+\infty)$.

In this case we set

$$Y = \mathbb{R} \times L^1(0,+\infty)$$

endowed with the usual product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| = |\alpha| + \|\varphi\|_{L^1(0,+\infty)}.$$

Define $B : D(B) \subset Y \to Y$ by

$$B \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix},$$

where

$$D(B) = \{0\} \times W^{1,1}(0,+\infty).$$

Then it is clear that

$$Y_0 = \overline{D(B)} = \{0\} \times L^1(0,+\infty),$$

so B is non-densely defined. Also define

$$\widetilde{L}\left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \right) = \begin{pmatrix} \int_0^{+\infty} b(a)\varphi(a)\mathrm{d}a \\ 0 \end{pmatrix}.$$

Then by identifying $u(t)$ and $v(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, the equation (39) can be rewritten as the following abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = Bv(t) + \widetilde{L}(v(t-r)), \; t \geq 0 \\ v_0 = \varphi \in C_B. \end{cases}$$

Here

$$C_B = \left\{ \begin{pmatrix} \alpha(.) \\ \varphi(.) \end{pmatrix} \in C([-r,0],Y) : \alpha(0) = 0 \right\}$$

and the operator $\widehat{L} : C_B \to Y$ is defined by

$$\widehat{L} \begin{pmatrix} \alpha(.) \\ \varphi(.) \end{pmatrix} = \widetilde{L}\left(\begin{pmatrix} 0 \\ \varphi(-r) \end{pmatrix} \right) = \begin{pmatrix} \alpha \int_0^{+\infty} b(a)\varphi(-r)(a)\mathrm{d}a \\ 0 \end{pmatrix}.$$

Now we can explicitly compute the resolvent of the operator B. Indeed, we have for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\mu$ that

$$\begin{aligned} &(\lambda I - B)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ &\Leftrightarrow \varphi(a) = \mathrm{e}^{-(\lambda+\mu)a}\alpha + \textstyle\int_0^a \mathrm{e}^{-(\lambda+\mu)(a-l)}\psi(l)\mathrm{d}l. \end{aligned}$$

Next, since the operator $\widehat{L}$ is a one-dimensional rank operator we obtain by using Lemma 3.10-(a) that

$$\omega_{0,\mathrm{ess}}((A+L)_0)) \le \omega_{0,\mathrm{ess}}(A_0) \le \omega_{0,\mathrm{ess}}(B_0) \le -\mu.$$

Setting

$$\Omega = \{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) > -\mu\}$$

and using Lemma 4.4, we obtain that

$$\sigma(A+L) \cap \Omega = \{\lambda \in \Omega : \ \lambda \text{ is a pole of } \Delta(\lambda)\},$$

where the operator $\Delta(\lambda) : D(B) \subset Y \to Y$ is defined by

$$\Delta(\lambda) = \lambda I - B - \mathrm{e}^{-\lambda r}\widetilde{L}.$$

We now derive the characteristic equation for this problem.

Define a characteristic function $\widehat{\Delta} : \Omega \to \mathbb{C}$ by

$$\widehat{\Delta}(\lambda) := 1 - \mathrm{e}^{-\lambda r} \int_0^{+\infty} b(a)\mathrm{e}^{-(\mu+\lambda)a}\mathrm{d}a, \ \forall \lambda \in \Omega. \tag{40}$$

By using Lemmas 4.1 and 4.2, we obtain the following result.

Lemma 6.5. *We have*

$$\sigma(A+L) \cap \Omega = \{\lambda \in \Omega : \Delta(\lambda) \textit{ is not invertible}\} = \{\lambda \in \Omega : \widehat{\Delta}(\lambda) = 0\}.$$

Moreover, for each $\lambda \in \sigma(A+L) \cap \Omega$,

$$\Delta(\lambda)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix}=\begin{pmatrix}0\\ \varphi\end{pmatrix}$$
$$\Leftrightarrow$$
$$\varphi(a)=\left[\mathrm{e}^{-(\lambda+\mu)a}\alpha+\int_0^a \mathrm{e}^{-(\lambda+\mu)(a-l)}\psi(l)\mathrm{d}l\right]$$
$$+\widehat{\Delta}(\lambda)^{-1}\mathrm{e}^{-(\lambda+\mu)a}\mathrm{e}^{-\lambda r}\int_0^{+\infty}b(\sigma)\left[\mathrm{e}^{-(\lambda+\mu)\sigma}\alpha+\int_0^\sigma \mathrm{e}^{-(\lambda+\mu)(\sigma-l)}\psi(l)\mathrm{d}l\right]\mathrm{d}\sigma$$

and

$$(\lambda I-(A+L))^{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{pmatrix}0_Y\\ \psi\end{pmatrix}$$
$$\Leftrightarrow$$
$$\psi(\theta)=\int_\theta^0 \mathrm{e}^{\lambda(\theta-s)}\varphi(s)\,\mathrm{d}s+\mathrm{e}^{\lambda\theta}\Delta(\lambda)^{-1}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_.^0 \mathrm{e}^{\lambda(.-s)}\varphi(s)\,\mathrm{d}s\right)\right].$$

In the case of simple eigenvalues we obtain the following result.

Proposition 6.6. *Assume that* $\sigma(A+L)\cap\Omega\neq\emptyset$. *Let* $\lambda_0\in\sigma(A+L)\cap\Omega$. *If*

$$\frac{\mathrm{d}\widehat{\Delta}(\lambda_0)}{\mathrm{d}\lambda}=r+\mathrm{e}^{-\lambda_0 r}\int_0^{+\infty}ab(a)\mathrm{e}^{-(\mu+\lambda_0)a}\mathrm{d}a\neq 0,$$

then λ_0 *is a pole of order* 1 *of* $(\lambda I-(A+L))^{-1}$ *and*

$$B_{-1}^{\widehat{\lambda}}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{pmatrix}0\\ \mathrm{e}^{\lambda\theta}\Delta_{-1}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_.^0 \mathrm{e}^{\lambda(.-s)}\varphi(s)\,\mathrm{d}s\right)\right]\end{pmatrix},$$

where the linear operator $\Delta_{-1}=\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\Delta(\lambda)^{-1}$ *is defined by*

$$\Delta_{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix}=\begin{pmatrix}0\\ \varphi\end{pmatrix}$$
$$\Leftrightarrow\varphi(a)=\frac{\mathrm{d}\widehat{\Delta}(\lambda_0)}{\mathrm{d}\lambda}^{-1}\mathrm{e}^{-(\lambda_0+\mu)a}\mathrm{e}^{-\lambda_0 r}\int_0^{+\infty}b(\sigma)$$
$$\times\left[\mathrm{e}^{-(\lambda_0+\mu)\sigma}\alpha+\int_0^\sigma \mathrm{e}^{-(\lambda_0+\mu)(\sigma-l)}\psi(l)\mathrm{d}l\right]\mathrm{d}\sigma.$$

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Global Convergence in Monotone and Uniformly Stable Recurrent Skew-Product Semiflows

Yejuan Wang and Xiao-Qiang Zhao

Dedicated to George Sell on the occasion of his 70th birthday

Abstract The 1-covering property of omega limit sets is established for monotone and uniformly stable skew-product semiflows with a minimal base flow. Then the convergence result for monotone and subhomogeneous semiflows is applied to obtain the asymptotic recurrence of solutions to a linear recurrent nonhomogeneous ordinary differential system and a nonlinear recurrent reaction-diffusion equation.

Mathematics Subject Classification (2010): Primary 37B55, 37C65; Secondary 34D23, 35K57

1 Introduction

It is well known that the skew-product semiflows approach is a powerful tool in the study of linear and nonlinear nonautonomous evolution systems (see, e.g., [4, 5, 8, 9]). This approach has also been used extensively to generalize certain important results on global dynamics of monotone autonomous and periodic systems (see, e.g., [1, 12]) to monotone almost periodic and recurrent systems, see, e.g., [2,6,10,11,14] and references therein. Recently, Jiang and Zhao [2] established the 1-covering property of omega limit sets for monotone and uniformly stable

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skew-product semiflows with a minimal and distal base flow, and obtained the asymptotic almost periodicity of bounded solutions for monotone almost periodic systems with a first integral or subhomogeneous nonlinearity. It is natural to expect that there are similar convergence results in a more general nonautonomous case. To be more precise, we take into account a nonautonomous ordinary differential system $x' = f(t,x)$ with $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and let $H(f)$ be the hull of f with respect to the compact open topology. The function f is said to be time recurrent if $H(f)$ is compact and the translation flow is minimal on $H(f)$. Note that any uniformly almost automorphic function in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$ is time recurrent (see [10, Sect. I.3.1]). Accordingly, one may ask under what conditions on f, every bounded solution of such a recurrent system is asymptotic to a recurrent full solution. Clearly, one can not use the abstract results in [2] to prove this kind of asymptotic recurrence since the base flow is only minimal in our current case. More recently, Novo, Obaya and Sanz [6] have presented some general results on the structure of uniformly stable and uniformly asymptotically stable compact invariant sets for skew-product semiflows with a minimal base flow. In particular, they proved that the omega limit set of a precompact and uniformly stable forward orbit admits a minimal and fiber distal flow extension. Further, we observe that if a compact invariant set K of such a skew-product semiflow admits a flow extension and its section map is continuous, then K is an 1-covering of the base space whenever its intersection with some fiber is a singleton (see Lemma 2.4).

The purpose of our current paper is to generalize two abstract convergence results in [2] to skew-product semiflows with a minimal base flow in such a way that we can obtain the asymptotic recurrence for monotone and subhomogeneous or uniformly stable recurrent evolution systems. For a monotone and subhomogeneous semiflow, we can not directly utilize the afore-mentioned results under the norm-induced metric. However, we can prove the 1-covering property by employing the uniform stability of compact and strongly positive invariant sets with respect to the part metric (Lemma 3.1). For a monotone and uniformly stable skew-product semiflow, we will introduce a new assumption of the strong componentwise separating property (see (A4)) to prove the convergence result without assuming that the positive cone P is solid (i.e., $Int(P) \neq \emptyset$). We should point out that our arguments were highly motivated by those in [2, 6, 14].

The remained part of this paper is organized as follows. In Sect. 2, we present some basic concepts and results in the theory of skew-product semiflows. In Sect. 3, we prove the 1-covering property of omega limit sets for subhomogeneous and strongly monotone skew-product semiflows in the case where $Int(P) \neq \emptyset$ (Theorem 3.3), and for monotone and uniformly stable skew-product semiflows with the strong componentwise separating property without assuming $Int(P) \neq \emptyset$ (Theorem 3.4). Section 4 is devoted to the application of the convergence result to a linear recurrent nonhomogeneous ordinary differential system (Theorem 4.1) and a nonlinear recurrent reaction-diffusion equation (Theorem 4.2).

2 Preliminaries

Let $\mathbb{R}_+ = [0,\infty)$, (M,d_M) be a complete metric space, and (Y,d_Y) be a compact metric space. We use d to denote the metric on the product space $M\times Y$, which is induced by d_M and d_Y.

A continuous flow $\sigma : Y\times\mathbb{R}\to Y$ is said to be minimal if Y contains no nonempty, proper, closed invariant subset. Clearly, a flow $\sigma : Y\times\mathbb{R}\to Y$ is minimal if and only if every full orbit is dense in Y.

Consider a continuous skew-product semiflow $\Pi : M\times Y\times\mathbb{R}_+ \to M\times Y$ defined by

$$\Pi(x,y,t) = (u(x,y,t),\, \sigma(y,t)), \quad \forall (x,y,t)\in M\times Y\times\mathbb{R}_+.$$

For convenience, we also use Π_t to denote $\Pi(\cdot,t)$. A subset K of $M\times Y$ is said to be Π-invariant if $\Pi_t(K) = K$ for all $t\geq 0$. Further, the skew-product semiflow Π is said to be recurrent if its base flow $(Y,\sigma,\mathbb{R})$ is minimal. Throughout this paper, we always assume that Π is recurrent.

Definition 2.1. Let Π be a skew-product semiflow on $M\times Y$. A forward orbit $\Pi_t(x_0,y_0), t\geq 0$, is said to be uniformly stable if for every $\varepsilon>0$, there is a $\delta=\delta(\varepsilon)>0$, called the modulus of uniform stability, such that

$$d(\Pi(x,y_0,\tau+t),\Pi(x_0,y_0,\tau+t)) < \varepsilon, \quad \forall t\geq 0$$

whenever $\tau\geq 0$ and $d_M(u(x,y_0,\tau),u(x_0,y_0,\tau))<\delta$. Let $K\subset M\times Y$ be a compact Π-invariant set. The semiflow $(K,\Pi,\mathbb{R}_+)$ is uniformly stable if for every $\varepsilon>0$, there is a $\delta=\delta(\varepsilon)>0$ such that $d(\Pi(x_1,y,t),\Pi(x_2,y,t))<\varepsilon$, $\forall t\geq 0$, whenever $(x_1,y),(x_2,y)\in K$ with $d_M(x_1,x_2)<\delta$.

A compact Π-invariant set $K\subset M\times Y$ which admits a flow extension is said to be fiber distal if for any $y\in Y$, any two distinct points (x_1,y), $(x_2,y)\in K$, we have

$$\inf_{t\in\mathbb{R}} d_M(u(x_1,y,t),\, u(x_2,y,t)) > 0.$$

For any given set $K\subset M\times Y$, we define $K_y := \{x\in M : (x,y)\in K\}$, $\forall y\in Y$, and

$$\begin{aligned}Y_0(K) := \{\tilde{y}\in Y : &\text{ for any } \tilde{x}\in K_{\tilde{y}},\, y\in Y \text{ and any sequence } \{t_i\}\subset\mathbb{R} \text{ with}\\ &\sigma_{t_i}(y)\to\tilde{y}, \text{ there is a sequence } \{x_i\}\subset K_y \text{ such that } u(x_i,y,t_i)\to\tilde{x}\}.\end{aligned}$$

Further, K is said to be an 1-covering of Y if K_y is a singleton for all $y\in Y$.

Theorem 2.2 [6, Theorem 3.4]. *Let $K\subset M\times Y$ be a compact Π-invariant set such that every point of K admits a backward orbit. If the semiflow $(K,\Pi,\mathbb{R}_+)$ is uniformly stable, then it admits a flow extension which is fiber distal and uniformly stable as $t\to-\infty$. Further, the section map for K, $y\in Y\mapsto K_y$, is continuous at every $y\in Y$ with respect to the Hausdorff metric.*

Theorem 2.3 [6, Proposition 3.6]. *Let $\{\Pi(\tilde{x},\tilde{y},t) : t \geq 0\}$ be a precompact forward orbit of the skew-product semiflow Π and let $\tilde{K}$ denote the omega-limit set of $(\tilde{x},\tilde{y})$. Then the following statements are valid:*

(1) If $\tilde{K}$ contains a minimal set K which is uniformly stable, then $\tilde{K} = K$ and it admits a fiber distal flow extension.
(2) If the forward orbit is uniformly stable, then its omega-limit set $\tilde{K}$ is a uniformly stable minimal set which admits a fiber distal flow extension.

Lemma 2.4. *Let K be a compact Π-invariant set such that it admits a flow extension. If the section map for K is continuous at every $y \in Y$ with respect to the Hausdorff metric, then $Y_0(K) = Y$, and K is an 1-covering of Y whenever K_{y_0} is a singleton for some $y_0 \in Y$.*

Proof. Clearly, it suffices to prove that $Y \subseteq Y_0(K)$. Let $\tilde{y} \in Y$ be given. For any $\tilde{x} \in K_{\tilde{y}}$, $y \in Y$ and any sequence $\{t_i\} \subset \mathbb{R}$ with $\sigma_{t_i}(y) \to \tilde{y}$, by the continuity of the section map $y \in Y \mapsto K_y$, we can deduce that there is a sequence $x'_i \in K_{\sigma_{t_i}(y)}$ such that $x'_i \to \tilde{x}$. Note that K is invariant under the flow extension Π, we have $K_{\sigma_{t_i}(y)} = u(K_y, y, t_i)$. It then follows that there is a sequence $\{x_i\} \subseteq K_y$ such that $x'_i = u(x_i, y, t_i)$ and $u(x_i, y, t_i) \to \tilde{x}$, and hence, $\tilde{y} \in Y_0(K)$. This proves that $Y_0(K) = Y$. Now we assume that $K_{y_0} = \{x_0\}$. For any given $y \in Y$, the minimality of the base flow implies that there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that $\sigma_{t_n}(y_0) \to y$ as $n \to \infty$. Since $y \in Y = Y_0(K)$, it follows from the definition of $Y_0(K)$ that for any $\tilde{x} \in K_y$, there holds $\lim_{n\to\infty} u(x_0, y_0, t_n) = \tilde{x}$. This implies that K_y is a singleton. □

Let (X,P) be an ordered Banach space. For $x_1, x_2 \in X$, we write $x_1 \leq x_2$ if $x_2 - x_1 \in P$; $x_1 < x_2$ if $x_2 - x_1 \in P \setminus \{0\}$; $x_1 \ll x_2$ if $Int(P) \neq \emptyset$ and $x_2 - x_1 \in Int(P)$. A subset U of X is said to be order convex if for any $a, b \in U$ with $a < b$, the order interval $[a,b]_X := \{x \in X : a \leq x \leq b\}$ is contained in U.

In the rest of this paper, we assume that V is a closed and order convex subset of the positive cone P.

Let $Q : V \times Y \to Y$ be the natural projection. For a skew-product semiflow, we always use the order relations on each fiber $Q^{-1}(y)$. We write $(x_1, y) \geq_y (>_y, \gg_y)(x_2, y)$ if $x_1 \geq x_2 (x_1 > x_2, x_1 \gg x_2)$. Without any confusion, we will drop the subscript "y".

A skew-product semiflow Π_t on $V \times Y$ is said to be monotone (strongly monotone) if

$$\Pi_t(x_1, y) \leq (\ll) \Pi_t(x_2, y)$$

whenever $t > 0$ and $(x_1, y) \leq (x_2, y)((x_1, y) < (x_2, y))$.

To study omega limit sets of a monotone skew-product semiflow Π, we need the following assumptions.

(A1) Every compact subset in V has both the greatest lower bound (g.l.b.) and the least upper bound (l.u.b.).
(A2) For every $(x,y) \in V \times Y$, there is a $t_0 = t_0(x,y)$ such that $\{\Pi_t(x,y) : t \geq t_0\}$ is precompact.

(A3) The skew-product semiflow $\Pi_t : V \times Y \to V \times Y$ is monotone, and every forward orbit of Π_t is uniformly stable.

Note that if (A2) holds, then every omega limit set $\omega(x,y)$, $(x,y) \in V \times Y$, is a nonempty, compact and Π-invariant subset in $V \times Y$.

Let $K \subset V \times Y$ be a compact Π-invariant set. For any given $y \in Y$, we define

$$(p(y),y) = g.l.b. \quad \text{of } K \cap Q^{-1}(y)$$

and

$$(q(y),y) = l.u.b. \quad \text{of } K \cap Q^{-1}(y).$$

By (A1), $p(y)$ and $q(y)$ are well defined.

In view of Theorems 2.2 and 2.3 and Lemma 2.4, it is easy to see that the proof of [2, Propositions 3.1 and 3.2] still works in the case where the base flow is only minimal. Thus, we have the following result.

Theorem 2.5 [2, Propositions 3.1 and 3.2]. *Assume that* $(A1)-(A3)$ *hold. Let* $K := \omega(x_0,y_0)$, $(x_0,y_0) \in V \times Y$, *be fixed. Then the following statements are valid:*

(1) For any $y \in Y$, *both* $\omega(p(y),y)$ *and* $\omega(q(y),y)$ *are* 1*-coverings of* Y.
(2) Let $(p_*(y),y) := \omega(p(y),y) \cap Q^{-1}(y)$ *and* $(q_*(y),y) := \omega(q(y),y) \cap Q^{-1}(y)$. *Then* $(p_*(y),y) \le (p(y),y) \le (z,y) \le (q(y),y) \le (q_*(y),y)$, $\forall (z,y) \in K$.
(3) For any $y \in Y$ *and* $t \in \mathbb{R}$, *there holds* $u(p_*(y),y,t) = p_*(\sigma_t(y))$.

3 Global Convergence

In this section, we establish the 1-covering property for omega limit sets of monotone and uniformly stable recurrent skew-product semiflows.

We first consider a monotone skew-product semiflow $(\Pi, V \times Y, \mathbb{R}_+)$ in the case where $Int(P) \neq \emptyset$. We use the notations

$$\gamma^+(x,y) := \{\Pi_t(x,y) : t \ge 0\}$$

and

$$\gamma(x,y) := \{\Pi_t(x,y) : t \in \mathbb{R}\}$$

to denote the forward orbit and full orbit (if it exists) through (x,y), respectively. Π is said to be subhomogeneous if

$$u(\alpha x,y,t) \ge \alpha u(x,y,t), \quad \forall (x,y) \in V \times Y, \alpha \in (0,1), t \ge 0. \tag{1}$$

As in [2, 7, 13, 14], we define the part metric ρ on $Int(P)$ by

$$\rho(x_1,x_2) := \inf\{\ln \alpha : \alpha \ge 1 \text{ and } \alpha^{-1} x_1 \le x_2 \le \alpha x_1\}, \quad \forall x_1, x_2 \in Int(P).$$

Then $(Int(P), \rho)$ is a metric space (see, e.g., [7], [13]). We denote the metrics of the product space $Int(P) \times Y$ induced by (ρ, d_Y) and $(\|\cdot\|, d_Y)$ as d_ρ and $d_{\|\cdot\|}$, respectively.

Lemma 3.1 [14, Claim 1]. *If $(\Pi, V \times Y, \mathbb{R}_+)$ is monotone and subhomogeneous, then*

$$\rho(u(x_1,y,t), u(x_2,y,t)) \le \rho(x_1,x_2)$$

for all $x_1, x_2 \in Int(P)$ and $(y,t) \in Y \times \mathbb{R}_+$ with $u(x_i,y,t) \in Int(P), i = 1,2$.

Clearly, Lemma 3.1 implies that for any two forward orbits $\gamma^+(x_i,y) \subset Int(P) \times Y, i = 1,2$, we have

$$d_\rho(\Pi_{t+\tau}(x_1,y), \Pi_{t+\tau}(x_2,y)) \le d_\rho(\Pi_\tau(x_1,y), \Pi_\tau(x_2,y)), \quad \forall t \ge 0, \tau \ge 0. \tag{2}$$

Given $x_0 \in Int(P)$, we can choose a real number $r > 0$ such that the closed norm ball $\overline{B}(x_0, 2r) := \{x \in X : \|x - x_0\| \le 2r\} \subset Int(P)$. Then for any $x \in \overline{B}(x_0, r)$, there holds $\overline{B}(x, r) \subset Int(P)$. By [3, Lemma 2.3 (i)], we have

$$\rho(x, x_0) \le \ln\left(1 + \frac{\|x - x_0\|}{r}\right), \quad \forall x \in \overline{B}(x_0, r). \tag{3}$$

Lemma 3.2 [2, Lemma 5.1]. *Assume that $\gamma^+(x_0,y_0) \subset Int(P) \times Y$ is precompact in $(Int(P) \times Y, d_{\|\cdot\|})$ and its omega limit set $\omega(x_0,y_0)_{d_{\|\cdot\|}} \subset Int(P) \times Y$. Then $\gamma^+(x_0,y_0)$ is also precompact in $(Int(P) \times Y, d_\rho)$, and $\omega(x_0,y_0)_{d_\rho} = \omega(x_0,y_0)_{d_{\|\cdot\|}}$.*

Theorem 3.3. *Assume that the skew-product semiflow $(\Pi, V \times Y, \mathbb{R}_+)$ is subhomogeneous and strongly monotone, and (A2) holds. If Π admits a forward orbit $\gamma^+(x_0,y_0) \subset Int(P) \times Y$ such that $\omega(x_0,y_0) \subset Int(P) \times Y$, then for any $(x,y) \in V \times Y$ with $x \gg 0$, $\omega(x,y)$ is an 1-covering of Y, and $\lim_{t\to\infty} \|u(x,y,t) - u(x^*,y,t)\| = 0$, where $(x^*,y) = \omega(x,y) \cap Q^{-1}(y)$.*

Proof. For any $(x,y) \in V \times Y$ with $x \gg 0$, we can choose a point $(\bar{x},y) \in \omega(x_0,y_0) \subset Int(P) \times Y$ such that $x \gg \alpha\bar{x}$ for some sufficiently small $\alpha \in (0,1)$. It then follows that

$$\Pi_t(x,y) = (u(x,y,t), \sigma_t(y)) \ge (u(\alpha\bar{x},y,t), \sigma_t(y)) \ge (\alpha u(\bar{x},y,t), \sigma_t(y)), \quad \forall t \ge 0,$$

and hence $K := \omega(x,y) \subset Int(P) \times Y$. Note that $\omega(x,y) = \omega(\tilde{x},\tilde{y})$ for any $(\tilde{x},\tilde{y}) \in \gamma^+(x,y)$. Without loss of generality, we then assume that $\gamma^+(x,y) \subset Int(P) \times Y$. Clearly, Lemma 3.2 implies that $K = \omega(x,y)_{d_\rho}$. In view of Lemma 3.1, (2) and (3), it is easy to check that Π is a semiflow on the compact metric space (K, d_ρ), and that any Π-invariant subset of K is uniformly stable in d_ρ. It then follows from Theorems 2.2 and 2.3 that K is a minimal set which admits a fiber distal flow extension, and the section map for K, $y \in Y \mapsto K_y$, is continuous at every $y \in Y$ with respect to d_ρ. Let $Y_0(K)$ be defined with respect to d_ρ. Thus, Lemma 2.4 implies that $Y_0 := Y_0(K) = Y$ in our current case. By [10, Theorem II.3.1], we then conclude that for any $y \in Y$, $K \cap Q^{-1}(y)$ contains no pair of strongly ordered distinct points. By the same contradiction argument as in [2, Theorem 5.1], it follows that $Card(K \cap Q^{-1}(y)) = 1$ for all $y \in Y$, that is, $\omega(x,y)$ is an 1-covering of Y.

To prove $\lim_{t\to\infty} \|u(x,y,t) - u(x^*,y,t)\| = 0$, we assume, by contradiction, that there exist an $\varepsilon_0 > 0$ and a sequence $t_n \to \infty$ such that $\|u(x,y,t_n) - u(x^*,y,t_n)\| \geq \varepsilon_0$, $\forall n \geq 1$. Clearly, $\omega(x^*,y) \subseteq K = \omega(x,y)$. In view of (A2), we can further assume, without loss of generality, that $\lim_{n\to\infty} \Pi(x,y,t_n) = (x_1^*, y^*) \in K$ and $\lim_{n\to\infty} \Pi(x^*,y,t_n) = (x_2^*, y^*) \in K$. Since $Card(K \cap Q^{-1}(y^*)) = 1$, we have $x_1^* = x_2^*$. Thus, $0 = \|x_1^* - x_2^*\| = \lim_{n\to\infty} \|u(x,y,t_n) - u(x^*,y,t_n)\| \geq \varepsilon_0$, a contradiction. □

Next, we establish the 1-covering property of omega limit sets for a monotone and uniformly stable skew-product semiflow with the strong componentwise separating property without assuming $Int(P) \neq \emptyset$.

Let (X_i, P_i), $1 \leq i \leq n$, be ordered Banach spaces. For each $I = \{j_1, j_2, \cdots, j_m\} \subset N := \{1, 2, \cdots, n\}$, we define

$$X_I := \prod_{k=1}^{m} X_{j_k}, \quad P_I := \prod_{k=1}^{m} P_{j_k}.$$

Then (X_I, P_I) is an ordered Banach space. Let $\leq_I$ and $<_I$ be the orders induced by P_I in X_I. In the case where $I = N$, we use (X, P) to denote the ordered Banach space (X_N, P_N), and omit the order subscripts to get the orders $\leq$ and $<$ in X, respectively. For each $1 \leq i \leq n$, let $Q_i : X \times Y \mapsto X_i$ be the projection mapping defined by $Q_i(x,y) = x_i$.

Let V be a closed and order convex subset of X. For the skew-product semiflow

$$\Pi : V \times Y \times \mathbb{R}_+ \to V \times Y,$$

we make the following additional assumption.

(A4) For each $1 \leq i \leq n$, there exists a continuous map $\mathcal{P}_i : X \mapsto Z_i$, where (Z_i, Z_i^+) is an order Banach space with $Int(Z_i^+) \neq \emptyset$, such that

$$\mathcal{P}_i u(x_1,y,t) \gg \mathcal{P}_i u(x_2,y,t),\ \forall t > 0,\ y \in Y,\ \text{whenever}\ x_1 \geq x_2\ \text{with}\ \mathcal{P}_i x_1 > \mathcal{P}_i x_2.$$

Theorem 3.4. *Assume that* $(A1)-(A4)$ *hold. Then for any* $(x_0, y_0) \in V \times Y$, $K = \omega(x_0, y_0)$ *is an* 1*-covering of* Y, *and* $\lim_{t\to\infty} \|u(x_0,y_0,t) - u(x_0^*,y_0,t)\| = 0$, *where* $(x_0^*, y_0) = \omega(x_0,y_0) \cap Q^{-1}(y_0)$.

Proof. By Theorem 2.2, Theorem 2.3 (2), and the assumptions (A2)-(A3), we can deduce that $K := \omega(x_0, y_0)$ has a flow extension which is minimal and fiber distal, and the section map for K, $y \in Y \mapsto K_y$, is continuous at every $y \in Y$. Thus, Lemma 2.4 implies that $Y_0 := Y_0(K) = Y$ in our current case. Invoking Theorem 2.5, we conclude that for each $\hat{y} \in Y$,

$$\omega(p(\hat{y}), \hat{y}) = K^* := \{(p_*(y), y) : y \in Y\}, \tag{4}$$

where $(p(\hat{y}),\hat{y}) = \text{g.l.b. of } K\cap Q^{-1}(\hat{y})$ and $(p_*(y),y) = \omega(p(\hat{y}),\hat{y})\cap Q^{-1}(y)$ for every $y\in Y$. Let $(x,y)\in K$ be given arbitrary. Since K is minimal, there is $\tau_n\to+\infty$ such that $\Pi_{\tau_n}(x,y)\to(x,y)$ as $n\to\infty$. Note that $p(y)\le x$, $(p_*(y),y)=\omega(p(y),y)\cap Q^{-1}(y)$. On the other hand, $(A2)$ implies that there exists a subsequence of τ_n, which we relabel as τ_n such that $\Pi_{\tau_n}(p(y),y)\to(p_*(y),y)$. Hence $p_*(y)\le x$. As in the proof of [2, Theorem 4.1], we now prove that there is a subset $J\subset\{1,\cdots,n\}$, denoted by $J=\{1,2,\cdots,m\}$ without loss of generality, such that

$$\mathcal{P}_i p_*(y) = \mathcal{P}_i x \quad \text{for each } (x,y)\in K \text{ and } i\notin J, \tag{5}$$

$$\mathcal{P}_i p_*(y) \ll \mathcal{P}_i x \quad \text{for each } (x,y)\in K \text{ and } i\in J, \tag{6}$$

and that there exists an $\varepsilon_0>0$ such that

$$\mathcal{P}_i(B_+^J(p_*(y),\varepsilon_0)) \ll \mathcal{P}_i x \quad \text{for each } (x,y)\in K \text{ and } i\in J, \tag{7}$$

where $B_+^J(p_*(y),\varepsilon_0)$ denotes the set

$$\{x=(x_1,\cdots,x_m,p_{*m+1},\cdots,p_{*n})\in V : x\ge p_*(y) \text{ and } \|x-p_*(y)\|\le\varepsilon_0\}.$$

We first show that if $\mathcal{P}_i p_*(\tilde{y}) = \mathcal{P}_i\tilde{x}$ for some $i\in\{1,\cdots,n\}$ and $(\tilde{x},\tilde{y})\in K$, then $\mathcal{P}_i p_*(y)=\mathcal{P}_i x$ for any $(x,y)\in K$. Thanks to Theorem 2.2, we know that K admits a flow extension and $\mathcal{P}_i u(p_*(\tilde{y}),\tilde{y},s)=\mathcal{P}_i u(\tilde{x},\tilde{y},s)$ for all $s\in(-\infty,0)$. Otherwise, there would be $s\in(-\infty,0)$ with $\mathcal{P}_i u(p_*(\tilde{y}),\tilde{y},s) < \mathcal{P}_i u(\tilde{x},\tilde{y},s)$. Then $(A4)$ implies that $\mathcal{P}_i p_*(\tilde{y}) \ll \mathcal{P}_i\tilde{x}$, a contradiction. Let $(x,y)\in K$ be given. Since K is minimal, there exists a sequence $s_n\downarrow-\infty$ such that $\sigma_{s_n}(\tilde{y})\to y$ and $u(\tilde{x},\tilde{y},s_n)\to x$. In view of Theorem 2.5 (3), we have

$$\mathcal{P}_i x = \lim_{n\to\infty}\mathcal{P}_i u(\tilde{x},\tilde{y},s_n) = \lim_{n\to\infty}\mathcal{P}_i u(p_*(\tilde{y}),\tilde{y},s_n) = \lim_{n\to\infty}\mathcal{P}_i p_*(\sigma_{s_n}(\tilde{y})) = \mathcal{P}_i p_*(y).$$

Thus, we have $\mathcal{P}_i x = \mathcal{P}_i p_*(y)$ for each $(x,y)\in K$.

Now we prove that if $\mathcal{P}_j p_*(\tilde{y}) < \mathcal{P}_j\tilde{x}$ for some $j\in\{1,\cdots,n\}$, then we have $\mathcal{P}_j p_*(y)\ll\mathcal{P}_j x$ for any $(x,y)\in K$. The flow extension on K implies $\mathcal{P}_j u(p_*(\tilde{y}),\tilde{y},s) < \mathcal{P}_j u(\tilde{x},\tilde{y},s)$ for all $s\in(-\infty,0)$. Suppose not, then there would be $s\in(-\infty,0)$ with $\mathcal{P}_j u(p_*(\tilde{y}),\tilde{y},s)=\mathcal{P}_j u(\tilde{x},\tilde{y},s)$. By Theorem 2.5 (3) and the above argument, we see that $\mathcal{P}_j p_*(\tilde{y})=\mathcal{P}_j\tilde{x}$, a contradiction. Note that for any $(x,y)\in K$, there exists a sequence $s_n\downarrow-\infty$ such that $\sigma_{s_n}(\tilde{y})\to y$ and $u(\tilde{x},\tilde{y},s_n)\to x$. Let $t_0>0$ be given, it is clear that $\sigma_{s_n-t_0}(\tilde{y})\to\sigma_{-t_0}(y)$, $u(\tilde{x},\tilde{y},s_n-t_0)\to\bar{x}\in K_{\sigma_{-t_0}(y)}$ and $u(\bar{x},\sigma_{-t_0}(y),t_0)=x$. By Theorem 2.5 (3) again, we have

$$\begin{aligned}\mathcal{P}_j\bar{x} &= \lim_{n\to\infty}\mathcal{P}_j u(\tilde{x},\tilde{y},s_n-t_0) \ge \lim_{n\to\infty}\mathcal{P}_j u(p_*(\tilde{y}),\tilde{y},s_n-t_0)\\ &= \lim_{n\to\infty}\mathcal{P}_j p_*(\sigma_{s_n-t_0}(\tilde{y})) = \mathcal{P}_j p_*(\sigma_{-t_0}(y)).\end{aligned}$$

By the above result, we can deduce that $\mathcal{P}_j\bar{x} > \mathcal{P}_j p_*(\sigma_{-t_0}(y))$. Hence, (A4) and Theorem 2.5 (3) imply that

$$\mathcal{P}_j x = \mathcal{P}_j u(\bar{x}, \sigma_{-t_0}(y), t_0) \gg \mathcal{P}_j u(p_*(\sigma_{-t_0}(y)), \sigma_{-t_0}(y), t_0) = \mathcal{P}_j p_*(y).$$

Since (6) implies (7), (5)–(7) follow immediately from the above arguments.

Let $(x,y) \in K$ and define $x_\alpha = (1-\alpha)p_*(y) + \alpha x$ for $\alpha \in [0,1]$, and

$$L = \{\alpha \in [0,1] : \omega(x_\alpha, y) = K^*\}.$$

Invoking Theorem 2.5, we see that $\omega(p_*(y), y) = K^*$. Combining this with the monotonicity of the semiflow, we deduce that if $0 < \alpha \in L$, then $[0,\alpha] \subset L$.

Next we show that L is closed, that is, if $[0,\alpha) \subset L$, then $\alpha \in L$. Since $\{\Pi(x_\alpha, y, t) : t \geq 0\}$ is uniformly stable, let $\delta(\varepsilon) > 0$ be the modulus of uniform stability for ε. Thus, we take $\beta \in [0,\alpha)$ with $\|x_\alpha - x_\beta\| < \delta(\varepsilon)$ and we obtain $\|u(x_\alpha, y, t) - u(x_\beta, y, t)\| < \varepsilon$ for each $t \geq 0$. Moreover, $\omega(x_\beta, y) = K^*$ and hence, there is a t_0 such that $\|u(x_\beta, y, t) - p_*(\sigma_t(y))\| < \varepsilon$ for each $t \geq t_0$. Then, we deduce that $\|u(x_\alpha, y, t) - p_*(\sigma_t(y))\| < 2\varepsilon$ for each $t \geq t_0$ and $\omega(x_\alpha, y) = K^*$, as claimed.

Now we prove that $L = [0,1]$. Assume, by contradiction, that $L = [0,\alpha]$ for some $0 \leq \alpha < 1$. Let ε_0 be the number defined in $B_+^J(p_*(y), \varepsilon_0)$ of (7). Then the uniform stability assumption implies that we can take $\alpha < \gamma < 1$ such that

$$\|u(x_\alpha, y, t) - u(x_\gamma, y, t)\| < \frac{\varepsilon_0}{2}, \quad \forall t \geq 0. \tag{8}$$

As above, from $\omega(x_\alpha, y) = K^*$ we deduce that there is a $t_1 \geq 0$ such that $\|u(x_\alpha, y, t) - p_*(\sigma_t(y))\| < \frac{\varepsilon_0}{2}$ for each $t \geq t_1$. Consequently, for each $t \geq t_1$,

$$\|u(x_\gamma, y, t) - p_*(\sigma_t(y))\| < \varepsilon_0. \tag{9}$$

Let $(\tilde{x}, \tilde{y}) \in \omega(x_\gamma, y)$, i.e., $(\tilde{x}, \tilde{y}) = \lim_{n\to\infty}(u(x_\gamma, y, t_n), \sigma_{t_n}(y))$ for some $t_n \uparrow \infty$. The monotonicity, $p_*(y) \leq x_\gamma$ and Theorem 2.5 (3) imply that $p_*(\sigma_{t_n}(y)) \leq u(x_\gamma, y, t_n)$, which yields to $p_*(\tilde{y}) \leq \tilde{x}$. Since $p_*(y) \leq x_\gamma \leq x$, we have $p_*(\sigma_{t_n}(y)) \leq u(x_\gamma, y, t_n) \leq u(x, y, t_n)$, and hence, by (5), we deduce that $\mathcal{P}_i p_*(\sigma_{t_n}(y)) = \mathcal{P}_i u(x_\gamma, y, t_n)$ for each $i \notin J$. This yields to $\mathcal{P}_i p_*(\tilde{y}) = \mathcal{P}_i \tilde{x}$ for each $i \notin J$. For any given $(z, \tilde{y}) \in K$, it follows from (5) that $\mathcal{P}_i p_*(\tilde{y}) = \mathcal{P}_i z$ for each $i \notin J$. By (9), we deduce that $\tilde{x} \in B_+^J(p_*(\tilde{y}), \varepsilon_0)$. In view of (7), we have $\mathcal{P}_i \tilde{x} \ll \mathcal{P}_i z$ for each $i \in J$. Thus, we can conclude that $p_*(\tilde{y}) \leq \tilde{x} \leq z$. Since this holds for each $(z, \tilde{y}) \in K$, the definition of p provides $p_*(\tilde{y}) \leq \tilde{x} \leq p(\tilde{y})$. From (4) we see that $\omega(p(\tilde{y}), \tilde{y}) = K^*$. It then follows from Theorem 2.5 that $\omega(\tilde{x}, \tilde{y}) = K^* \subseteq \omega(x_\gamma, y)$. By (A3) and Theorem 2.3, we conclude that $\omega(\tilde{x}, \tilde{y}) = \omega(x_\gamma, y) = K^*$, and hence $\gamma \in L$, a contradiction.

Since $L = [0,1]$, we have $\omega(x, y) = K^*$, and hence, the minimality of K implies that $K = K^*$ and $J = \emptyset$. Thus, K is an 1-covering of Y. As in the proof of Theorem 3.3, we can deduce that $\lim_{t\to\infty} \|u(x_0, y_0, t) - u(x_0^*, y_0, t)\| = 0$, where $(x_0^*, y_0) = \omega(x_0, y_0) \cap Q^{-1}(y_0)$. □

4 Applications

In this section, we applied the results in Sect. 3 to study the asymptotic recurrence of solutions to two recurrent evolution systems.

First, we consider

$$\frac{du}{dt} = A(t)u + f(t), \quad u(0) \in \mathbb{R}^n, \tag{10}$$

where $A(t)$ is a continuous $n \times n$ matrix function, and $f = (f_1, \dots, f_n) : \mathbb{R} \to \mathbb{R}^n$ is a continuous function. We assume that there exists $\delta_0 > 0$ such that

$$A_{ij}(t) \geq \delta_0, \quad f_i(t) \geq \delta_0, \; \forall 1 \leq i \neq j \leq n.$$

Define $A_s(t) = A(s+t)$ and $f_s(t) = f(s+t)$. We further assume that the set $\{(A_s, f_s) : s \in \mathbb{R}\}$ has compact closure $H(A, f)$ with respect to the compact open topology, and that the flow $\sigma : H(A, f) \times \mathbb{R} \to H(A, f)$, defined by $\sigma((B, g), t) = (B_t, g_t), t \in \mathbb{R}$, is minimal. For any given $(B, g) \in H(A, f)$, let $u(t, x, B, g), t \geq 0$, be the unique solution of the linear system $\frac{du}{dt} = A(t)u + f(t)$ satisfying $u(0) = x \in \mathbb{R}^n$.

Theorem 4.1. *Assume that for any $(B,g) \in H(A, f)$ and $x \in \mathbb{R}^n_+$, the solution $u(t,x,B,g)$ is bounded. Then (10) has a unique positive, recurrent and bounded full solution $u^*(t)$ such that* $\lim_{t\to\infty} |u(t,x,A,f) - u^*(t)| = 0$ *for any $x \in \mathbb{R}^n_+$.*

Proof. For each $x \in \mathbb{R}^n_+$ and $(B,g) \in H(A, f)$, let $u(t,x,B,g)$ be the unique solution of (10) with (A, f) replaced by (B,g). By the comparison theorem for cooperative systems, each $u(t,x,B,g)$ exists globally on $[0,\infty)$ and $u(t,x,B,g) \geq 0, \forall t \geq 0$. We define the skew-product semiflow Π_t on $\mathbb{R}^n_+ \times H(A, f)$ by $\Pi_t(x,(B,g)) = (u(t,x,B,g), \sigma_t(B,g))$. By the comparison theorem for cooperative and irreducible systems and the variation of constants formula for inhomogeneous linear systems, it then follows that the skew-product semiflow Π_t is strongly monotone and strongly subhomogeneous. By our assumption on f, we see that the omega limit set $\omega(x,A,f)$ is compact and $\omega(x,A,f) \subset Int(\mathbb{R}^n_+) \times H(A,f)$ for any $x \in \mathbb{R}^n_+$. Thus, Theorem 3.3 implies that for any $x \in \mathbb{R}^n_+$, $\omega(x,A,f)$ is a 1-covering of $H(A, f)$. Clearly, $\Pi_t : \omega(x,A,f) \to \omega(x,A,f)$ is a compact, minimal and fiber distal flow.

Let $x^0 \in \mathbb{R}^n_+$ be given, and define $u^*(t) := u(t,x^*,A,f)$, where $(x^*,(A,f)) = \omega(x^0,A,f) \cap Q^{-1}(A,f)$. It then follows that $u^*(t)$ is a positive, recurrent and bounded full solution of (10). In order to prove that $\omega(x,A,f) = \omega(x^0,A,f)$ for any $x \in \mathbb{R}^n_+$, by the minimality of both $\Pi_t : \omega(x,A,f) \to \omega(x,A,f)$ and $\Pi_t : \omega(x^0,A,f) \to \omega(x^0,A,f)$, it suffices to prove that $\omega(x,A,f) \cap \omega(x^0,A,f) \neq \emptyset$. Assume, by contradiction, that $\omega(x,A,f) \cap \omega(x^0,A,f) = \emptyset$. Then we have $d(\omega(x,A,f), \omega(x^0,A,f)) > 0$, where d is the metric on the product space $\mathbb{R}^n_+ \times H(A,f)$. Let $(x_1,B,g) = \omega(x,A,f) \cap Q^{-1}(B,g)$ and $(x_2,B,g) = \omega(x^0,A,f) \cap Q^{-1}(B,g)$. On the other hand, by the minimality and the 1-covering property, there exists $t_n \to \infty$ such that $\lim_{n\to\infty} \Pi_{t_n}(x_i,(B,g)) = (x_i,(B,g))$, and hence $\lim_{n\to\infty} u(t_n,x_i,B,g) = x_i$, $i = 1,2$. For any fixed $t_0 > 0$, let

$$(x_1',(B,g)') = \Pi_{t_0}(x_1,(B,g)) \in \omega(x,A,f) \subset Int(\mathbb{R}^n_+) \times H(A,f),$$

and

$$(x_2',(B,g)') = \Pi_{t_0}(x_2,(B,g)) \in \omega(x^0,A,f) \subset Int(\mathbb{R}^n_+) \times H(A,f).$$

Then

$$\Pi_{t_n}(x_i,(B,g)) = \Pi_{t_n - t_0}(x_i',(B,g)'), \quad \forall i = 1,2, n \geq 1.$$

By the strong monotonicity and strong subhomogeneity of Π_t, Claim 2 in the proof [14, Theorem 2.1], and [14, Remarks 2.1–2.2], we then obtain

$$\begin{aligned} \rho(x_1,x_2) &= \lim_{n\to\infty} \rho(u(t_n,x_1,B,g), u(t_n,x_2,B,g)) \\ &= \lim_{n\to\infty} \rho(u(t_n - t_0,x_1',(B,g)'), u(t_n - t_0,x_2',(B,g)')) \\ &\leq \rho(x_1',x_2') = \rho(u(t_0,x_1,B,g), u(t_0,x_2,B,g)) \\ &< \rho(x_1,x_2), \end{aligned}$$

a contradiction. Therefore, $\omega(x,A,f) = \omega(x^0,A,f)$ for any $x \in \mathbb{R}^n_+$. It then follows from Theorem 3.3 that $\lim_{t\to\infty} |u(t,x,A,f) - u^*(t)| = 0$ for any $x \in \mathbb{R}^n_+$. □

Next, we consider the scalar nonautonomous Kolmogorov parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = d(t)\triangle u + uf(x,t,u) & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{11}$$

where Ω is a bounded and open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the Laplacian operator on $\mathbb{R}^N$, $Bv = v$ or $Bv = \frac{\partial v}{\partial n} + \alpha v$ for some nonnegative function $\alpha \in C^{1+\theta}(\partial\Omega,\mathbb{R})$, $d(\cdot) \in C(\mathbb{R},\mathbb{R})$, and $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+,\mathbb{R})$.

Let $H(d,f)$ be the closure of $\{(d_s,f_s) : s \in \mathbb{R}\}$ with respect to the compact open topology, where $(d_s,f_s) \in C(\mathbb{R},\mathbb{R}) \times C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+,\mathbb{R})$ is defined by

$$d_s(t) = d(s+t), \quad f_s(x,t,u) = f(x,t+s,u), \forall (x,t,u) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+.$$

Define $\sigma_t(\mu,g) = (\mu_t,g_t)$, $(\mu,g) \in H(d,f)$, $t \in \mathbb{R}$. We assume that

(B1) $H(d,f)$ is compact with respect to the compact open topology and the flow σ_t on $H(d,f)$ is minimal.

(B2) $d(\cdot) \in C(\mathbb{R},\mathbb{R})$ is bounded with $d(t) \geq d_0$, $\forall t \in \mathbb{R}$, for some $d_0 > 0$, and $d(t)$ is Hölder continuous in $t \in \mathbb{R}$; $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+,\mathbb{R})$ is bounded, $f_u'(x,t,u) \leq 0$, $\forall (x,t,u) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+$, and $f(x,t,0)$ is uniformly Hölder continuous in $(t,x) \in \mathbb{R} \times \bar{\Omega}$.

(B3) There exists $M_0 > 0$ such that $f(x,t,M_0) \leq 0$, $\forall (x,t) \in \bar{\Omega} \times \mathbb{R}$.

Let $p \in (N, \infty)$ be fixed. For each $\beta \in (1/2 + N/(2p), 1)$, let X_β be the fractional power space of $X = L^p(\Omega)$ with respect to $(-\Delta, B)$. Then X_β is an ordered Banach space with the cone X_β^+ consisting of all nonnegative functions in X_β, and X_β^+ has nonempty interior $Int(X_\beta^+)$. Moreover, $X_\beta \subset C^{1+\nu}(\bar{\Omega})$ with continuous inclusion for $\nu \in [0, 2\beta - 1 - N/p)$. We denote the norms in X_β and $L^2(\Omega)$ by $\|\cdot\|_\beta$ and $\|\cdot\|_2$, respectively.

By the theory of semilinear parabolic differential equations (see, e.g., [1, Sect. III. 20]), it follows that for every $\phi \in X_\beta^+$ and $(\mu, g) \in H(d, f)$, the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mu(t)\triangle u + ug(x,t,u) & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot, 0) = \phi \end{cases} \tag{12}$$

has a unique regular solution $u(x,t,\phi,\mu,g)$ with the maximal interval of existence $I(\phi,\mu,g) \subset [0,\infty)$, and $I(\phi,\mu,g) = [0,\infty)$ provided $u(\cdot,t,\phi,\mu,g)$ has an L^∞-bound on $I(\phi,\mu,g)$.

According to [4,5], the principal spectrum of the linear nonautonomous parabolic problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = d(t)\Delta v + f(x,t,0)v, & x \in \Omega,\ t \in \mathbb{R}, \\ Bv = 0, \qquad x \in \partial\Omega,\ t \in \mathbb{R} \end{cases} \tag{13}$$

is defined to be the dynamical (Sacker-Sell) spectrum of its associated linear skew-product flow restricted to the one-dimensional subbundle of $X_\beta \times H(d, f(\cdot,\cdot,0))$. By [4, Theorem 2.6 and Proposition 2.11 (i)], it follows that the principal spectrum of (13) is a nonempty and compact interval $[\lambda_{\inf}(d, f(\cdot,\cdot,0)), \lambda_{\sup}(d, f(\cdot,\cdot,0))]$, and (13) admits a unique strongly positive full solution $v(t)$, $t \in \mathbb{R}$, with $\|v(0)\|_2 = 1$ such that

$$\lambda_{\inf}(d, f(\cdot,\cdot,0)) = \liminf_{t-s\to\infty} \frac{\log\|v(t)\|_2 - \log\|v(s)\|_2}{t-s},$$

and

$$\lambda_{\sup}(d, f(\cdot,\cdot,0)) = \limsup_{t-s\to\infty} \frac{\log\|v(t)\|_2 - \log\|v(s)\|_2}{t-s}.$$

We are now in a position to prove the following result on the global convergence for (11).

Theorem 4.2. *Let (B1)–(B3) hold. Then the following two statements are valid:*

(1) If $\lambda_{\sup}(d, f(\cdot,\cdot,0)) < 0$*, then* $\lim_{t\to\infty} \|u(\cdot,t,\phi,d,f)\|_\beta = 0$ *for every* $\phi \in X_\beta^+$*;*

(2) If $\lambda_{\inf}(d, f(\cdot,\cdot,0)) > 0$, then for every $\phi \in X_\beta^+ \setminus \{0\}$, there exists a positive, recurrent and bounded full solution $u(x,t,\phi^*,d,f)$ of (11) such that $\lim_{t\to\infty} \|u(\cdot,t,\phi,d,f) - u(\cdot,t,\phi^*,d,f)\|_\beta = 0$.

Proof. For any $(\mu,g) \in H(d,f)$, both $(B2)$ and $(B3)$ imply that $u = M$, $M \geq M_0$, is an upper-solution of (12), and hence, by the comparison theorem and a priori estimates of parabolic equations (see, e.g., [1]), each solution $u(x,t,\phi,\mu,g)$ exists globally on $[0,\infty)$, and for any $t_0 > 0$, the set $\{u(\cdot,t,\phi,\mu,g) : t \geq t_0\}$ is precompact in X_β^+. We define the skew-product semiflow $\Pi_t : X_\beta^+ \times H(d,f) \to X_\beta^+ \times H(d,f)$ by $\Pi_t(\phi,\mu,g) = (u(\cdot,t,\phi,\mu,g),\mu_t,g_t)$. Then for each $(\phi,\mu,g) \in X_\beta^+ \times H(d,f)$, the omega limit set $\omega(\phi,\mu,g)$ of the forward orbit $\gamma^+(\phi,\mu,g) := \{\Pi_t(\phi,\mu,g) : t \geq 0\}$ is well defined, compact and invariant under Π_t, $t \geq 0$. Moreover, the maximum principle for parabolic equations implies that $\Pi_t((X_\beta^+ \setminus \{0\}) \times H(d,f)) \subset Int(X_\beta^+) \times H(d,f)$, $\forall t > 0$.

In the case where $\lambda_{\sup}(d, f(\cdot,\cdot,0)) < 0$, we choose a sufficiently small number $\delta_1 > 0$ such that $\lambda_{\sup}(d, f(\cdot,\cdot,0)) + \delta_1 < 0$. It then follows that $\|v(t)\|_2 \leq e^{(\lambda_{\sup}(d,f(\cdot,\cdot,0))+\delta_1)t}$ for sufficiently large t, and hence $\lim_{t\to\infty} \|v(t)\|_2 = 0$. For any $\phi \in X_\beta^+$, there exists a sufficiently large number $K > 0$ such that $\phi \leq Kv(0)$. Note that $u(x,t,\phi,d,f)$ satisfies the following differential inequality

$$\begin{cases} \dfrac{\partial u}{\partial t} \leq d(t)\triangle u + f(x,t,0)u & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{14}$$

By the comparison principle, we see that

$$u(\cdot,t,\phi,d,f) \leq Kv(t),\ \forall t \geq 0,$$

and hence, $\lim_{t\to\infty} \|u(\cdot,t,\phi,d,f)\|_2 = 0$. Now we prove $\lim_{t\to\infty} \|u(\cdot,t,\phi,d,f)\|_\beta = 0$. For any $(\psi,\mu,g) \in \omega(\phi,d,f)$, there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \Pi_{t_n}(\phi,d,f) = (\psi,\mu,g)$, and hence, $\lim_{n\to\infty} \|u(\cdot,t_n,\phi,d,f) - \psi\|_\beta = 0$. Since $X_\beta \subset C^1(\bar{\Omega})$ with continuous inclusion, we have $\lim_{n\to\infty} u(x,t_n,\phi,d,f) = \psi(x)$ uniformly for $x \in \bar{\Omega}$, and hence, $\|\psi\|_2 = 0$. Since $\psi(x)$ is nonnegative and continuous on $\bar{\Omega}$, we further obtain that $\psi(\cdot) \equiv 0$. It then follows that $\lim_{t\to\infty} \|u(\cdot,t,\phi,d,f)\|_\beta = 0$.

In the case where $\lambda_{\inf}(d, f(\cdot,\cdot,0)) > 0$, we fix a positive number $\varepsilon < \lambda_{\inf}(d, f(\cdot,\cdot,0))$. Then we have the following claim.

Claim. There exists $\delta > 0$ such that $\limsup_{t\to\infty} \|u(\cdot,t,\phi,\mu,g)\|_\beta \geq \delta$ for all $(\phi,\mu,g) \in (X_\beta^+ \setminus \{0\}) \times H(d,f)$.

Indeed, since $H(d,f)$ is compact and the translation flow σ_t is minimal on $H(d,f)$, there exists $\delta_0 > 0$ such that

$$|g(x,t,u) - g(x,t,0)| < \varepsilon, \quad \forall x \in \bar{\Omega},\ t \in \mathbb{R},\ u \in [0,\delta_0],\ (\mu,g) \in H(d,f).$$

Since $X_\beta \subset C^1(\bar{\Omega})$ with continuous inclusion, there exists $\delta > 0$ such that for any $\phi \in X_\beta$, $\|\phi\|_\beta \le \delta$ implies that $\|\phi\|_\infty \le \delta_0$. Suppose for contradiction that for some $(\phi,\mu,g) \in (X_\beta^+ \setminus \{0\}) \times H(d,f)$, there holds $\limsup_{t\to\infty} \|u(\cdot,t,\phi,\mu,g)\|_\beta < \delta$. Then there is $t_0 > 0$ such that $\|u(\cdot,t,\phi,\mu,g)\|_\beta < \delta$ for all $t \ge t_0$, and hence $\|u(\cdot,t,\psi,\gamma,h)\|_\beta < \delta$ for all $t \ge 0$, where $(\psi,\gamma,h) = (u(\cdot,t_0,\phi,\mu,g),\mu_{t_0},g_{t_0}) \in Int(X_\beta^+) \times H(d,f)$. By the choice of δ_0 and δ, it follows that $u(x,t,\psi,\gamma,h)$ satisfies the following differential inequality

$$\begin{cases} \dfrac{\partial u}{\partial t} \ge \gamma(t)\triangle u + (h(x,t,0)-\varepsilon)u & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{15}$$

It is easy to check that $(\gamma,h(\cdot,\cdot,0)) \in H(d,f(\cdot,\cdot,0))$ and the translation flow σ_t is minimal on $H(d,f(\cdot,\cdot,0))$. Thus, we have $H(\gamma,h(\cdot,\cdot,0)) = H(d,f(\cdot,\cdot,0))$. It then follows that

$$\lambda_{\inf}(\gamma,h(\cdot,\cdot,0)) = \lambda_{\inf}(d,f(\cdot,\cdot,0)) > \varepsilon.$$

Let $w(t)$ be the unique strongly positive full solution of the linear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma(t)\triangle u + h(x,t,0)u & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{16}$$

We choose a sufficiently small number $\delta_2 > 0$ such that $\lambda_{\inf}(\gamma,h(\cdot,\cdot,0)) > \varepsilon + \delta_2$. By [4, Proposition 2.11 (i)], it then follows that $\|w(t)\|_2 \ge e^{(\lambda_{\inf}(\gamma,h(\cdot,\cdot,0))-\delta_2)t}$ for sufficiently large t, and hence, $\lim_{t\to\infty} \|e^{-\varepsilon t}w(t)\|_2 = \infty$. Since $\psi \gg 0$, there exists a sufficiently small number $k > 0$ such that $\psi \ge kw(0)$. By (15) and the comparison principle, it follows that

$$u(\cdot,t,\psi,\gamma,h) \ge ke^{-\varepsilon t}w(t),\ \forall t \ge 0,$$

and hence, $\lim_{t\to\infty} \|u(\cdot,t,\psi,\gamma,h)\|_2 = \infty$, which contradicts the boundedness of $u(\cdot,t,\psi,\gamma,h)$ in $C(\bar{\Omega})$.

By the claim above and the same arguments as in [14, Theorem 3.1], we further have

$$\omega(\phi,\mu,g) \subset Int(X_\beta^+) \times H(d,f), \quad \forall(\phi,\mu,g) \in (X_\beta^+ \setminus \{0\}) \times H(d,f).$$

Let $u(\phi,\mu,g,t) := u(\cdot,t,\phi,\mu,g)$, $t \ge 0$. By the standard comparison theorem, it then follows that $u(\cdot,\mu,g,t)$ is strongly monotone on X_β^+ for each $(\mu,g,t) \in H(d,f) \times (0,\infty)$. It is easy to see from $(B2)$ that each function $ug(x,t,u)$, $(\mu,g) \in H(d,f)$, is subhomogeneous in u for any fixed $(x,t) \in \bar{\Omega} \times \mathbb{R}_+$. By the integral version of parabolic (12) (see, e.g., [1]), it then follows that $u(\cdot,\mu,g,t)$ is subhomogeneous on X_β^+ for each $(\mu,g,t) \in H(d,f) \times \mathbb{R}_+$. Thus, the skew-product semiflow Π_t is subhomogeneous and strongly monotone on $X_\beta^+ \times H(d,f)$. By Theorem 3.3, it follows that for every $\phi \in X_\beta^+ \setminus \{0\}$, $\omega(\phi,d,f)$ is a 1-covering of $H(d,f)$, and $\lim_{t\to\infty} \|u(\cdot,t,\phi,d,f) - u(\cdot,t,\phi^*,d,f)\|_\beta = 0$, where $(\phi^*,d,f) \in \omega(\phi,d,f)$. Since $\Pi_t : \omega(\phi,d,f) \to \omega(\phi,d,f)$ is a minimal flow, $u(\cdot,t,\phi^*,d,f)$ is a positive, recurrent and bounded solution of (11). □

Finally, we remark that Theorem 3.4 can be applied to establish the global asymptotic recurrence of bounded solutions for n-dimensional monotone and recurrent nonautonomous differential systems with a first integral, which generalizes the results on the almost periodicity for these systems obtained in [2, 11]. For such a system with infinite time delay, one may choose the phase space X to be an appropriate subset of $C((-\infty,0],\mathbb{R}^n)$ and the ordered Banach space (Z_i,Z_i^+) to be $(\mathbb{R},\mathbb{R}^+)$, and define $\mathcal{P}_i(\phi) = \phi_i(0)$ in order to verify the assumption (A4) for the associated skew-product semiflow.

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The Infinite Hierarchy of Elastic Shell Models: Some Recent Results and a Conjecture

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This paper is dedicated to George Sell

Abstract We summarize some recent results of the authors and their collaborators, regarding the derivation of thin elastic shell models (for shells with mid-surface of arbitrary geometry) from the variational theory of 3d nonlinear elasticity. We also formulate a conjecture on the form and validity of infinitely many limiting 2d models, each corresponding to its proper scaling range of the body forces in terms of the shell thickness.

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1 Introduction

Elastic materials exhibit qualitatively different responses to different kinematic boundary conditions or body forces. A sheet of paper may crumple under compressive forces, but it shows a more rigid behavior in a milder regime. A cylinder buckles in presence of axial loads. A clamped convex shell enjoys great resistance to bending and stretching, but if a hole is pierced into it, the whole structure might

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easily collapse. Growing tissues, such as leaves, attain non-flat elastic equilibrium configurations with non-zero stress, even in the absence of any external forces.

Such observations gave rise to many interesting questions in the mathematical theory of elasticity. Its main goal is to explain these apparently different phenomena based on some common mathematical ground. Among others, the variational approach to the nonlinear theory has been very effective in rigorously deriving models pertaining to different scaling regimes of the body forces [9]. The strength of this approach lies in its ability to predict the appropriate model together with the response of the plate without any a priori assumptions other than the general principles of 3d nonlinear elasticity.

The purpose of this paper is to introduce some new results and conjectures on the variational derivation of shell theories. They can be considered as generalizations of the results in [9], justifying a hierarchy of theories for nonlinearly elastic plates. This hierarchy corresponds to the scaling of the elastic energy in terms of thickness h, in the limit as $h \to 0$. Some of the derived models were absent from the physics and engineering literature before.

2 The Set-Up and a Glance at Previously known Results

2.1 Three Dimensional Nonlinear Elasticity and the Limiting Lower Dimensional Theories

The equations for the balance of linear momentum for the deformation $u = u(t,x) \in \mathbb{R}^3$ of the reference configuration $\Omega \subset \mathbb{R}^3$ of an elastic body with constant temperature and density read [1]:

$$\partial_{tt} u - \operatorname{div} DW(\nabla u) = f, \tag{1}$$

where DW is the Piola-Kirchhoff stress tensor, f is the external body force, and the elastic energy density W is assumed to satisfy the following fundamental properties of frame indifference (with respect to the group of proper rotations $SO(3)$), normalization and non-degeneracy:

$$\begin{aligned} \forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \qquad & W(RF) = W(F), \quad W(R) = 0, \\ & W(F) \geq c \cdot \operatorname{dist}^2(F, SO(3)), \end{aligned} \tag{2}$$

with a uniform constant $c > 0$.

The steady state solutions to (1) satisfy the equilibrium equations: $-\operatorname{div} DW(\nabla u) = f$ which, expressed in their weak form, yield the formal Euler-Lagrange equations for the critical points of the total energy functional:

$$J(u) = \int_\Omega W(\nabla u) - \int_\Omega f u, \tag{3}$$

defined for deformation $u : \Omega \longrightarrow \mathbb{R}^3$. We will refer to the term $E(u) = \displaystyle\int_\Omega W(\nabla u)$ as the elastic energy of the deformation u.

As a first step towards understanding the dynamical problem (1) it is natural to study the minimizers of (3), in an appropriate function space. The questions regarding existence and regularity of these minimizers are vastly considered in the literature. However, due to the loss of convexity of W, caused by the frame indifference assumption, these problems cannot be dealt with the usual techniques in the calculus of variations; see [1] for a review of results and open problems.

One advantageous direction of research has been to restrict the attention to domains Ω which are thin in one or two directions, and hence practically reduce the theory to a 2d or 1d problem. Indeed, the derivation of lower dimensional models for thin structures (such as membranes, shells, or beams) has been one of the fundamental questions since the beginning of research in elasticity [20]. The classical approach is to propose a formal asymptotic expansion for the solutions (in other words an *Ansatz*) and derive the corresponding limiting theory by considering the first terms of the 3d equations under this expansion [2]. The more rigorous variational approach of Γ-*convergence* was more recently applied by LeDret and Raoult [13] in this context, and then significantly furthered by Friesecke, James and Müller [9], leading to the derivation of a hierarchy of limiting plate theories. Among other features, it provided a rigorous justification of convergence of minimizers of (3) to minimizers of suitable lower dimensional limit energies.

2.2 Γ-Convergence

Recall [6] that a sequence of functionals $F_n : X \longrightarrow [-\infty, +\infty]$ defined on a metric space X, Γ-converges to the limit functional $F : X \longrightarrow [-\infty, +\infty]$ whenever:

(i) (the Γ-liminf inequality) For any sequence $x_n \to x$ in X, one has $F(x) \leq \liminf_{n\to\infty} F_n(x_n)$.
(ii) (the Γ-limsup inequality) For any $x \in X$, there exits a sequence x_n (called a *recovery sequence*) converging to $x \in X$, such that $\limsup_{n\to\infty} F_n(x_n) \leq F(x)$.

It is straightforward that $\lim_{n\to\infty} F_n(x_n) = F(x)$ for any recovery sequence $x_n \to x$.

When X is only a topological space, the definition of Γ-convergence involves, naturally, systems of neighborhoods rather than sequences. However, when the functionals F_n are equi-coercive and X is a reflexive Banach space equipped with weak topology, one can still use (i) and (ii) above (for weakly converging sequences), as an equivalent version of this definition.

A fundamental consequence of the above definition is the following. If x_n is a sequence of approximate minimizers of F_n in X:

$$\lim_{n\to\infty} \left\{ F_n(x_n) - \inf_X F_n \right\} = 0,$$

and if $x_n \to x$, then x is a minimizer of F. In turn, any recovery sequence associated to a minimizer of F is an approximate minimizing sequence for F_n. The convergence of (a subsequence of) x_n is usually independently established through a compactness argument.

2.3 A Glance at Previously known Results.

Let S be a 2d surface embedded in $\mathbb{R}^3$, which is compact, connected, oriented, of class $\mathcal{C}^{1,1}$ and whose boundary ∂S is the union of finitely many (possibly none) Lipschitz curves. By $\mathbf{n}$ we denote the unit normal vector to S, and $\pi : S^{h_0} \to S$ is the usual orthogonal projection of the tubular neighborhood onto S.

Consider a family $\{S^h\}_{h>0}$ of thin shells of thickness h around S:

$$S^h = \{z = x + t\mathbf{n}(x);\ x \in S,\ -h/2 < t < h/2\}, \qquad 0 < h < h_0,$$

The elastic energy per unit thickness of a deformation $u \in W^{1,2}(S^h, \mathbb{R}^3)$ is given by:

$$E^h(u) = \frac{1}{h}\int_{S^h} W(\nabla u), \tag{4}$$

On above the properties where the stored-energy density $W : \mathbb{R}^{3\times 3} \longrightarrow [0,\infty]$ is assumed to satisfy (2) and to be $\mathcal{C}^2$ regular in some open neighborhood of $SO(3)$.

In presence of applied forces $f^h \in L^2(S^h, \mathbb{R}^3)$, the (scaled) total energy reads:

$$J^h(u) = E^h(u) - \frac{1}{h}\int_{S^h} f^h u. \tag{5}$$

It can be shown that if the forces f^h scale like h^α, then the elastic energy $E^h(u^h)$ at (approximate) minimizers u^h of J^h scale like h^β, where $\beta = \alpha$ if $0 \le \alpha \le 2$ and $\beta = 2\alpha - 2$ if $\alpha > 2$. The main part of the analysis consists therefore of identifying the Γ-limit I_β of the energies $h^{-\beta}E^h(\cdot, S^h)$ as $h \to 0$, for a given scaling $\beta \ge 0$. No a priori assumptions are made on the form of the deformations u^h in this context.

In the case when S is a subset of $\mathbb{R}^2$ (i.e. a plate), such Γ-convergence was first established for $\beta = 0$ [12], and later [8, 9] for all $\beta \ge 2$. This last scaling regime corresponds to a rigid behavior of the elastic material, since the limiting admissible deformations are isometric immersions (if $\beta = 2$) or infinitesimal isometries (if $\beta > 2$) of the mid-plate S. One particular case is $\beta = 4$, where the derived limiting theory turns out to be the von Kármán theory [11]. A totally clamped plate exhibits a very rigid behavior already for $\beta > 0$ [5]. In case $0 < \beta < 5/3$, the Γ-convergence was recently obtained in [4], while the regime $5/3 \le \beta < 2$ remains open and is conjectured to be relevant for crumpling of elastic sheets [27].

Much less is known in the general case when S is a surface of arbitrary geometry. The first result in [13] relates to scaling $\beta = 0$ and models *membrane shells*: the limit I_0 depends only on the stretching and shearing produced by the deformation on S. Another study [7] analyzed the case $\beta = 2$, corresponding to a *flexural shell model* [2], or a geometrically nonlinear purely bending theory, where the only admissible deformations are isometric immersions, that is those preserving the metric on S (see Sect. 2). The energy I_2 depends then on the change of curvature produced by the deformation.

All the above mentioned theories should be put in contrast with a large body of literature, devoted to derivations starting from 3d *linear* elasticity (see [2] and references therein). In the present setting one allows for large deformations, i.e. not necessarily close to a rigid motion. The basic assumption of the linear elasticity is not taken for granted in our context.

Also, let us mention that some other closely related studies have been recently put forward in [18, 19, 21].

3 The Kirchhoff Theory for Shells: $\beta = 2$ and Arbitrary S

The limiting theory for $\beta = 2$ is precisely described in the following result:

Theorem 3.1. *[7] (a)* ***Compactness and the Γ-liminf inequality.*** *Let $u^h \in W^{1,2}(S^h,\mathbb{R}^3)$ be a sequence of deformations such that $E^h(u^h)/h^2$ is uniformly bounded. Then there exists a sequence $c^h \in \mathbb{R}^3$ such that the rescaled deformations:*

$$y^h(x+t\mathbf{n}) = u^h(x+th/h_0\mathbf{n}) - c^h : S^{h_0} \longrightarrow \mathbb{R}^3,$$

converge (up to a subsequence) in $W^{1,2}$ to $y \circ \pi$, where $y \in W^{2,2}(S,\mathbb{R}^3)$ and it satisfies:

$$(\nabla y)^T \nabla y = \mathrm{Id} \qquad a.e.\ in\ S. \tag{6}$$

Moreover:

$$I_2(y) = \int_S \mathcal{Q}_2(x, \Pi(y) - \Pi) \leq \liminf_{h\to 0} \frac{1}{h^2} E^h(u^h).$$

(b) ***The recovery sequence and the Γ-limsup inequality.*** *Given any isometric immersion $y \in W^{2,2}(S,\mathbb{R}^3)$ satisfying (6), there exists a sequence $u^h \in W^{1,2}(S^h,\mathbb{R}^3)$ such that the rescaled deformations $y^h(x+t\mathbf{n}) = u^h(x+th/h_0\mathbf{n})$ converge to $y \circ \pi$ in $W^{1,2}$ and:*

$$I_2(y) \geq \limsup_{h\to 0} \frac{1}{h^2} E^h(u^h).$$

In the definition of the limit functional I_2, the quadratic forms $\mathcal{Q}_2(x,\cdot)$ are defined as follows:

$$\mathcal{Q}_2(x, F_{tan}) = \min\{\mathcal{Q}_3(\tilde{F});\ (\tilde{F} - F)_{tan} = 0\}, \qquad \mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F,F). \tag{7}$$

The form $\mathcal{Q}_3$ is defined for all $F \in \mathbb{R}^{3\times 3}$, while $\mathcal{Q}_2(x,\cdot)$, for a given $x \in S$ is defined on tangential minors F_{tan} of such matrices. Recall that the tangent space to $SO(3)$ at Id is $so(3)$. As a consequence, both forms depend only on the symmetric parts of their arguments and are positive definite on the space of symmetric matrices [8].

The functional $I_2(y)$ measures the total change of curvature (bending) induced by the deformation y of the mid-surface S. In the form of the integrand Π denotes the shape operator on S, while $\Pi(y)$ is the pull back of the shape operator of the surface $y(S)$ under y. For any orthonormal tangent frame $\tau, \eta \in T_xS$ there holds:

$$\eta \cdot \Pi\tau = \eta \cdot \partial_\tau \mathbf{n} \quad \text{and} \quad \eta \cdot \Pi(y)\tau = \eta \cdot \partial_\tau \mathbf{N},$$

where $\mathbf{N} : S \to \mathbb{R}^3$ is the unit normal to $y(S)$: $\mathbf{N}(x) = \partial_\tau y \times \partial_\eta y$.

4 The von-Kármán Theory for Shells: $\beta = 4$ and Arbitrary S

For the range of scalings $\beta > 2$ a rigidity argument [8, 9, 15] shows that the admissible deformations u are only those which are close to a rigid motion R and whose first order term in the expansion of $u - R$ with respect to h is given by RV. The displacement field V is an element of the class $\mathcal{V}_1$ of *infinitesimal isometries* on S [26]. The space $\mathcal{V}_1$ consists of vector fields $V \in W^{2,2}(S,\mathbb{R}^3)$ for whom there exists a matrix field $A \in W^{1,2}(S,\mathbb{R}^{3\times 3})$ so that:

$$\partial_\tau V(x) = A(x)\tau \quad \text{and} \quad A(x)^T = -A(x) \qquad \text{a.e. } x \in S \quad \forall \tau \in T_x S. \tag{8}$$

In other terms, V is a (first order) infinitesimal isometry if the change of metric induced by the deformation $\mathrm{id} + \varepsilon V$ is at most of order ε^2 (Fig. 1).

For $\beta = 4$ the Γ-limit turns out to be the generalization of the von Kármán functional [9] to shells, and it consists of two terms:

$$I_4(V, B_{tan}) = \frac{1}{2}\int_S \mathcal{Q}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right) + \frac{1}{24}\int_S \mathcal{Q}_2\left(x, (\nabla(A\mathbf{n}) - A\Pi)_{tan}\right).$$

The quadratic form $\mathcal{Q}_2$ is defined as in (7) and A is as in (8). The second term above measures bending, that is the first order change in the second fundamental form of S, produced by V. The first term measures stretching, that is the second order change in the metric of S. It involves a symmetric matrix field B_{tan} belonging to the *finite strain space*:

$$\mathcal{B} = \mathrm{cl}_{L^2}\left\{\mathrm{sym}\nabla w;\ w \in W^{1,2}(S,\mathbb{R}^3)\right\}.$$

The space $\mathcal{B}$ emerges as well in the context of linear elasticity and ill-inhibited surfaces [10, 25].

Theorem 4.1. *[15] (a) Let $u^h \in W^{1,2}(S^h,\mathbb{R}^3)$ be a sequence of deformations whose scaled energies $E^h(u^h)/h^4$ are uniformly bounded. Then there exist a sequence $Q^h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that for the normalized rescaled deformations:*

$$y^h(x + t\mathbf{n}) = Q^h u^h(x + h/h_0 t\mathbf{n}) - c^h : S^{h_0} \longrightarrow \mathbb{R}^3 \tag{9}$$

the following holds.

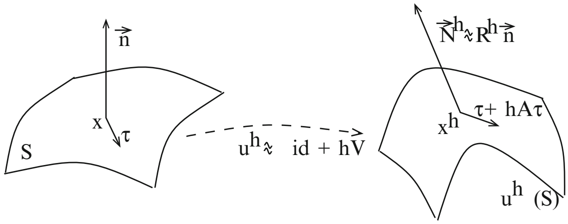

Fig. 1 The mid-surface S and its deformation

(i) y^h *converge in* $W^{1,2}(S^{h_0})$ *to* π.

(ii) *The scaled average displacements:*

$$V^h(x) = \frac{1}{h}\int_{-h_0/2}^{h_0/2} y^h(x+t\mathbf{n}) - x \,\mathrm{d}t \tag{10}$$

converge (up to a subsequence) in $W^{1,2}(S)$ *to some* $V \in \mathcal{V}_1$.

(iii) *The scaled strains* $\frac{1}{h}\,\mathrm{sym}\nabla V^h$ *converge weakly in* L^2 *to a symmetric matrix field* $B_{tan} \in \mathcal{B}$.

(iv) $I_4(V, B_{tan}) \leq \liminf_{h\to 0} 1/h^4 E^h(u^h)$.

(b) *For every* $V \in \mathcal{V}_1$ *and* $B_{tan} \in \mathcal{B}$ *there exists a sequence of deformations* $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ *such that:*

(i) *the rescaled deformations* $y^h(x+t\mathbf{n}) = u^h(x+th/h_0\mathbf{n})$ *converge in* $W^{1,2}(S^{h_0})$ *to* π.

(ii) *the scaled average displacements* V^h *given above converge in* $W^{1,2}(S)$ *to* V.

(iii) *the scaled linearized strains* $\frac{1}{h}\,\mathrm{sym}\nabla V^h$ *converge weakly in* L^2 *to* B_{tan}.

(iv) $I_4(V, B_{tan}) \geq \limsup_{h\to 0} 1/h^4 E^h(u^h)$.

The special case of this theorem for plates, that is when $S \subset \mathbb{R}^2$, was already proved in [9]. It can be shown that for a flat surface, the infinitesimal isometries coincide essentially with the out-of-plane displacements. Also, the space $\mathcal{B}$ becomes then the set of all linearized strains associated with the in-plane displacements in S, so the functional I_4 can be written directly in terms of an out-of-plane and an in-plane displacement. The Euler-Lagrange equations derived from this limit functional lead to the von-Kármán equations [11]. An extension of Theorem 4.1 for thin shells with variable thickness has been formulated in [16]. Convergence of equilibria in the same context, following the case of plates in [23], has been studied in [14].

5 The Linear Theory for Shells: $\beta > 4$ and Arbitrary S; $\beta = 4$ and Approximately Robust S

It was shown in [15] that for a certain class of surfaces, referred to as *approximately robust surfaces*, the limiting theory for $\beta = 4$ reduces to the purely linear bending functional:

$$I_{lin}(V) = \frac{1}{24}\int_S \mathcal{Q}_2\left(x, (\nabla(A\mathbf{n}) - A\Pi)_{tan}\right) \,\mathrm{d}x \qquad \forall V \in \mathcal{V}_1, \tag{11}$$

This class of surfaces is given by the property that any first order infinitesimal isometry $V \in \mathcal{V}_1$ can be modified to be arbitrarily close to a $W^{1,2}$ second order isometry:

$$\tilde{\mathcal{V}}_2 = \{V \in \mathcal{V}_1;\ (A^2)_{tan} \in \mathcal{B}\} = \mathcal{V}_1.$$

Convex surfaces, surfaces of revolution and developable surfaces belong to this class [15].

In [15], it was also proved that the Γ-limit of E^h/h^β for the scaling regime $\beta > 4$ is also given by the functional (11). This corresponds to the linear pure bending theory derived in [2] from linearized elasticity. The important qualitative difference between this theory and the limiting theory for $\beta = 4$ and an approximately robust surface is in the type of convergences one establishes for a sequence u^h satisfying $E^h(u^h) \leq Ch^\beta$. Indeed, if $\beta > 4$, the best one can prove is the convergence in $W^{1,2}$, up to a subsequence, of the rescaled displacement fields:

$$V^h(x) = \frac{1}{h^{\beta/2-1}} \int_{-h_0/2}^{h_0/2} y^h(x+t\mathbf{n}) - x \,\mathrm{d}t$$

to an element $V \in \mathcal{V}_1$. Note the finer rescaling parameter $h^{\beta/2-1}$ with respect to (10).

6 Intermediate Theories for Plates and Convex Shells: $\beta \in (\mathbf{2}, \mathbf{4})$

In paper [17] we focused on the range of scalings $2 < \beta < 4$, looking hence for an intermediate theory between those corresponding to $\beta = 2$ and $\beta \geq 4$. On one hand, modulo a rigid motion, the deformation of the mid-surface must look like $\mathrm{id} + \varepsilon V$, up to its first order of expansion. On the other hand, the closer β is to 2, the closer the mid-surface deformation must be to an exact isometry of S. To overcome this apparent disparity between first order infinitesimal isometries and exact isometries in this context, one is immediately drawn to consider higher order infinitesimal isometries which lay somewhat between these two categories. This will be the subject of discussion in Sect. 7. Another angle of approach, which turns out to be useful in special cases, is to study conditions under which, given $V \in \mathcal{V}_1$, one can construct an exact isometry of the form $\mathrm{id} + \varepsilon V + \varepsilon^2 w_\varepsilon$, with equibounded corrections w_ε. This is what we refer to as a *matching property*.

If $S \subset \mathbb{R}^2$ represents a plate, the above issues have been answered in [9]. In this case:

(i) The limit displacement V must necessarily belong to the space of second order infinitesimal isometries: $\tilde{\mathcal{V}}_2 = \{V \in \mathcal{V}_1;\ (A^2)_{tan} \in \mathcal{B}\}$, where the matrix field A is as in (8).
(ii) Any Lipschitz second order isometry $V \in \tilde{\mathcal{V}}_2$ satisfies the matching property.

Combining these two facts with the density of Lipschitz second order infinitesimal isometries in $\tilde{\mathcal{V}}_2$ for a plate [22], one concludes through the Γ-convergence arguments that the limiting plate theory is given by the functional (11) over $\tilde{\mathcal{V}}_2$. Note that, for a plate, $V \in \tilde{\mathcal{V}}_2$ means that there exists an in-plane displacement $w \in W^{1,2}(S, \mathbb{R}^2)$ such that the change of metric due to $\mathrm{id} + \varepsilon V + \varepsilon^2 w$ is of order ε^3. Also, in this case, an equivalent analytic characterization for $V = (V^1, V^2, V^3) \in \tilde{\mathcal{V}}_2$ is given by $(V^1, V^2) = (-\omega y, \omega x) + (b_1, b_2)$ and: $\det \nabla^2 V^3 = 0$.

Towards analyzing more general surfaces S, we derived a matching property and the corresponding density of isometries, for elliptic surfaces. We say that S is elliptic if its shape operator Π is strictly positive (or strictly negative) definite up to the boundary:

$$\forall x \in \bar{S} \quad \forall \tau \in T_x S \qquad \frac{1}{C}|\tau|^2 \leq \Big(\Pi(x)\tau\Big) \cdot \tau \leq C|\tau|^2. \tag{12}$$

The novelty here is the fact that for an elliptic surface, all sufficiently smooth infinitesimal isometries satisfy the matching property:

Theorem 6.1 ([17]). *Let S be elliptic as in (12), homeomorphic to a disk and let for some $\alpha > 0$, S and ∂S be of class $\mathcal{C}^{3,\alpha}$. Given $V \in \mathcal{V}_1 \cap \mathcal{C}^{2,\alpha}(\bar{S})$, there exists a sequence $w_\varepsilon : \bar{S} \longrightarrow \mathbb{R}^3$, equibounded in $\mathcal{C}^{2,\alpha}(\bar{S})$, and such that for all small $\varepsilon > 0$ the map $u_\varepsilon = \mathrm{id} + \varepsilon V + \varepsilon^2 w_\varepsilon$ is an (exact) isometry.*

We apply this result to construct the recovery sequence in the Γ-limsup inequality. Clearly, Theorem 6.1 is not sufficient for this purpose as the elements of $\mathcal{V}_1$ are only $W^{2,2}$ regular. In most Γ-convergence results, a key step is to prove density of suitably regular mappings in the space of mappings admissible for the limit problem. Results in this direction, for Sobolev spaces of isometries and infinitesimal isometries, have been shown and applied in the context of derivation of plate theories. The interested reader can refer to [22, 24] for statements of these density theorems and their applications in [3, 9].

In general, even though $\mathcal{V}_1$ is a linear space, and assuming S to be $\mathcal{C}^\infty$, the usual mollification techniques do not guarantee that elements of $\mathcal{V}_1$ can be approximated by smooth infinitesimal isometries. An interesting example, discovered by Cohn-Vossen [26], is a closed smooth surface of non-negative curvature for which $\mathcal{C}^\infty \cap \mathcal{V}_1$ consists only of trivial fields $V : S \longrightarrow \mathbb{R}^3$ with constant gradient, whereas $\mathcal{C}^2 \cap \mathcal{V}_1$ contains non-trivial infinitesimal isometries. Therefore $\mathcal{C}^\infty \cap \mathcal{V}_1$ is not dense in $\mathcal{V}_1$ for this surface. We however have:

Theorem 6.2 ([17]). *Assume that S is elliptic, homeomorphic to a disk, of class $\mathcal{C}^{m+2,\alpha}$ up to the boundary and that ∂S is $\mathcal{C}^{m+1,\alpha}$, for some $\alpha \in (0,1)$ and an integer $m > 0$. Then, for every $V \in \mathcal{V}_1$ there exists a sequence $V_n \in \mathcal{V}_1 \cap \mathcal{C}^{m,\alpha}(\bar{S}, \mathbb{R}^3)$ such that:*

$$\lim_{n\to\infty} \|V_n - V\|_{W^{2,2}(S)} = 0.$$

Ultimately, and as a consequence of Theorems 6.1 and 6.2, the main result of [17] states that for elliptic surfaces of sufficient regularity, the Γ-limit of the nonlinear elastic energy (4) for the scaling regime $2 < \beta < 4$ (and hence for all $\beta > 2$) is still given by the functional (11) over the linear space $\mathcal{V}_1$:

Theorem 6.3 ([17]). *Let S be as in Theorem 6.1 and let $2 < \beta < 4$.*

(a) Assume that for a sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ their scaled energies $E^h(u^h)/h^\beta$ are uniformly bounded. Then there exist a sequence $Q^h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that for the normalized rescaled deformations in (9) the following holds.

(i) y^h *converge in* $W^{1,2}(S^{h_0})$ *to* π.

(ii) *The scaled average displacements:*

$$V^h(x) = \frac{1}{h^{\beta/2-1}} \int_{-h_0/2}^{h_0/2} y^h(x+t\mathbf{n}) - x \, \mathrm{d}t$$

converge (up to a subsequence) in $W^{1,2}(S)$ *to some* $V \in \mathcal{V}_1$.

(iii) $I_{lin}(V) \leq \liminf_{h\to 0} 1/h^\beta E^h(u^h)$.

(b) *For every* $V \in \mathcal{V}_1$ *there exists a sequence of deformations* $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ *such that:*

(i) *the rescaled deformations* $y^h(x+t\mathbf{n}) = u^h(x+th/h_0\mathbf{n})$ *converge in* $W^{1,2}(S^{h_0})$ *to* π.

(ii) *the scaled average displacements* V^h *given above converge in* $W^{1,2}(S)$ *to* V.

(iii) $I_{lin}(V) \geq \limsup_{h\to 0} 1/h^\beta E^h(u^h)$.

One can actually prove that for $2 < \beta < 4$ and surface S of arbitrary geometry, the part (a) or Theorem 6.3 remains valid, and moreover $1/h^{\beta/2-1}$ sym ∇V^h converge (up to a subsequence) in $L^2(S)$ to $1/2(A^2)_{tan}$, where A is related to V by (8). The novelty with respect to the equivalent result for $\beta = 4$ in the first part of Theorem 4.1 is the constraint $V \in \tilde{\mathcal{V}}_2$. If S is an elliptic surface of sufficient regularity, the set $\mathcal{B}$ coincides with the whole space $L^2_{sym}(S, \mathbb{R}^{2\times 2})$ [15], hence the constraint is automatically satisfied for all $V \in \mathcal{V}_1$. In the general case where S is an arbitrary surface, a characterization of this constraint and the exact form of $\mathcal{B}$ may be complicated.

The case of the scaling range $2 < \beta < 4$ is still open for general shells. The following section is dedicated to the presentation of a conjecture on this problem, stating that other constraints, similar to the inclusion $V \in \tilde{\mathcal{V}}_2$, should be present for values of β closer to 2. Heuristically, the closer β is to 2, we expect V to be an infinitesimal isometry of higher order.

7 A Conjecture on the Infinite Hierarchy of Shell Models

If the deformations u^h of S^h satisfy a simplified version of Kirchhoff-Love assumption:

$$u^h(x+t\mathbf{n}) = u^h(x) + t\mathbf{N}^h(x),$$

vector $\mathbf{N}^h$ being the unit normal to the image surface $u^h(S)$, then formal calculations show that:

$$E^h(u^h) \approx \int_S |\delta g_S|^2 + h^2 \int_S |\delta \Pi_S|^2. \tag{13}$$

Here by δg_S and $\delta \Pi_S$ we denote, respectively, the change in the metric (first fundamental form) and in the shape operator (second fundamental form), between the surface $u^h(S)$ and the reference mid-surface S. For a more rigorous treatment of this observation see e.g. [4]. The two terms in (13) correspond to the stretching and bending energies, and the factor h^2 in the bending term points to the fact that a shell undergoes bending more easily than stretching. For a plate, the latter energy is known in the literature as the Föppl-von Kármán functional.

Another useful observation is that for minimizers u^h, the energy should be distributed equi-partedly between the stretching and bending terms. When $E^h(u^h) \approx h^2$, then equating the order of both terms in (13) we obtain in the limit of $h \to 0$:

$$\int_S |\delta g_S|^2 \approx 0 \quad \text{and} \quad \frac{1}{h^2} E^h(u^h) \approx \int_S |\delta \Pi_S|^2.$$

This indeed corresponds to the Kirchhoff model, as in Theorem 3.1 [7], where the limiting energy I_2 is given by the bending term (measuring the change in the second fundamental form) under the constraint of zero stretching: $\delta g_S = 0$. As we have seen, the limiting deformation u must be an isometry $(\nabla u)^T \nabla u = \mathrm{Id}$ and hence preserve the metric.

To discuss higher energy scalings, assume that:

$$E^h(u^h) \approx h^\beta, \qquad \beta > 2. \tag{14}$$

Then, as mentioned before, by the rigidity estimate [8], the restrictions of u^h to S have, modulo appropriate rigid motions, the following expansions:

$$u^h_{|S} = \mathrm{id} + \sum_{i=1}^{\infty} \varepsilon^i w_i.$$

Thus, $\delta\Pi$ is of the order ε and after equating the order of the bending term in (13) by (14), we arrive at: $h^2\varepsilon^2 = h^\beta$, that is:

$$\varepsilon = h^{\beta/2-1}. \tag{15}$$

On the other hand, the stretching term has the form: $\delta g_S = (\nabla u^h)\nabla u^h - \mathrm{Id} = \sum_{i=1}^{\infty} \varepsilon^i A_i$, with:

$$A_i = \sum_{j+k=i} \mathrm{sym}\Big((\nabla w_j)^T \nabla w_k\Big),$$

indicating the i-th order change of metric. Taking into account (14), this yields: $\varepsilon^{2i}\int_S |A_i|^2 \approx h^\beta$, and so in view of (15): $\|A_i\|^2_{L^2} \approx h^{\beta - i(\beta-2)}$. A first consequence is that A_1 must vanish in the limit as $h \to 0$, that is the limiting deformation is a first order infinitesimal isometry. For $i > 1$, we observe that $\|A_i\|^2_{L^2} \approx h^{(i-1)(\beta_i - \beta)}$, where:

$$\beta_i = 2 + \frac{2}{i-1}.$$

We conclude that if $\beta < \beta_N$, then $\|A_i\|_{L^2} \approx 0$ for $i \leq N$, and if $\beta = \beta_N$, then $\|A_N\|_{L^2} = O(1)$. The study of the asymptotic behavior of the energy $1/h^\beta E^h$ leads us hence to the following conjecture.

Conjecture 7.1. The limiting theory of thin shells with midsurface S, under the elastic energy scaling $\beta > 2$ as in (14) is given by the following functional I_β below, defined on the space $\mathcal{V}_N$ of N-th order infinitesimal isometries, where:

$$\beta \in [\beta_{N+1}, \beta_N).$$

The space $\mathcal{V}_N$ is identified with the space of N-tuples $(V_1, \ldots, V_N)$ of displacements $V_i : S \longrightarrow \mathbb{R}^3$ (having appropriate regularity), such that the deformations of S:

$$u_\varepsilon = \mathrm{id} + \sum_{i=1}^{N} \varepsilon^i V_i$$

preserve its metric up to order ε^N. We have:

(i) When $\beta = \beta_{N+1}$ then $I_\beta = \int_S \mathcal{Q}_2(x, \delta_{N+1} g_S) + \int_S \mathcal{Q}_2(x, \delta_1 \Pi_S)$, where $\delta_{N+1} g_S$ is the change of metric on S of the order ε^{N+1}, generated by the family of deformations u_ε and $\delta_1 \Pi_S$ is the corresponding first order change in the second fundamental form.
(ii) When $\beta \in (\beta_{N+1}, \beta_N)$ then $I_\beta = \int_S \mathcal{Q}_2(x, \delta_1 \Pi_S)$.
(iii) The constraint of N-th order infinitesimal isometry $\mathcal{V}_N$ may be relaxed to that of $\mathcal{V}_M$, $M < N$, if S has the following matching property. For every $(V_1, \ldots V_M) \in \mathcal{V}_M$ there exist sequences of corrections $V_{M+1}^\varepsilon, \ldots V_N^\varepsilon$, uniformly bounded in ε, such that:

$$\tilde{u}_\varepsilon = \mathrm{id} + \sum_{i=1}^{M} \varepsilon^i V_i + \sum_{i=M+1}^{N} \varepsilon^i V_i^\varepsilon$$

preserve the metric on S up to order ε^N.

This conjecture is consistent with the so far established results in [15] for $N = 1$ (i.e. $\beta \geq \beta_2 = 4$) and arbitrary surfaces. Note that in the case of approximately robust surfaces, any element of $\mathcal{V}_1$ can be matched with an element of $\mathcal{V}_2$, and hence the term $\int_S |\delta_2 g_S|^2$ in the limit energy can be dropped. The second order infinitesimal isometry constraint $\mathcal{V}_2$ is established for all surfaces when $2 < \beta < 4$. In the particular case of plates, any second order isometry in a dense subset of $\mathcal{V}_2$, can be matched with an exact isometry [9]. As a consequence, the theory reduces to minimizing the bending energy under the second order infinitesimal isometry constraint. A similar matching property for elliptic surfaces, this time for elements of $\mathcal{V}_1$, is given in [17] (see Theorems 6.1 and 6.2). As a consequence, for elliptic surfaces, the limiting theory for the whole range $\beta > 2$ reduces to the linear bending.

The case $2 < \beta < 4$ remains open for all other types of surfaces. The main difficulty lies in obtaining the appropriate convergences and the limiting nonlinear constraints:

$$\sum_{j+k=i} \mathrm{sym}\Big((\nabla V_j)^T \nabla V_k\Big) = 0, \qquad 1 \leq i \leq N,$$

for the elements of $\mathcal{V}_N$ when $\beta < \beta_N$. The above nonlinearity, implies a rapid loss of Sobolev regularity of V_i as i increases. Moreover, applying methods of [17] to surfaces changing type leads in this context to working with mixed-type PDEs.

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Traveling Wavefronts in Lattice Differential Equations with Time Delay and Global Interaction

Shiwang Ma and Xingfu Zou

Abstract In this paper, we study the existence of traveling wave solutions in lattice differential equations with time delay and global interaction

$$u_n'(t) = D \sum_{i \in Z^q \setminus \{0\}} J(i)[u_{n-i}(t) - u_n(t)] + F\left(u_n(t), \sum_{i \in Z^q} K(i) \int_{-r}^{0} \mathrm{d}\eta(\theta) g(u_{n-i}(t+\theta))\right).$$

Following an idea in [10], we are able to relate the existence of traveling wavefront to the existence of heteroclinic connecting orbits of the corresponding ordinary delay differential equations

$$u'(t) = F\left(u(t), \int_{-r}^{0} \mathrm{d}\eta(\theta) g(u(t+\theta))\right).$$

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1 Introduction

In a recent work [10], Faria et al. considered the existence of traveling wavefront for the following general class of delayed reaction–diffusion systems with nonlocal interaction:

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + F\left(u(x,t), \int_{-r}^{0}\int_{\Omega} \mathrm{d}\eta(\theta)\mathrm{d}\mu(y) g(u(x+y,t+\theta))\right), \quad (1)$$

where $x \in R^m$ is the spatial variable, $t \geq 0$ is the time variable, $u(x,t) \in R^n$ is the unknown vector function, and $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i > 0$, $i = 1, 2, \ldots, n$, $\Delta = \sum_{i=1}^{m} \partial^2/\partial x_i^2$ is the Laplacian operator. They treated the wave profile equation for (1) as a perturbation of the following corresponding ordinary delay differential equation

$$u'(t) = F\left(u(t), \int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_{\Omega} g(u(t+\theta))\right), \quad (2)$$

where $\mu_\Omega = \int_\Omega d\mu$. Then by choosing some appropriate Banach space and applying the perturbation theory to the associated Fredholm operator with some careful estimation of the nonlinear perturbation, the authors were able to relate the existence of traveling wave solution of (1) to the existence of heteroclinic connecting orbits of (2).

In this paper, we apply the novel approach used in [10] to tackle the existence of traveling wavefront for a very general class of lattice differential equations with time delay and global interaction:

$$\begin{aligned} u_n'(t) = D &\sum_{i \in Z^q \setminus \{0\}} J(i)[u_{n-i}(t) - u_n(t)] \\ &+ F\left(u_n(t), \sum_{i \in Z^q} K(i) \int_{-r}^{0} \mathrm{d}\eta(\theta) g(u_{n-i}(t+\theta))\right), \end{aligned} \quad (3)$$

where $n \in Z^q$, q is a positive integer, $t \geq 0$, $u_n(t) \in R^N$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ with $d_j \geq 0, j = 1, \ldots, N$, $r \geq 0$ and $\eta : [-r, 0] \to R^{N \times N}$ is of bounded variation, $F : R^N \times R^N \to R^N$ and $g : R^N \to R^N$ are C^k-smooth functions, $k \geq 2$. Here, the first term in (3) accounts for diffusion to point n in the lattice from all other points, while the second term explains global nonlinear interactions. System (3) includes, as special cases, many model systems arising from various fields among which is population biology where the mobility of the immature individuals is responsible for the nonlocality of the interaction term (see, e.g., [22] and the references therein). In such a context, the choice of spatially discrete domain corresponds to a patch environment in which the species lives. Due to the biological background, traveling wave solutions to such equations are an important type of solutions since they explain spatial spread/invasion of the species within the lattice (patch) environment.

In recent years, this topic has attracted the attention of the mathematical community and has resulted in many research papers; see, e.g., [5, 6, 8, 11–15, 22, 24] and the reference therein.

We point out that as far as traveling waves are concerned, a system of lattice differential equations may demonstrate essentially different behavior from that of its continuous version (reaction–diffusion system). For example, pinning phenomenon may occur in a lattice differential system, while this phenomenon would be impossible in its spatially continuous version (a reaction–diffusion equation); see, e.g., [11, 15]. Another example is that the direction of the waves play a role in the existence of traveling wavefront for a system on a lattice with a higher dimension, but in the case of continuous reaction–diffusion equation with a higher spatial dimension, the direction has no such impact; see, e.g., [3, 16, 25]. Therefore, one can not expect that a method that works for (2) would automatically work for (1). This motivates us to see if the ideas used in [10] for (1) could be applied to (3) for the existence of traveling wavefront. It turns out that after some nontrivial and careful explorations on properties of some operators resulted from the wave equation for (1) and the associated ordinary functional differential equation

$$u'(t) = F\left(u(t), \int_{-r}^{0} d\eta(\theta) g(u(t+\theta))\right), \tag{4}$$

we can also establish a similar result to that in [10], that is, relating the existence of traveling wavefront of (3) to the existence of heteroclinic connecting orbits of (4).

To proceed further, and also from the practical background of (3), we assume throughout this paper that the kernel functions J and K satisfies

$$\sum_{i\in Z^q\setminus\{0\}} J(i) = 1, \qquad \sum_{i\in Z^q\setminus\{0\}} |J(i)|\cdot|i| < +\infty,$$

and

$$\sum_{i\in Z^q} K(i) = 1, \qquad \sum_{i\in Z^q} |K(i)|\cdot|i| < +\infty,$$

where $|i| = \sum_{j=1}^{q} |i_j|$ for $i = (i_1, \ldots, i_q) \in Z^q$.

Let $F_u(u,v)$ and $F_v(u,v)$ denote the partial derivatives of F with respect to the variables $u \in R^N$ and $v \in R^N$, respectively, and let $g_u(u)$ be the derivative of g with respect to the variable $u \in R^N$. Suppose that (4) has two equilibria E_j, $j = 1,2$ (i.e., $F(E_i, E_i) = 0$, $i = 1,2$), and let

$$A_j = F_u\left(E_j, \int_{-r}^{0} d\eta(\theta) g(E_j)\right), \quad B_j = F_v\left(E_j, \int_{-r}^{0} d\eta(\theta) g(E_j)\right).$$

For a complex number λ, denote

$$\Lambda_j(\lambda) = \det\left[\lambda I - A_j - B_j \int_{-r}^{0} d\eta(\theta) g_u(E_j) e^{\lambda\theta}\right].$$

We assume that the following hypotheses hold:

(H1) E_1 is hyperbolic and the unstable manifold at the equilibrium E_1 is M-dimensional with $M \geq 1$. In other words, $\Lambda_1(i\beta) \neq 0$ for all $\beta \in R$ and $\Lambda_1(\lambda) = 0$ has exactly M roots with positive real parts, where the multiplicities are taken into account.

(H2) All eigenvalues corresponding to the equilibrium E_2 have negative real parts, that is, $\sup\{\Re\lambda : \Lambda_2(\lambda) = 0\} < 0$.

(H3) Equation (2) has a heteroclinic solution $u_* : R \to R^N$ from E_1 to E_2. Namely, (2) has a solution $u_*(t)$ defined for all $t \in R$ such that

$$u_*(-\infty) := \lim_{t\to-\infty} u_*(t) = E_1, \quad u_*(+\infty) := \lim_{t\to+\infty} u_*(t) = E_2.$$

As usual, a traveling wave solution of (3) is a solution of the form $u_n(t) = U(\nu \cdot n + ct)$, where $U(\cdot)$ is called the profile of the wave and c is the wave speed. If U satisfy

$$U(-\infty) = E_1, \quad U(\infty) = E_2, \tag{5}$$

then the traveling wave solution is called a wavefront.

We can now formulate our main result as follows, which states that the existence of traveling wave solutions with large wave speeds for(1) is related to the existence of heteroclinic orbit of (2) connecting the two equilibria.

Theorem 1.1. *Assume that (H1), (H2), and (H3) hold. Then, there exists* $c^* > 0$ *such that:*

(i) For each fixed unit vector $\nu \in R^q$ *and* $c > c^*$, *(1) has a traveling wavefront* $u_n(t) = U(\nu \cdot n + ct)$ *connecting* E_1 *to* E_2 *(i.e., (5) holds).*

(ii) If restricted to a small neighborhood of the heteroclinic solution u_* *in the space* $BC(R, R^N)$ *of bounded continuous functions equipped with the sup-norm, then for each* $c > c^*$ *and* $\nu \in R^q$, *the set of all traveling wave solutions connecting* E_1 *to* E_2 *in the neighborhood forms a* M*-dimensional manifold* $\mathsf{M}_\nu(c)$.

(iii) $\mathsf{M}_\nu(c)$ *is a* C^{k-1}*-smooth manifold which is also* C^{k-1}*-smooth with respect to c. More precisely, there is a* C^{k-1}*-function* $h : O \times (c^*, +\infty) \to C(\mathrm{R}, \mathrm{R}^N)$, *where* O *is an open set in* R^N, *such that* $\mathsf{M}_\nu(c)$ *has the form*

$$\mathsf{M}_\nu(c) = \{\psi : \ \psi = h(z,c), \ z \in O\}.$$

Let $\nu \cdot n + ct = s \in R$ and $u_n(t) = U(\nu \cdot n + ct)$. Then, by straightforward substitution, one finds that the profile function $U(s)$ satisfies the following associated wave equation:

$$\begin{aligned} cU'(s) = {} & D \sum_{i\neq 0} J(i)[U(s - \nu \cdot i) - U(s)] \\ & + F\left(U(s), \sum_i K(i) \int_{-r}^{0} \mathrm{d}\eta(\theta) g(U(s - \nu \cdot i + c\theta))\right). \end{aligned} \tag{6}$$

Let $V(s) = U(cs)$ and $\varepsilon = 1/c$; then (3) leads to

$$V'(s) = D\sum_{i\neq 0} J(i)[V(s-\varepsilon\nu\cdot i) - V(s)] + F\left(V(s), \sum_i K(i)\int_{-r}^{0} \mathrm{d}\eta(\theta) g(V(s-\varepsilon\nu\cdot i+\theta))\right). \tag{7}$$

Thus, existence of traveling wavefront solutions to (3) is equivalent to existence of solutions to (6) or (7) with the asymptotically boundary conditions (5). In Sect. 2, we will further transform (7) into some operational equation, and in Sect. 3, we will explore the properties of the operators in the equations obtained in Sect. 2. After the preparation in Sects. 2 and 3, we give the proof of Theorem 1.1 in Sect. 4. Section 5 is devoted to applications of the main theorem to some cases where the heteroclinic orbits of the corresponding ODE (4) can be guaranteed by the connecting orbit theorem for monotone dynamical systems.

2 Operational Equations for Profile of Traveling Waves

We denote by $C = C(R, R^N)$ the space of continuous and bounded functions from R to R^N equipped with the standard sup-norm: $\|\psi\|_C = \sup\{|\psi(t)| : t \in R\}$, where $|\cdot|$ is the Euclid norm in R^N.

Using the idea in [10], we relate (4) to an equivalent operational equation in a suitable Banach space. For this purpose, we further transform (4) by introducing the variable $w(s) = V(s) - u_*(s)$ for $s \in R$. Then we obtain the equation for w as follows:

$$\begin{aligned}
w'(s) &= V'(s) - u_*'(s)\\
&= D\sum_{i\neq 0} J(i)[V(s-\varepsilon\nu\cdot i) - V(s)]\\
&\quad + F(V(s), \sum_i K(i)\int_{-r}^{0} \mathrm{d}\eta(\theta) g(V(s-\varepsilon\nu\cdot i+\theta)))\\
&\quad - F(u_*(s), \int_{-r}^{0} \mathrm{d}\eta(\theta) g(u_*(s+\theta)))\\
&= D\sum_{i\neq 0} J(i)[[w+u_*](s-\varepsilon\nu\cdot i) - [w+u_*](s)]\\
&\quad + F([w+u_*](s), \sum_i K(i)\int_{-r}^{0} \mathrm{d}\eta(\theta) g([w+u_*](s-\varepsilon\nu\cdot i+\theta)))\\
&\quad - F(u_*(s), \int_{-r}^{0} \mathrm{d}\eta(\theta) g(u_*(s+\theta)))\\
&= L_0 w(s) + J_\varepsilon w(s) + G(\varepsilon, s, w) + J_\varepsilon u_*(s),
\end{aligned} \tag{8}$$

where $[w+u_*](s)=w(s)+u_*(s)$ for $s\in R$, and the linear operators $L_0:C\to C$ and $J_\varepsilon:C\to C$ are defined by

$$L_0\psi(s)=A(s)\psi(s)+B(s)\int_{-r}^{0}\mathrm{d}\eta(\theta)g_u(u_*(s+\theta))\psi(s+\theta), \tag{9}$$

with

$$A(s)=F_u(u_*(s),\int_{-r}^{0}\mathrm{d}\eta(\theta)g(u_*(s+\theta))), \tag{10}$$

$$B(s)=F_v(u_*(s),\int_{-r}^{0}\mathrm{d}\eta(\theta)g(u_*(s+\theta))), \tag{11}$$

and

$$J_\varepsilon\psi(s)=D\sum_{i\neq 0}J(i)[\psi(s-\varepsilon\nu\cdot i)-\psi(s)], \tag{12}$$

respectively, and the nonlinear operator $G(\varepsilon,\cdot,\cdot):C\to C$ is defined by

$$G(\varepsilon,s,\psi)=F(\psi(s)+u_*(s),\sum_i K(i)\int_{-r}^{0}\mathrm{d}\eta(\theta)g([\psi+u_*](s-\varepsilon\nu\cdot i+\theta))) - F(u_*(s),\int_{-r}^{0}\mathrm{d}\eta(\theta)g(u_*(s+\theta)))-L_0\psi(s). \tag{13}$$

Next we further transform (8) into an integral equation as follows. We first rewrite (8) as

$$w'(s)+w(s)=w(s)+L_0w(s)+J_\varepsilon w(s)+G(\varepsilon,s,w)+J_\varepsilon u_*(s). \tag{14}$$

Clearly $w:R\to R^N$ is a bounded solution of (14) if and only if it is a bounded solution of the integral equation

$$w(s)=\int_{-\infty}^{s}\mathrm{e}^{-(s-t)}\{[Id+L_0]w(t)+J_\varepsilon w(t)+G(\varepsilon,t,w)+J_\varepsilon u_*(t)\}\mathrm{d}t.$$

Therefore, w is a bounded solution of (14) if and only if it solves the operational equation

$$\mathcal{L}w=\mathcal{G}(\cdot,w,\varepsilon), \tag{15}$$

where the linear operator $\mathcal{L}:C\to C$ is defined by

$$\mathcal{L}w(s)=w(s)-\int_{-\infty}^{s}\mathrm{e}^{-(s-t)}[Id+L_0]w(t)\mathrm{d}t, \tag{16}$$

and the nonlinear operator $\mathcal{G}(\cdot,\cdot,\varepsilon):C\to C$ is defined by

$$\mathcal{G}(s,w,\varepsilon)=\int_{-\infty}^{s}\mathrm{e}^{-(s-t)}[J_\varepsilon w(t)+G(\varepsilon,t,w)+J_\varepsilon u_*(t)]\mathrm{d}t. \tag{17}$$

3 Properties of the Operators $\mathcal{L}$ and $\mathcal{G}$

Let $C^1(R,R^N) = \{\psi \in C: \ \psi' \in C\}$ be the Banach space equipped with the standard norm $\|\psi\|_{C^1} = \|\psi\|_C + \|\psi'\|_C$. Let $C_0 = \{\psi \in C: \ \lim_{t\to\pm\infty} \psi(t) = 0\}$ and $C_0^1 = \{\psi \in C_0: \ \psi' \in C_0\}$ equipped with the same norms as C and C^1, respectively.

If restricted to the subspace C_0, we then have $\mathcal{L}: C_0 \to C_0$. Let $\mathcal{N}(\mathcal{L})$ and $\mathcal{R}(\mathcal{L})$ denote the kernel and the range of the operator $\mathcal{L}$, and then we have the following result.

Proposition 3.1. $\dim\mathcal{N}(\mathcal{L}) = M$ *and* $\mathcal{R}(\mathcal{L}) = C_0$.

For a proof of the Proposition 3.1, we refer the reader to the recent paper due to Faria et al. [10].

In order to complete the proof of Theorem 1.1, we need further information about the behavior of the nonlinear operator $\mathcal{G}(\cdot,w,\varepsilon)$ when $\varepsilon \geq 0$ is small and w is near the origin. To simplify the presentation, for any $\varepsilon \geq 0$, we let $R(\varepsilon,\cdot): C \to C$ be defined by

$$R(\varepsilon,\psi)(s) = \sum_i K(i)\int_{-r}^{0} \mathrm{d}\eta(\theta) g(\psi(s-\varepsilon\nu\cdot i+\theta)), \tag{18}$$

and let the linear operator $L_\varepsilon: C_0 \to C$ be defined by

$$L_\varepsilon\psi(s) = A^\varepsilon(s)\psi(s) + B^\varepsilon(s)\sum_i K(i)\int_{-r}^{0} \mathrm{d}\eta(\theta) g_u(u_*(s-\varepsilon\nu\cdot i+\theta))\psi(s-\varepsilon\nu\cdot i+\theta), \tag{19}$$

with

$$A^\varepsilon(s) = F_u(u_*(s), R(\varepsilon,u_*)(s)), \tag{20}$$

$$B^\varepsilon(s) = F_v(u_*(s), R(\varepsilon,u_*)(s)). \tag{21}$$

Then

$$\begin{aligned} G(\varepsilon,s,\psi) &= F(\psi(s)+u_*(s), R(\varepsilon,\psi+u_*)(s)) - F(u_*(s), R(0,u_*)(s)) - L_0\psi(s) \\ &= [L_\varepsilon - L_0]\psi(s) + G^1(\varepsilon,s,\psi) + G^2(\varepsilon,s), \end{aligned} \tag{22}$$

where

$$G^1(\varepsilon,s,\psi) = F(\psi(s)+u_*(s), R(\varepsilon,\psi+u_*)(s)) - F(u_*(s), R(\varepsilon,u_*)(s)) - L_\varepsilon\psi(s), \tag{23}$$

and

$$G^2(\varepsilon,s) = F(u_*(s), R(\varepsilon,u_*)(s)) - F(u_*(s), R(0,u_*)(s)). \tag{24}$$

Therefore, we can express $\mathcal{G}$ as

$$\mathcal{G}(s,\psi,\varepsilon) = \mathcal{J}_\varepsilon\psi(s) + \mathcal{L}_\varepsilon\psi(s) + \mathcal{G}^1(s,\psi,\varepsilon) + \mathcal{G}^2(s,\varepsilon) + \mathcal{J}_\varepsilon u_*(s), \tag{25}$$

where

$$\mathcal{J}_\varepsilon \psi(s) = \int_{-\infty}^{s} \mathrm{e}^{-(s-t)} J_\varepsilon \psi(t)\mathrm{d}t,$$

$$\mathcal{L}_\varepsilon \psi(s) = \int_{-\infty}^{s} \mathrm{e}^{-(s-t)} [L_\varepsilon - L_0]\psi(t)\mathrm{d}t,$$

$$\mathcal{G}^1(s,\psi,\varepsilon) = \int_{-\infty}^{s} \mathrm{e}^{-(s-t)} G^1(\varepsilon,t,\psi)\mathrm{d}t,$$

and

$$\mathcal{G}^2(s,\varepsilon) = \int_{-\infty}^{s} \mathrm{e}^{-(s-t)} G^2(\varepsilon,t)\mathrm{d}t.$$

Lemma 3.2. *Let $\{f_j(x)\}$, $j \in Z^q$, $x \in R$, be a sequence of functions such that $\sum_j f_j(x)$ exits for any $x \in R$ and $f_j(x) \to \bar{f}_j$ as $x \to x_0 \in \{R, -\infty, +\infty\}$ for all $j \in Z^q$. If there exists a summable sequence $\{g_j\}$ such that $|f_j(x)| \le g_j$ for all $j \in Z^q$ and $x \in R$, then*

$$\sum_j f_j(x) \to \sum_j \bar{f}_j, \quad \text{as } x \to x_0.$$

The proof of Lemma 3.2 is similar to that of the Lebesgue' dominated convergence theorem and is omitted.

Proposition 3.3. *For each $\varepsilon \ge 0$ and $\psi \in C_0$, $\mathcal{G}(\cdot,\psi,\varepsilon) \in C_0$. In other words, $\mathcal{G}(\cdot,C_0,\varepsilon) \subseteq C_0$ for each $\varepsilon \ge 0$.*

Proof. At first, we note that for each $\varepsilon \ge 0$ and each $\varphi \in C$, if $\lim_{s\to\pm\infty}\varphi(s) = \varphi(\pm\infty)$ exist, then it follows from Lemma 3.2 that

$$\lim_{s\to\pm\infty} J_\varepsilon\varphi(s) = \lim_{s\to\pm\infty} D\sum_{i\neq 0} J(i)[\varphi(s-\varepsilon\nu\cdot i) - \varphi(s)] = 0.$$

Therefore, we have

$$\lim_{s\to\pm\infty} \mathcal{J}_\varepsilon u_*(s) = \lim_{s\to\pm\infty} J_\varepsilon u_*(s) = 0, \tag{26}$$

and

$$\lim_{s\to\pm\infty} \mathcal{J}_\varepsilon \psi(s) = \lim_{s\to\pm\infty} J_\varepsilon \psi(s) = 0, \tag{27}$$

for all $\varepsilon \ge 0$ and $\psi \in C_0$.

In a similar way, let $\varphi \in C$ be such that $\lim_{s\to\pm\infty}\varphi(s) = \varphi(\pm\infty)$ exist, then it follows from Lemma 3.2 that

$$\lim_{s\to\pm\infty} R(\varepsilon,\varphi)(s) = \lim_{s\to\pm\infty} \sum_i K(i)\int_{-r}^{0} \mathrm{d}\eta(\theta) g(\varphi(s-\varepsilon\nu\cdot i+\theta)) = \int_{-r}^{0} \mathrm{d}\eta(\theta) g(\varphi(\pm\infty)), \tag{28}$$

for all $\varepsilon \geq 0$. Therefore, we have

$$\lim_{s\to-\infty} A^{\varepsilon}(s) = F_u(E_1, \int_{-r}^{0} \mathrm{d}\eta(\theta)g(E_1)), \quad \lim_{s\to+\infty} A^{\varepsilon}(s) = F_u(E_2, \int_{-r}^{0} \mathrm{d}\eta(\theta)g(E_2)),$$

and

$$\lim_{s\to-\infty} B^{\varepsilon}(s) = F_v(E_1, \int_{-r}^{0} \mathrm{d}\eta(\theta)g(E_1)), \quad \lim_{s\to+\infty} B^{\varepsilon}(s) = F_v(E_2, \int_{-r}^{0} \mathrm{d}\eta(\theta)g(E_2)).$$

Hence, it follows from Lemma 3.2 that for each $\varepsilon \geq 0$ and $\psi \in C_0$,

$$\lim_{s\to\pm\infty} L_{\varepsilon}\psi(s) = 0, \tag{29}$$

and hence

$$\begin{aligned}
&\lim_{s\to\pm\infty} G^1(\varepsilon, s, \psi) \\
&\quad = \lim_{s\to\pm\infty} [F(\psi(s) + u_*(s), R(\varepsilon, \psi + u_*)(s)) - F(u_*(s), R(\varepsilon, u_*)(s))] \\
&\qquad - \lim_{s\to\pm\infty} L_{\varepsilon}\psi(s) \\
&\quad = 0.
\end{aligned} \tag{30}$$

Notice that for each $\varepsilon \geq 0$

$$\lim_{s\to\pm\infty} G^2(\varepsilon, s) = \lim_{s\to\pm\infty} [F(u_*(s), R(\varepsilon, u_*)(s)) - F(u_*(s), R(0, u_*)(s))] = 0; \tag{31}$$

it follows from (26) to (27) and (29) to (31) that for each $\varepsilon \geq 0$ and $\psi \in C_0$

$$\begin{aligned}
&\lim_{s\to\pm\infty} \mathcal{G}(s, \psi, \varepsilon) \\
&\quad = \lim_{s\to\pm\infty} \mathcal{J}_{\varepsilon}\psi(s) + \lim_{s\to\pm\infty} \mathcal{L}_{\varepsilon}\psi(s) \\
&\qquad + \lim_{s\to\pm\infty} \mathcal{G}^1(s, \psi, \varepsilon) + \lim_{s\to\pm\infty} \mathcal{G}^2(s, \varepsilon) + \lim_{s\to\pm\infty} \mathcal{J}_{\varepsilon}u_*(s) \\
&\quad = \lim_{s\to\pm\infty} J_{\varepsilon}\psi(s) + \lim_{s\to\pm\infty} [L_{\varepsilon} - L_0]\psi(s) \\
&\qquad + \lim_{s\to\pm\infty} G^1(\varepsilon, s, \psi) + \lim_{s\to\pm\infty} G^2(\varepsilon, s) + \lim_{s\to\pm\infty} J_{\varepsilon}u_*(s) \\
&\quad = 0.
\end{aligned}$$

This completes the proof. □

Proposition 3.4. *For each* $\varepsilon \geq 0$, $\|\mathcal{J}_{\varepsilon}\psi\|_{C_0} \leq 2\varepsilon\|D\|\sum_{i\neq 0}|J(i)|\cdot|i|\cdot\|\psi\|_{C_0}$ *for* $\psi \in C_0$ *and* $\|\mathcal{J}_{\varepsilon}u_*\|_{C_0} \leq 2\varepsilon\|D\|\sum_{i\neq 0}|J(i)|\cdot|i|\cdot\|u_*\|_C$.

Proof. If $\psi \in C_0^1$, by exchanging the order of integration and integration by parts, we get

$$\begin{aligned}\mathcal{J}_\varepsilon \psi(s) &= \int_{-\infty}^s \mathrm{e}^{-(s-t)} J_\varepsilon \psi(t)\mathrm{d}t \\ &= \int_{-\infty}^s \mathrm{e}^{-(s-t)} \{D\textstyle\sum_{i\neq 0} J(i)[\psi(t-\varepsilon\nu\cdot i)-\psi(t)]\}\mathrm{d}t \\ &= -\int_{-\infty}^s \mathrm{e}^{-(s-t)} \{D\textstyle\sum_{i\neq 0} J(i)\int_0^1 \psi'(t-\tau\varepsilon\nu\cdot i)\varepsilon\nu\cdot i\mathrm{d}\tau\}\mathrm{d}t \\ &= -\varepsilon D\textstyle\sum_{i\neq 0} J(i)\nu\cdot i\int_0^1\int_{-\infty}^s \mathrm{e}^{-(s-t)}\psi'(t-\tau\varepsilon\nu\cdot i)\mathrm{d}t\mathrm{d}\tau \\ &= -\varepsilon D\textstyle\sum_{i\neq 0} J(i)\nu\cdot i\int_0^1[\psi(s-\tau\varepsilon\nu\cdot i)-\int_{-\infty}^s \mathrm{e}^{-(s-t)}\psi(t-\tau\varepsilon\nu\cdot i)\mathrm{d}t]\mathrm{d}\tau,\end{aligned}$$

which yields

$$\|\mathcal{J}_\varepsilon \psi\|_{C_0} \leq 2\varepsilon\|D\|\sum_{i\neq 0}|J(i)|\cdot|i|\cdot\|\psi\|_{C_0}. \tag{32}$$

Since $\mathcal{J}_\varepsilon : C_0 \to C_0$ is a bounded linear operator and C_0^1 is dense in C_0, the last inequality holds for all $\psi \in C_0$. This completes the proof. □

Proposition 3.5. *There exists $M_0 > 0$ such that for all $\varepsilon \geq 0$ and $\psi \in C_0$*

$$\|\mathcal{L}_\varepsilon \psi\|_{C_0} \leq \varepsilon M_0\|\psi\|_{C_0}.$$

Proof. We first note that

$$\begin{aligned}&[L_\varepsilon - L_0]\psi(s) \\ &\quad= [A^\varepsilon(s)-A(s)]\psi(s) \\ &\qquad+[B^\varepsilon(s)-B(s)]\sum_i K(i)\int_{-r}^0 \mathrm{d}\eta(\theta)g_u(u_*(s-\varepsilon\nu\cdot i+\theta))\psi(s-\varepsilon\nu\cdot i+\theta) \\ &\qquad+B(s)\sum_i K(i)\int_{-r}^0 \mathrm{d}\eta(\theta)[g_u(u_*(s-\varepsilon\nu\cdot i+\theta))-g_u(u_*(s+\theta))] \\ &\qquad\quad\times\psi(s-\varepsilon\nu\cdot i+\theta) \\ &\qquad+B(s)\sum_i K(i)\int_{-r}^0 \mathrm{d}\eta(\theta)g_u(u_*(s+\theta))[\psi(s-\varepsilon\nu\cdot i+\theta)-\psi(s+\theta)],\end{aligned}$$

where

$$\begin{aligned}&A^\varepsilon(s)-A(s) \\ &\quad= F_u(u_*(s),R(\varepsilon,u_*)(s))-F_u(u_*(s),R(0,u_*)(s)) \\ &\quad= \int_0^1 F_{uv}(u_*(s),R(0,u_*)(s) \\ &\qquad+\tau[R(\varepsilon,u_*)(s)-R(0,u_*)(s)])\mathrm{d}\tau\cdot[R(\varepsilon,u_*)(s)-R(0,u_*)(s)],\end{aligned}$$

and

$$
\begin{aligned}
&B^{\varepsilon}(s)-B(s)\\
&\quad = F_v(u_*(s),R(\varepsilon,u_*)(s))-F_v(u_*(s),R(0,u_*)(s))\\
&\quad = \int_0^1 F_{vv}(u_*(s),R(0,u_*)(s)\\
&\qquad + \tau[R(\varepsilon,u_*)(s)-R(0,u_*)(s)])\mathrm{d}\tau\cdot[R(\varepsilon,u_*)(s)-R(0,u_*)(s)].
\end{aligned}
$$

Since

$$
\begin{aligned}
&R(\varepsilon,\psi)(s)-R(0,\psi)(s)\\
&\quad = \sum_i K(i)\int_{-r}^{0}\mathrm{d}\eta(\theta)[g(\psi(s-\varepsilon\nu\cdot i+\theta))-g(\psi(s+\theta))]\\
&\quad = -\sum_i K(i)\int_{-r}^{0}\mathrm{d}\eta(\theta)\int_0^1 g_u(\psi(s-\tau\varepsilon\nu\cdot i+\theta))\psi'(s-\tau\varepsilon\nu\cdot i+\theta)\varepsilon\nu\cdot i\mathrm{d}\tau,
\end{aligned}
$$

we have

$$
\|R(\varepsilon,u_*)-R(0,u_*)\|_{C_0}\le\varepsilon\sum_i|K(i)|\cdot|i|\|\eta\|\|g_u\|\|u_*'\|_{C_0}, \tag{33}
$$

where $\|\eta\|=\bigvee_{[-r,0]}\eta$ and $\|g_u\|=\max\{\|g_u(u_*(s))\|:\ s\in R\}$.

Since F is C^2-smooth, there exist $\varepsilon_0>0$ and $M>0$ such that for all $s\in R$ and $\varepsilon\in[0,\varepsilon_0]$

$$
\left\|\int_0^1 F_{uv}(u_*(s),R(0,u_*)(s)+\tau[R(\varepsilon,u_*)(s)-R(0,u_*)(s)])\mathrm{d}\tau\right\|\le M \tag{34}
$$

and

$$
\left\|\int_0^1 F_{vv}(u_*(s),R(0,u_*)(s)+\tau[R(\varepsilon,u_*)(s)-R(0,u_*)(s)])\mathrm{d}\tau\right\|\le M. \tag{35}
$$

Therefore, we have

$$
\|A^{\varepsilon}(s)-A(s)\|\le\varepsilon M\sum_i|K(i)|\cdot|i|\|\eta\|\|g_u\|\|u_*'\|_{C_0}
$$

and

$$
\|B^{\varepsilon}(s)-B(s)\|\le\varepsilon M\sum_i|K(i)|\cdot|i|\|\eta\|\|g_u\|\|u_*'\|_{C_0}.
$$

Therefore, it follows that for all $s\in R$ and $\psi\in C_0$

$$
\left\|\int_{-\infty}^{s}\mathrm{e}^{-(s-t)}[A^{\varepsilon}(t)-A(t)]\psi(t)\mathrm{d}t\right\|\le\varepsilon M_1\|\psi\|_{C_0} \tag{36}
$$

and

$$\Big\|\int_{-\infty}^{s} \mathrm{e}^{-(s-t)}[B^{\varepsilon}(t)-B(t)]\sum_i K(i)\int_{-r}^{0}\mathrm{d}\eta(\theta)g_u(u_*(t-\varepsilon\nu\cdot i+\theta))$$
$$\times\psi(t-\varepsilon\nu\cdot i+\theta)\mathrm{d}t\Big\| \le \varepsilon M_2\|\psi\|_{C_0}, \tag{37}$$

where

$$M_1 = M\|\eta\|\|g_u\|\|u_*'\|_{C_0}\sum_i |K(i)|\cdot|i|,$$
$$M_2 = M\|\eta\|^2\|g_u\|^2\|u_*'\|_{C_0}\sum_i |K(i)||i|\cdot\sum_i |K(i)|.$$

Since

$$g_u(u_*(s-\varepsilon\nu\cdot i))-g_u(u_*(s)) = -\varepsilon\nu\cdot i\int_0^1 g_{uu}(u_*(s-\tau\varepsilon\nu\cdot i))u_*'(s-\tau\varepsilon\nu\cdot i)d\tau,$$

we have

$$\|g_u(u_*(s-\varepsilon\nu\cdot i))-g_u(s)\| \le \varepsilon|i|\|g_{uu}\|\|u_*'\|_{C_0},$$

where $\|g_{uu}\| = \max\{\|g_{uu}(u_*(s))\| :\ s\in R\}$. Therefore, for all $s\in R$ and $\psi\in C_0$, we have

$$\left\|\int_{-\infty}^{s} e^{-(s-t)}B(t)\sum_i K(i)\int_{-r}^{0} d\eta(\theta)[g_u(u_*(t-\varepsilon\nu\cdot i+\theta))\right.$$
$$-g_u(u_*(t+\theta))]\psi(t-\varepsilon\nu\cdot i+\theta)dt\|$$
$$\le \varepsilon M_3\|\psi\|_{C_0}, \tag{38}$$

where

$$M_3 = \sup_{t\in R}\|B(t)\|\|\eta\|\|g_{uu}\|\|u_*'\|_{C_0}\sum_i |K(i)|\cdot|i|.$$

If $\psi\in C_0^1$, by exchanging the order of integration and integration by parts, we have

$$\int_{-\infty}^{s} \mathrm{e}^{-(s-t)}B(t)\sum_i K(i)\int_{-r}^{0}\mathrm{d}\eta(\theta)g_u(u_*(t+\theta))[\psi(t-\varepsilon\nu\cdot i+\theta)-\psi(t+\theta)]\mathrm{d}t$$
$$=\int_{-\infty}^{s} \mathrm{e}^{-(s-t)}B(t)\sum_i K(i)\int_{-r}^{0}\mathrm{d}\eta(\theta)g_u(u_*(t+\theta))$$
$$\times\int_0^1 \psi'(t-\tau\varepsilon\nu\cdot i+\theta)(-\varepsilon\nu\cdot i)\mathrm{d}\tau\mathrm{d}t$$
$$=-\varepsilon\int_0^1\int_{-\infty}^{s} \mathrm{e}^{-(s-t)}B(t)\sum_i K(i)(\nu\cdot i)\int_{-r}^{0}\mathrm{d}\eta(\theta)g_u(u_*(t+\theta))$$

$$
\begin{aligned}
&\times \psi'(t-\tau\varepsilon\nu\cdot i+\theta)\mathrm{d}t\mathrm{d}\tau \\
&= -\varepsilon\int_0^1 \Big\{B(s)\sum_i K(i)(\nu\cdot i)\int_{-r}^0 \mathrm{d}\eta(\theta)g_u(u_*(s+\theta))\psi(s-\tau\varepsilon\nu\cdot i+\theta) \\
&\quad -\int_{-\infty}^s \mathrm{e}^{-(s-t)}(B(t)+B'(t)\sum_i K(i)(\nu\cdot i)\int_{-r}^0 \mathrm{d}\eta(\theta)g_u(u_*(t+\theta)) \\
&\quad \times \psi(t-\tau\varepsilon\nu\cdot i+\theta)\mathrm{d}t \\
&\quad -\int_{-\infty}^s \mathrm{e}^{-(s-t)}B(t)\sum_i K(i)(\nu\cdot i)\int_{-r}^0 \mathrm{d}\eta(\theta)g_{uu}(u_*(t+\theta))u_*'(t+\theta) \\
&\quad \times \psi(t-\tau\varepsilon\nu\cdot i+\theta)\Big\}\mathrm{d}\tau.
\end{aligned}
$$

Therefore, we have that for all $s\in R$

$$
\begin{aligned}
&\left\|\int_{-\infty}^s \mathrm{e}^{-(s-t)}B(t)\sum_i K(i)\int_{-r}^0 \mathrm{d}\eta(\theta)g_u(u_*(t+\theta))[\psi(t-\varepsilon\nu\cdot i+\theta)-\psi(t+\theta)]\mathrm{d}t\right\| \\
&\le \varepsilon M_4\|\psi\|_{C_0},
\end{aligned} \tag{39}
$$

where

$$M_4=\sup_{t\in R}(\|B(t)\|+\|B'(t)\|)(2\|g_u\|+\|g_{uu}\|\|u_*'\|_{C_0})\|\eta\|\sum_i |K(i)|\cdot|i|.$$

Thus, for $\varepsilon\in[0,\varepsilon_0]$ and $\psi\in C_0^1$,

$$\|\mathcal{L}_\varepsilon\psi\|_{C_0}=\sup_{s\in R}\left\|\int_{-\infty}^s \mathrm{e}^{-(s-t)}[L_\varepsilon-L_0]\psi(t)\mathrm{d}t\right\|\le\varepsilon M_0\|\psi\|_{C_0},\quad M_0=\sum_{j=1}^4 M_j. \tag{40}$$

Since $\mathcal{L}_\varepsilon: C_0\to C_0$ is a bounded linear operator and C_0^1 is dense in C_0, the inequality (40) holds for all $\psi\in C_0$. This completes the proof. □

Proposition 3.6. $\mathcal{G}^1(\cdot,0,\varepsilon)=0$ *and for each* $\delta>0$*, there is a* $\sigma>0$ *such that*

$$\|\mathcal{G}^1(\varepsilon,\cdot,\phi)-\mathcal{G}^1(\varepsilon,\cdot,\psi)\|_{C_0}\le\delta\|\phi-\psi\|_{C_0}$$

for all $\varepsilon\in[0,1]$ *and all* $\phi,\psi\in B(\sigma)$*, where* $B(\sigma)$ *is the ball in* C_0 *with radius* σ *and center at the origin.*

Proof. From the definition of $G^1(\varepsilon,\cdot,\psi)$, it is obvious that $G^1_\psi(\varepsilon,\cdot,\psi)$ and $G^1_{\psi\psi}(\varepsilon,\cdot,\psi)$ are continuous for $\varepsilon\in[0,1]$ and for ψ in a neighborhood of the origin in C_0. Moreover, we have $G^1_\psi(\varepsilon,\cdot,0)=0$ for all $\varepsilon\in[0,1]$. It therefore follows that

$$\|G^1(\varepsilon,\cdot,\psi)\|_{C_0}=O(\|\psi\|^2_{C_0}),\quad \text{as}\quad \|\psi\|_{C_0}\to 0 \tag{41}$$

uniformly for $\varepsilon\in[0,1]$, and Proposition 3.6 follows from (41) and the definition of $\mathcal{G}^1(\varepsilon,\cdot,\cdot)$. □

Proposition 3.7. $\|\mathcal{G}^2(\varepsilon,\cdot)\|_{C_0} = O(\varepsilon)$ *as* $\varepsilon \to 0$.

Proof. Note that

$$\begin{aligned}
&F(u_*(s), R(\varepsilon,u_*)(s)) - F(u_*(s), R(0,u_*)(s)) \\
&\quad = \int_0^1 F_v(u_*(t), R(0,u_*)(t) + \tau[R(\varepsilon,u_*)(t) - R(0,u_*)(t)])\mathrm{d}\tau \\
&\qquad \times [R(\varepsilon,u_*)(t) - R(0,u_*)(t)].
\end{aligned}$$

Since

$$\|R(\varepsilon,u_*) - R(0,u_*)\|_{C_0} \leq \varepsilon \sum_i |K(i)| \cdot |i| \|\eta\| \|g_u\| \|u_*'\|_{C_0},$$

and there exist $\varepsilon_0 > 0$ and $M > 0$ such that for all $s \in R$ and $\varepsilon \in [0, \varepsilon_0]$

$$\left\| \int_0^1 F_v(u_*(s), R(0,u_*)(s) + \tau[R(\varepsilon,u_*)(s) - R(0,u_*)(s)])\mathrm{d}\tau \right\| \leq M,$$

where $\|\eta\| = \bigvee_{[-r,0]} \eta$ and $\|g_u\| = \max\{\|g_u(u_*(s))\| : \ s \in R\}$. Therefore, we have

$$\|G^2(\varepsilon,\cdot)\|_{C_0} = \|F(u_*(\cdot), R(\varepsilon,u_*)(\cdot)) - F(u_*(\cdot), R(0,u_*)(\cdot))\|_{C_0} = O(\varepsilon), \quad \text{as} \quad \varepsilon \to 0. \tag{42}$$

Thus, Proposition 3.7 follows from (42) and the definition of $\mathcal{G}^2(\varepsilon,\cdot)$, and the proof is completed. □

By Proposition 3.1, there exist functions $\upsilon_1, \ldots, \upsilon_M \in C_0$ which give a basis of $\mathcal{N}(\mathcal{L})$. Hence, there exist linear functionals $h_1, \ldots, h_M : C_0 \to R$ such that

$$h_j(\upsilon_j) = 1, \quad h_j(\upsilon_i) = 0, \quad i \neq j, \ i,j = 1, \ldots, M.$$

Let $X = \{\phi \in C_0 : \ h_j(\phi) = 0, \ \ j = 1, \ldots, M\}$. Clearly $X \subset C_0$ is a Banach space and

$$C_0 = X \oplus \mathcal{N}(\mathcal{L}). \tag{43}$$

Moreover, if we let $\mathcal{S} = \mathcal{L}|_X$ be the restriction of $\mathcal{L}$ on X, then $\mathcal{S} : X \to C_0$ is one-to-one and onto, since $\mathcal{R}(\mathcal{L}) = C_0$ by Proposition 3.1. Therefore, $\mathcal{S}$ has an inverse $\mathcal{S}^{-1} : C_0 \to X$ which is a bounded linear operator.

4 Proof of the Main Theorem

We shall complete the proof of our main theorem 1.1 in this section. The proof is similar to that of the main result in [10], and for the reader's convenience, we present the details here.

Proof of Theorem 1.1. Firstly, by Proposition 3.1, there exist functions $\upsilon_1, \ldots, \upsilon_M \in C_0$ which give a basis of $\mathcal{N}(\mathcal{L})$. Hence, there exist linear functionals $h_1, \ldots, h_M : C_0 \to R$ such that

$$h_j(\upsilon_j)=1, \quad h_j(\upsilon_i)=0, \quad i\neq j,\ i,j=1,\ldots,M.$$

Let $X=\{\phi\in C_0:\ h_j(\phi)=0,\ \ j=1,\ldots,M\}$. Clearly $X\subset C_0$ is a Banach space and

$$C_0=X\oplus\mathcal{N}(\mathcal{L}). \tag{44}$$

Moreover, if we let $\mathcal{S}=\mathcal{L}|_X$ be the restriction of $\mathcal{L}$ on X, then $\mathcal{S}:X\to C_0$ is one-to-one and onto, since $\mathcal{R}(\mathcal{L})=C_0$ by Proposition 3.1. Therefore, $\mathcal{S}$ has an inverse $\mathcal{S}^{-1}:C_0\to X$ which is a bounded linear operator.

For each $\psi\in C_0$, there exist unique $\xi\in\mathcal{N}(\mathcal{L})$ and $\phi\in X$ such that $\psi=\phi+\xi$. Hence, ψ is a solution of (15) if and only if

$$\mathcal{L}\phi=\mathcal{G}(\cdot,\phi+\xi,\varepsilon) \tag{45}$$

or equivalently, if and only if ϕ is a solution of the equation

$$\phi=\mathcal{S}^{-1}\mathcal{G}(\cdot,\phi+\xi,\varepsilon). \tag{46}$$

It follows from Propositions 3.3–3.7 that there exist $\sigma>0$, $\varepsilon^*>0$ and $\rho\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon^*]$, and $\psi,\varphi\in\overline{B(\sigma)}\subset C_0$

$$\|\mathcal{G}(\cdot,\psi,\varepsilon)\|_{C_0}\leq\frac{1}{3\|\mathcal{S}^{-1}\|}(\|\psi\|_{C_0}+\sigma) \tag{47}$$

and

$$\|\mathcal{G}(\cdot,\psi,\varepsilon)-\mathcal{G}(\cdot,\varphi,\varepsilon)\|_{C_0}\leq\frac{\rho}{\|\mathcal{S}^{-1}\|}\|\psi-\varphi\|_{C_0}. \tag{48}$$

For each fixed $\xi\in\mathcal{N}(\mathcal{L})\cap\overline{B(\sigma)}$, (47) implies that

$$\|\mathcal{S}^{-1}\mathcal{G}(\cdot,\phi+\xi,\varepsilon)\|_{C_0}\leq\frac{1}{3}(\|\phi+\xi\|_{C_0}+\sigma)\leq\sigma, \quad \text{for } \varepsilon\in(0,\varepsilon^*], \quad \phi\in X\cap\overline{B(\sigma)}. \tag{49}$$

Therefore, from (48) and (50), we conclude that the mapping

$$\mathcal{F}:(X\cap\overline{B(\sigma)})\times(\mathcal{N}(\mathcal{L})\cap\overline{B(\sigma)})\times(0,\varepsilon^*)\to X\times\overline{B(\sigma)}$$

given by

$$\mathcal{F}(\phi,\xi,\varepsilon)=\mathcal{S}^{-1}\mathcal{G}(\cdot,\phi+\xi,\varepsilon)$$

is a uniform contraction mapping of $\phi\in X\cap\overline{B(\sigma)}$. Hence, for each $(\xi,\varepsilon)\in(\mathcal{N}(\mathcal{L})\cap\overline{B(\sigma)})\times(0,\varepsilon^*)$, there is a unique fixed point $\phi(\xi,\varepsilon)\in X\cap\overline{B(\sigma)}$ of the mapping $\mathcal{F}(\cdot,\xi,\varepsilon)$. In other words, $\phi(\xi,\varepsilon)$ is the unique solution in $X\cap\overline{B(\sigma)}$ of (46). Thus, for $\varepsilon\in(0,\varepsilon^*)$ fixed, $\psi(\xi,\varepsilon)=\phi(\xi,\varepsilon)+\xi$ is a solution of (15). Notice that $\mathcal{N}(\mathcal{L})\cap\overline{B(\sigma)}$ is M-dimensional. It follows that for each $\varepsilon\in(0,\varepsilon^*)$ and for each unit vector $\nu\in R^q$, the set

$$\Gamma_\nu(\varepsilon)=\{\psi(\xi,\varepsilon):\ \xi\in\mathcal{N}(\mathcal{L})\cap\overline{B(\sigma)}\}$$

is an M-dimensional manifold. This proves that claims (i) and (ii) in the statement of the theorem.

To prove (iii), we first note that if F, g are C^k $(k \geq 2)$, then $\mathcal{G}(\cdot, \psi, \varepsilon)$ is continuous on (ψ, ε) and C^{k-1}-smooth with respect to ψ. Hence, $\mathcal{F}(\phi, \xi, \varepsilon)$ is continuous on $(\phi, \psi, \varepsilon)$ and C^{k-1}-smooth with respect to ϕ and ξ. The uniform contraction mapping principle (see Chow and Hale [7]) implies that the fixed point $\phi(\xi, \varepsilon)$ is a continuous mapping on (ξ, ε) and C^{k-1}-smooth with respect to ξ. Therefore, in addition, we conclude that for each $\varepsilon \in (0, \varepsilon^*)$ and for each unit vector $\nu \in R^q$, $\Gamma_\nu(\varepsilon)$ is a C^{k-1}-manifold. It is locally given as the graph of a C^{k-1}-mapping that is also continuous with respect to ε.

Let $c = 1/\varepsilon$ with $\varepsilon \in (0, \varepsilon^*)$ and

$$\mathsf{M}_\nu(c) = \{U : U(s) = \psi_\xi(s/c) + u_*(s/c), \quad s \in R,\ \psi_\xi \in \Gamma_\nu(1/c)\}.$$

Then $\mathsf{M}_\nu(c)$ is an M-dimensional manifold in a neighborhood of u_* consisting of traveling wave solutions of (1) with wave speed c and direction ν. Moreover, for each $c > c^* := 1/\varepsilon^*$ and for each unit vector $\nu \in R^q$, $\mathsf{M}_\nu(c)$ is a C^{k-1}-manifold that is given by the graph of a C^{k-1}-mapping that is continuous on c.

Recall that F and g are assumed to be C^k-smooth. It remains to prove that the above fixed point $\phi(\xi, \varepsilon)$ is also C^{k-1}-smooth with respect to ε. We will achieve this in several steps.

Assume the functions F and g are $C^k (k \geq 2)$. For $p \in N$, define $C_0^p := \{\phi \in C_0 : \phi \text{ is } C^p - \text{smooth}\}$.

Claim 1. From the definition of L_0 in (9), it is clear that $L_0 : C_0 \to C_0$ is linear bounded and that $L_0(C_0^p) \subset C_0^p$, for $0 \leq p \leq k-1$.

Claim 2. From the definition of $\mathcal{L}$ in (16), $\mathcal{L} : C_0 \to C_0$ is linear bounded and $\mathcal{L}(C_0^p) \subset C_0^p$, for $0 \leq p \leq k$.

Claim 3. From the definition of $\mathcal{G}$ in (17), we have $\mathcal{G}(\cdot, C_0^{p-1}, \varepsilon) \subset C_0^p$ for $\varepsilon > 0$ and $p = 1, 2, \ldots, k$, where $C_0^0 = C_0$.

Claim 4. $\mathcal{N}(\mathcal{L}) \subset C^k$.

In fact, by definition, $\phi \in C_0$ and $\mathcal{L}\phi = 0$ if and only if

$$\phi(s) = \int_{-\infty}^{s} \mathrm{e}^{-(s-t)}[\phi(t) + L_0\phi(t)]\mathrm{d}t, \quad s \in R.$$

Hence, ϕ is continuously differentiable. By differentiating the last equation, we find that $\mathcal{L}\phi = 0$ if and only if $\phi'(s) = L_0\phi(s), \quad s \in R$. Therefore, $\mathcal{N}(\mathcal{L}) = \{\phi \in C^1 : \phi'(t) = L_0\phi(t),\ t \in R\}$. From Claim 1, by induction we conclude that $\mathcal{N}(\mathcal{L}) \subset C_0^k$.

Claim 5. For each $(\xi, \varepsilon) \in (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0, \varepsilon^*)$, the fixed point $\phi^* := \phi(\xi, \varepsilon) \in C_0^1$.

To prove this claim, we fix $(\xi, \varepsilon) \in (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0, \varepsilon^*)$ and define $\psi^* = \phi^* + \xi$. From $\phi^* = \mathcal{F}(\phi^*, \xi, \varepsilon)$, we obtain $\mathcal{L}\psi^* = \mathcal{G}(\cdot, \psi^*, \varepsilon)$, or equivalently

$$\psi^*(s) = \mathcal{G}(s, \psi^*, \varepsilon) + \int_{-\infty}^{s} \mathrm{e}^{-(s-t)}[\psi^*(t) + L_0\psi^*(t)]\mathrm{d}t, \quad s \in R.$$

Hence, $\psi^* \in C_0^1$. From Claim 4, we conclude that $\phi^* \in C_0^1$.

Claim 6. The fixed point $\phi^* = \phi(\xi,\varepsilon)$ is C^1-smooth with respect to ε.

Consider $\mathcal{F}$ restricted to $\phi \in X \cap \overline{B(\sigma)} \cap C_0^1$, more precisely, using Claim 2 and Claim 3, we consider

$$\mathcal{F}^1 : (X \cap \overline{B(\sigma)} \cap C_0^1) \times (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0,\varepsilon^*) \to X \cap \overline{B(\sigma)} \cap C_0^1,$$
$$\mathcal{F}^1(\phi,\xi,\varepsilon) = \mathcal{F}(\phi,\xi,\varepsilon).$$

Notice that $\mathcal{F}^1$ is a uniform contraction of $\phi \in X \cap \overline{B(\sigma)} \cap C_0^1$ for the norm $\|\cdot\|_{C_0}$ and that $\mathcal{F}^1$ is a C^1-mapping on (ϕ,ξ,ε). In fact, for $\psi(s) = \phi(s) + \xi(s)$ C^1-smooth on s, from the definition of $\mathcal{G}$ in (17), we conclude that $\frac{\partial \mathcal{G}}{\partial \varepsilon}(s,\psi,\varepsilon)$ exists and is continuous. In Claim 5, we have proved that there exists a fixed point $\phi^* = \phi(\xi,\varepsilon)$ of $\mathcal{F}^1$. By repeating the arguments used to prove the differentiability of the fixed point in the uniform contraction principle (see Chow and Hale [7]), we conclude that $\phi(\xi,\varepsilon)$ is a C^1-smooth mapping on (ξ,ε).

Claim 7. The fixed point $\phi^* = \phi(\xi,\varepsilon)$ is C^{k-1}-smooth with respect to ε.

As in Claim 5, by induction, we prove that $\phi(\xi,\varepsilon) \in C_0^p, p = 1,2,\ldots,k$. By using Claim 2 and Claim 3, we consider

$$\mathcal{F}^p : (X \cap \overline{B(\sigma)} \cap C_0^p) \times (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0,\varepsilon^*) \to X \cap \overline{B(\sigma)} \cap C_0^p,$$
$$\mathcal{F}^p(\phi,\xi,\varepsilon) = \mathcal{F}(\phi,\xi,\varepsilon), \quad p = 2,\ldots,k-1.$$

As in the proof of the uniform contraction principle, by an inductive argument, we conclude that $\phi^* = \phi(\xi,\varepsilon)$ is C^{k-1}-smooth with respect to ε.

5 Applications

Among the conditions for Theorem 1.1, (H1) and (H2) are verified by analyzing the characteristic equations of (2) at E_1 and E_2. To verify (H3), the connecting orbit theorem in monotone dynamical system theory is useful, which is stated below. See, e.g., Wu et al. [23], Dance and Hess [9], and Smith [17, 18] and the reference therein.

Let X be an ordered Banach space with a closed cone K. For $u,v \in X$, we write $u \geq v$ if $u - v \in K$, and $u > v$ if $u \geq v$ but $u \neq v$.

Lemma 5.1. *Let U be a subset of X and $\Phi : [0,+\infty) \times U \to U$ be a semiflow such that*

(i) *Φ is strictly order-preserving, that is, $\Phi(t,u) > \Phi(t,v)$ for $t \geq 0$ and for all $u,v \in U$ with $u > v$*
(ii) *For some $t_0 > 0$, $\Phi(t_0,\cdot) : U \to U$ is set-condensing with respect to a measure of non-compactness*

Suppose $u_2 > u_1$ are two equilibria of Φ and assume $[u_1, u_2] := \{u: u_2 \geq u \geq u_1\}$ contains no other equilibria. Then there exists a full orbit connecting u_1 and u_2. Namely, there is a continuous function $\phi: \mathrm{R} \to U$ such that $\Phi(t, \phi(s)) = \phi(t+s)$ for all $t \geq 0$ and all $s \in \mathrm{R}$, either (a): $\phi(t) \to u_1$ as $t \to +\infty$ and $\phi(t) \to u_2$ as $t \to -\infty$ or (b): $\phi(t) \to u_1$ as $t \to -\infty$ and $\phi(t) \to u_2$ as $t \to +\infty$.

Returning to the system (2), we use the standard phase space for (2). In this section, C will denote the Banach space $C([-r,0];R^N)$ of continuous R^N-valued functions on $[-r,0]$ with the usual supremum norm. Under the smoothness condition on F and g, the system (2) generates a (local) semiflow on C given by

$$\Phi(t,\phi) = u(\phi)(t+\cdot), \quad t \geq 0,\ \phi \in C,$$

for all those t for which a unique solution $u(\phi)$ of (2) with $u(\phi)(\theta) = \phi(\theta)$ for $\theta \in [-r,0]$ is defined. Let B be an $N \times N$ quasi-positive matrix, that is, $B + \lambda I \geq 0$ for all sufficiently large λ. Here and in what follows, we write $A \geq B$ for $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ if and only if $a_{ij} \geq b_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Define

$$K_B = \left\{ \phi \in C:\ \phi \geq 0, \phi(t) \geq e^{B(t-s)}\phi(s), -r \leq s \leq t \leq 0 \right\}.$$

Then K_B is a closed cone in C, and this induces a partial order on C, denoted by $\geq_B$. Namely, $\phi \geq_B \psi$ if and only if $\phi - \psi \in K_B$.

We shall need the following conditions:

(E_B) $\hat{E}_2 \geq_B \hat{E}_1$, here $\hat{E}_j$ is the constant mapping on $[-r,0]$ with value E_j, $j = 1,2$.

(M_B) Whenever $\phi, \psi \in C$ with $\phi \geq_B \psi$, then

$$F\left(\phi(0), \int_{-r}^{0} \mathrm{d}\eta(\theta) g(\phi(\theta))\right) - F\left(\psi(0), \int_{-r}^{0} \mathrm{d}\eta(\theta) g(\psi(\theta))\right) \geq B[\phi(0) - \psi(0)].$$

Under the above assumptions, Smith and Thieme [20] proved the following.

Lemma 5.2. *Assume that there exists an $N \times N$ quasi-positive matrix B such that (E_B) and (M_B) are satisfied. Then*

(i) $[E_1,E_2]_B := \{\phi \in C:\ \hat{E}_2 \geq_B \phi \geq_B \hat{E}_1\}$ is positive invariant for the semiflow Φ.

(ii) The semiflow $\Phi: [0,+\infty) \times [E_1,E_2]_B \to [E_1,E_2]_B$ is strictly monotone with respect to $\geq_B$ in the sense that if $\phi, \psi \in [E_1,E_2]_B$ with $\phi >_B \psi$, then $\Phi(t,\phi) >_B \Phi(t,\psi)$ for all $t \geq 0$.

In Smith and Thieme [20], it was also shown that (M_B) holds if for all $u, v \in R^N$ with $\hat{u}, \hat{v} \in [E_1,E_2]_B$, the following is satisfied:

$$\begin{cases} F_u(u, \int_{-r}^{0} \mathrm{d}\eta(\theta) g(v)) \geq B, \\ [F_u(u, \int_{-r}^{0} \mathrm{d}\eta(\theta) g(v)) - B]e^{Br} + F_v(u, \int_{-r}^{0} \mathrm{d}\eta(\theta) g(v)) g'(v) \geq 0. \end{cases}$$

In the case $N = 1$, it was shown that in Smith and Thieme [19] that (M_B) holds for some $B < 0$ if

(S_B) $\quad L_2 < 0, L_1 + L_2 < 0, r|L_2| < 1$ and $rL_1 - \ln(rL_2|) > 1$,

where

$$L_1 = \inf_{E_1 \le u,v \le E_2} F_u(u, \int_{-r}^{0} d\eta(\theta)g(v)), \quad L_2 = \inf_{E_1 \le u,v \le E_2} F_v(u, \int_{-r}^{0} d\eta(\theta)g(v))g'(v).$$

Note that $[E_1, E_2]_B$ is a bounded set in C and that $\Phi(t,\cdot) : C \to C$ is compact for $t > r$. Therefore, for $t_0 > r$, the mapping $\Phi(t_0,\cdot) : [E_1,E_2]_B \to [E_1,E_2]_B$ is compact and hence is set-condensing. This observation allows us to derive from Lemmas 5.1 and 5.2 and Theorem 1.1 the following result.

Theorem 5.1. *Assume that:*

(i) (H1) and (H2) are satisfied.
(ii) There exists an $N \times N$ quasi-positive matrix B such that (E_B) and (M_B) are satisfied.
(iii) There exist no other equilibria in $[E_1,E_2]_B$.
Then, the conclusions of Theorem 1.1 hold.

As a first application of our main result, we consider the following lattice differential equation:

$$u_n'(t) = D \sum_{i \in Z\setminus\{0\}} J(i)[u_{n-i}(t) - u_n(t)] - du_n(t) + \sum_{i \in Z} K(i)b(u_{n-i}(t-r)), \tag{50}$$

where $n \in Z, t \ge 0, u_n(t) \in R, D > 0, r \ge 0, d > 0$ and $b(\cdot)$ is of class C^2. We assume that $b(0) = dK - b(K) = 0$ for some $K > 0$ and

$$\sum_{i \in Z\setminus\{0\}} J(i) = 1, \qquad \sum_{i \in Z\setminus\{0\}} |J(i)| \cdot |i| < +\infty, \tag{51}$$

$$\sum_{i \in Z} K(i) = 1, \qquad \sum_{i \in Z} |K(i)| \cdot |i| < +\infty. \tag{52}$$

When $\sum_{|i|\ge 2} J(i) = 0$, (50) has been derived by Weng et al. [22] as a discrete nonlocal model parallel to the continuous nonlocal model in So et al. [21]. Clearly the auxiliary ordinary delay differential equation reads as

$$u'(t) = -du(t) + b(u(t-r)), \tag{53}$$

and it is easily seen that the corresponding characteristic equations at the equilibria $E_1 = 0$ and $E_2 = K$ of (53) are

$$\Lambda_1(\lambda) := \lambda + d - b'(0)e^{-\lambda r}, \tag{54}$$

and

$$\Lambda_2(\lambda) := \lambda + d - b'(K)e^{-\lambda r}, \tag{55}$$

respectively.

Theorem 5.2. *Assume that* $b'(0) > d > b'(K)$ *and* $b(u) > \mathrm{d}u$ *for all* $u \in (0,K)$. *Let* $I_0 = [0,r^1) \cap [0,r^2) \cap [0,r_1)$ *and* $I_j = [0,r^1) \cap [0,r^2) \cap (r_j, r_{j+1}), j \in \mathrm{N}$, *where*

$$r^1 := \sup\left\{ r \geq 0 : \ re^{dr} \min\{b'(u) : \ u \in [0,K]\} \geq -\mathrm{e}^{-1} \right\},$$

$$r^2 := \begin{cases} +\infty, & \text{if } -d \leq b'(K) < d, \\ \frac{\arccos(\frac{d}{b'(K)})}{\sqrt{(b'(K))^2 - d^2}}, & \text{if } \ b'(K) < -d, \end{cases}$$

and

$$r_j := \frac{2j\pi - \arccos(\frac{d}{b'(0)})}{\sqrt{[b'(0)]^2 - d^2}}, \quad j \in \mathrm{N}.$$

Then for any $j \in \mathrm{N}$ *with* $I_{j-1} \neq \emptyset$ *and for any* $r \in I_{j-1}$, *there exists* $c^* > 0$ *such that for every* $c > c^*$, *the set of all traveling wave solutions* $u_n(t) = U(n+ct)$ *with* $U(-\infty) = 0$ *and* $U(+\infty) = K$ *of (50) forms a* $(2j-1)$*-dimensional* C^1*-manifold, which is also* C^1*-smooth with respect to c.*

Theorem 5.2 is a direct consequence of Theorem 5.1 and the following three lemmas.

Lemma 5.3. *If* $r \in I_{j-1}, j \in \mathrm{N}$, *then the equilibrium* $E_1 = 0$ *of (53) is hypobolic and its unstable manifold is exactly* $2j-1$*-dimensional.*

Proof. Clearly, if $r \in [0, r_1)$, then (54) has a simple eigenvalue $\lambda > 0$. A straightforward calculation shows that $E_1 = 0$ is hyperbolic if $r \neq r_j, j \in N$, and (54) has a pair of simple eigenvalues $\lambda = \pm i\beta_j$ with $\beta_j > 0$ at $r = r_j, j \in N$. For any $r \geq 0$, suppose that $\lambda = \lambda(r) = \alpha(r) + i\beta(r)$ with $\beta(r) \geq 0$ is a eigenvalue of (54). It suffices to show that $\alpha'(r) > 0$ whenever $|\alpha(r)|$ is small enough.

Substituting $\lambda = \lambda(r) = \alpha(r) + i\beta(r)$ into (54), we get

$$\begin{cases} (\alpha + d)\mathrm{e}^{\alpha r} = b'(0)\cos\beta r, \\ \beta \mathrm{e}^{\alpha r} = -b'(0)\sin\beta r. \end{cases} \tag{56}$$

Therefore, we have

$$(\alpha + d)^2 + \beta^2 = [b'(0)]^2 \mathrm{e}^{-2\alpha r},$$

and hence

$$\beta\beta' = -\{\alpha + d + r[b'(0)]^2 \mathrm{e}^{-2\alpha r}\}\alpha'. \tag{57}$$

On the other hand, differentiating (54) with respect to r to get

$$\begin{cases} \alpha'\mathrm{e}^{\alpha r} + (\alpha + d)[\alpha' r + \alpha]\mathrm{e}^{\alpha r} = -b'(0)[\beta' r + \beta]\sin\beta r, \\ \beta'\mathrm{e}^{\alpha r} + \beta[\alpha' r + \alpha]\mathrm{e}^{\alpha r} = -b'(0)[\beta' r + \beta]\cos\beta r, \end{cases}$$

which yields

$$\alpha'\beta - \beta'(\alpha+d) = b'(0)\mathrm{e}^{-\alpha r}[\beta' r + \beta]Q,$$

where $Q = (\alpha+d)\cos\beta r - \beta\sin\beta r = \frac{1}{b'(0)}[(\alpha+d)^2+\beta^2]\mathrm{e}^{\alpha r} > 0$. Multiplying the above equality by β, then (57) implies that

$$\begin{aligned}\alpha'\beta^2 &= \beta\beta'[\alpha+d+b'(0)\mathrm{e}^{-\alpha r}rQ] + b'(0)\mathrm{e}^{-\alpha r}\beta^2 Q \\ &= -[\alpha+d+r(b'(0))^2\mathrm{e}^{-2\alpha r}][\alpha+d+b'(0)\mathrm{e}^{-\alpha r}rQ]\alpha' + b'(0)\mathrm{e}^{-\alpha r}\beta^2 Q.\end{aligned}$$

Therefore, we have

$$\alpha' = \alpha'(r) = \frac{b'(0)\mathrm{e}^{-\alpha r}\beta^2 Q}{\beta^2 + [\alpha+d+r(b'(0))^2\mathrm{e}^{-2\alpha r}][\alpha+d+b'(0)\mathrm{e}^{-\alpha r}rQ]} > 0.$$

This completes the proof. □

Lemma 5.4. *There exists $B < 0$ such that (E_B) and (M_B) are satisfied.*

Proof. In the case where $b'_{\min} := \min\{b'(u) : \ u \in [0,K]\} \geq 0$. Choose $B = -d$, then for any $u, v \in [0,K]$, we have $F_u(u, g(v)) = -d \geq B$ and $[F_u(u,g(v)) - B]\mathrm{e}^{Br} + F_v(u,g(v))g'(v) = b'(v) \geq 0$. Therefore, ($M_B$) holds for $B = -d < 0$.

In the case where $b'_{\min} < 0$, we have

$$L_1 := \inf_{E_1 \leq u,v \leq E_2} F_u(u,g(v)) = -d,$$

and

$$L_2 := \inf_{E_1 \leq u,v \leq E_2} F_v(u,g(v))g'(v) = b'_{\min}.$$

Therefore, $L_2 < 0, L_1 + L_2 < 0$. Thus, there is some $B < 0$ so that (S_B) (and hence (M_B)) holds if

$$0 < -rb'_{\min} < 1,$$

and

$$\ln(-rb'_{\min}) < -\mathrm{d}r - 1,$$

from which we conclude that (M_B) holds if $re^{dr}b'_{\min} > -\mathrm{e}^{-1}$. Thus, ($M_B$) holds for all $r \in [0, r^1)$ and some $B < 0$.

Since $B < 0$, we see that (E_B) also holds. This completes the proof. □

Lemma 5.5. *If $r \in [0, r^2)$, then the equilibrium $E_2 = K$ of (53) is asymptotic stable.*

Proof. We claim that if $r \in [0, r^2)$, then all zeros of $\Lambda_2(\lambda) = 0$ have negative real parts. Since $b'(K) < d$, we first note that if $\Lambda_2(\lambda) = 0$, then $\lambda \neq 0$. Suppose otherwise that there exists $\lambda = \alpha + i\beta$ with $\alpha \geq 0$ and $\beta \geq 0$ such that $\Lambda_2(\lambda) = \Lambda_2(\alpha + i\beta) = 0$. Then we have

$$\begin{cases} \alpha = -d + b'(K)e^{-\alpha r}\cos\beta r, \\ \beta = -b'(K)e^{-\alpha r}\sin\beta r. \end{cases}$$

If $-d < b'(K) < d$, then $d > b'(K)\cos\beta r = (\alpha + d)\mathrm{e}^{-\alpha r} \geq d$, which leads to a contradiction. If $b'(K) = -d$, then $\alpha > 0$. Suppose otherwise that $\alpha = 0$, we then have $\beta > 0$ and $\cos\beta r = -1$, and hence $\sin\beta r = 0$, which yields $\beta = -b'(k)\mathrm{e}^{-\alpha r}\sin\beta r = 0$, a contradiction. Therefore, $d \geq b'(K)\cos\beta r = (\alpha + d)\mathrm{e}^{\alpha r} > d$, which is also a contradiction. Thus, the assertion is valid for $-d \leq b'(K) < d$.

In the case where $b'(K) < -d$, let $\lambda = i\beta$ with $\beta > 0$ be such that $\Lambda_2(\lambda) = 0$. Then we have $d = b'(K)\cos\beta r$ and $\beta = -b'(K)\sin\beta r$, from which we find that $\arccos(d/b'(K))/\sqrt{[b'(K)]^2 - d^2}$ is the minimal value of r so that $\Lambda_2(i\beta) = 0$ has a solution $\beta > 0$. This completes the proof of the lemma. □

As another application of our main result, we consider the following lattice differential equation:

$$u_n' = \sum_{i \in Z \setminus \{0\}} J(i) u_{n-i} - u_n - f(u_n), \quad n \in Z, \tag{58}$$

where f is in C^2 and $f(-1) = f(1) = 0$, and the kernel $J(i)$ satisfies (51). Equation (58) was derived in [1] as an l_2-gradient flow for a Helmholtz free energy functional with general long-range interaction (see [2] for its continuum form). In [1], the authors constructed traveling waves and stationary solutions and obtained the uniqueness of traveling wavefronts with nonzero speed and the multiplicity of stationary solutions in the case where f is bistable. In a very recent paper, Carr and Chmaj [4] established the uniqueness of traveling wavefronts in the case where f is monostable. As a direct consequence of Theorem 5.1, we have the following.

Theorem 5.3. *Assume that $f'(-1) < 0$, $f'(1) > 0$ and $f(u) < 0$ for $u \in (-1, 1)$. Then there exists $c^* > 0$ such that for any $c > c^*$, (58) has a traveling wave solution $u_n(t) = U(n + ct)$ satisfying $U(-\infty) = -1$ and $U(+\infty) = 1$.*

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Bifurcation of Limit Cycles from a Non-Hamiltonian Quadratic Integrable System with Homoclinic Loop

Yulin Zhao and Huaiping Zhu

Dedicated to George Sell on the occasion of his 70th birthday.

Abstract In this chapter, we study quadratic perturbations of a non-Hamiltonian quadratic integrable system with a homoclinic loop. We prove that the perturbed system has at most two limit cycles in the finite phase plane, and the bound is exact. The proof relies on an estimation of the number of zeros of related Abelian integrals.

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1 Introduction

This chapter deals with the bifurcation of limit cycles in quadratic integrable systems under small quadratic perturbations, i.e., the study of limit cycles occurring in the quadratic vector fields of the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{H_y(x,y)}{M(x,y)} + \varepsilon F_2(x,y,\varepsilon), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\dfrac{H_x(x,y)}{M(x,y)} + \varepsilon G_2(x,y,\varepsilon), \end{cases} \tag{1)_\varepsilon (}$$

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where ε is a small parameter, $F_2(x,y,\varepsilon)$ and $G_2(x,y,\varepsilon)$ are quadratic polynomials in x and y with coefficients depending analytically on ε, and $H_y(x,y)/M(x,y)$ and $H_x(x,y)/M(x,y)$ are also quadratic polynomials. We call $H(x,y)$ a first integral of the integrable system $(1)_0$ with the integrating factor $M(x,y)$. By Zoladek's classification [29], the integrable systems $(1)_0$ with at least one center are classified into four cases: Q_3^H, Q_3^R, Q_3^{LV}, and Q_4, called Hamiltonian case, reversible case, Lotka–Volterra case, and codimension four case, respectively.

There has been a substantial amount of work devoted to the study of limit cycles in $(1)_\varepsilon$. Zoladek [29] proposed a conjecture about the maximal number of limit cycles which can occur in $(1)_\varepsilon$ for each case. Iliev [14] determined a set of essential perturbations which can realize the maximum number of limit cycles produced by the whole class of system $(1)_\varepsilon$ and computed the corresponding bifurcation functions for every case. For system $(1)_0$, if $M(x,y) \equiv 1$, then $(1)_0 \in Q_3^H$, the existence of the exact upper bound for the number of limit cycles that can be bifurcated from the period annulus of a Hamiltonian system $(1)_0 \in Q_3^H$ has been solved [4, 7, 8, 10, 12, 13, 16, 27, 28]. Yet, if the unperturbed system $(1)_0$ belongs $(Q_3^R \cup Q_3^{LV} \cup Q_4)\backslash Q_3^H$, then $H(x,y)$ and $M(x,y)$ are in general no longer polynomials, which poses some technical difficulties. Some results concerned with certain specific non-Hamiltonian quadratic integrable systems can be found in [2, 23, 26] and references therein.

Zoladek pointed out in [29] that the bifurcation of limit cycles of stratum Q_3^R is the most complicated case. One can verify that any system $(1)_0 \in Q_3^R$ can be transformed into the form

$$\begin{cases} \dot{x} = -y + \bar{a}x^2 + \bar{b}y^2, \\ \dot{y} = x(1+cy). \end{cases} \tag{2}$$

System (2) with $\bar{a}/c = 3/2$ was studied in [5, 15, 25]. After a Poincáre transformation and a change of time, system (2) with $\bar{a}/c = 3/2$ becomes a quadratic Hamiltonian vector filed Q_3^H in which the perturbation remains polynomials.

Iliev classified the reversible systems $Q_3^R\backslash Q_3^H$ into several subcases (see Appendix of [14]). One of them, called the general case, has a first integral as follows:

$$H(X,y) = X^{-\frac{a+b+2}{a-b}} \left[\frac{y^2}{2} + \frac{1}{8(a-b)^2}\left(\frac{a+b-2}{a-3b-2}X^2 + 2\frac{b-1}{b+1}X + \frac{a-3b+2}{a+b+2}\right)\right] \tag{3}$$

with the integrating factor $M(X,y) = X^{-\frac{2a+2}{a-b}}$, where $X = 1+2(a-b)x$, $b \neq -1$. It is not difficult to verify that system (2) with $\bar{a}/c = 3/2$ can be reduced to the non-Hamiltonian system which has the first integral (3) with $-(a+b+2)/(a-b) = -3$. In a recent paper [3], the authors studied quadratic perturbations of the quadratic integrable system whose first integral is (3) with $-(a+b+2)/(a-b) = -4$.

In this chapter, we consider a system of the form (2) which belongs to $Q_3^R \backslash Q_3^H$ whose orbits are almost all quartics and $-(a+b+2)/(a-b) = -3/2$ and $b \neq -1$. In other words, we will consider the perturbation of the following system:

$$\begin{cases} \dot{x} = y + 8(b+1)xy, \\ \dot{y} = -x - 2(3b+1)x^2 + 6(b+1)y^2, \end{cases} \tag{4}$$

where we use the normal form in Page 127 of [14]. One can verify that system (4) has a center at the origin $(0,0)$, and the types of all other critical points (if exist) in the finite phase plane fall into the following cases:

(1) A saddle point and two nodes if $b \in (-3,-1)$
(2) A degenerate critical point if $b = -3$
(3) A center if $b \in (-\infty,-3) \cup (-1/3,+\infty)$
(4) A saddle point if $b \in (-1,-1/3)$
(5) No other critical point if $b = -1/3$

The bifurcation of limit cycles for types (2) to (5) under quadratic perturbations have been studied in the papers [20, 21]. The purpose of this chapter is to investigate the bifurcation of limit cycles for type (1), i.e., to study system (4) with $b \in (-3,-1)$. We state our main result is the following:

Theorem 1.1. *If $b \in (-3,-1)$, then system*

$$\begin{cases} \dot{x} = y + 8(b+1)xy + \varepsilon F_2(x,y,\varepsilon) \\ \dot{y} = -x - 2(3b+1)x^2 + 6(b+1)y^2 + \varepsilon G_2(x,y,\varepsilon) \end{cases} \tag{5}$$

has at most two limit cycles in the finite phase plane for small ε. This bound is exact. Here $F_2(x,y,\varepsilon)$ and $G_2(x,y,\varepsilon)$ are defined as in $(1)_\varepsilon$.

The rest of this chapter is organized as follows. In Sect. 2, system (5) is reduced to a quartic non-Hamiltonian integrable system with nonpolynomial perturbation, and the related bifurcation function is derived. In Sect. 3, we obtain the asymptotic expansions of Abelian integrals near its endpoints of the domain of definition. The upper bound for the number of zeros of Abelian integrals is estimated in Sect. 4. Finally, the main result is proved in Sect. 5.

2 Reduction of the System and Bifurcation Function

In this section, we will consider changes of variables and then give the bifurcation function associated with system (5).

Proposition 2.1. *If $b \in (-3,-1)$, then the exact upper bound for the number of limit cycles of system (5) is equal to the maximum number of limit cycles in the right half-plane (i.e., $x > 0$) occurring in the system*

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = -(A-4)+2(A-2)x^2-Ax^4+\frac{3}{2}y^2+\varepsilon(\mu_1+\mu_2x^2+\mu_3x^{-2}), \end{cases} \tag{6$_\varepsilon$}$$

where $A=\frac{3b+1}{b+1}$. *Moreover, if* $b\in(-3,-1)$, *then* $A\in(4,+\infty)$.

Proof. Letting

$$X=1+8(b+1)x,\ Y=8\sqrt{2}(b+1)y,\ \mathrm{d}\tau=\frac{1}{\sqrt{2}}\mathrm{d}t,$$

then system (5) is transformed into

$$\begin{cases} \dot{X} = XY+\varepsilon\bar{F}_2(X,Y,\varepsilon), \\ \dot{Y} = \dfrac{b+3}{2(b+1)}+\dfrac{b-1}{b+1}X-\dfrac{3b+1}{2(b+1)}X^2+\dfrac{3}{4}Y^2+\varepsilon\bar{G}_2(X,Y,\varepsilon), \end{cases} \tag{7$_\varepsilon$}$$

where $\bar{F}_2(X,Y,\varepsilon)$ and $\bar{G}_2(X,Y,\varepsilon)$ are quadratic polynomials in X and Y with coefficients depending analytically on ε. We note that the integrable system $(7)_0$ has the first integral of the form

$$\bar{H}(X,Y)=X^{-3/2}\left(\frac{1}{2}Y^2+\frac{3b+1}{b+1}X^2+\frac{2(b-1)}{b+1}X+\frac{b+3}{3(b+1)}\right)=h$$

with the integrating factor $X^{-5/2}$. It follows from Theorem 2, Case (ii) in [14] that the first order Melnikov integral is defined as

$$M_1(h)=\int\int_{\bar{H}(X,Y)\le h}X^{-5/2}\left(\mu_1+\mu_2X+\mu_3X^{-1}\right)\mathrm{d}X\mathrm{d}Y,$$

where $M_1(h)$ is the coefficient at ε in the expansion of the displacement function

$$\mathrm{d}(h,\varepsilon)=\varepsilon M_1(h)+O(\varepsilon^2).$$

To get the maximum number of limit cycles for system $(7)_\varepsilon$, we only need to consider the number of zeros of $M_1(h)$ because system $(7)_0$ belongs to generic cases [14]. Therefore, the upper bound for the number of limit cycles can be realized by the following systems:

$$\begin{cases} \dot{X} = XY, \\ \dot{Y} = \dfrac{b+3}{2(b+1)}+\dfrac{b-1}{b+1}X-\dfrac{3b+1}{2(b+1)}X^2+\dfrac{3}{4}Y^2+\varepsilon(\mu_1+\mu_2X+\mu_3X^{-1})Y. \end{cases} \tag{8$_\varepsilon$}$$

Since $X=0$ is an invariant line and system $(8)_0$ has a unique center at $(1,0)$ if $b\in(-3,-1)$, we suppose $X>0$ in $(8)_\varepsilon$ without loss of generality. Finally, we get $(6)_\varepsilon$ by using the rescaling of both the variables and parameters

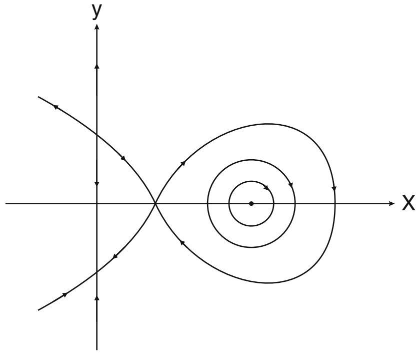


Fig. 1 The phase portrait of system (6)$_0$

$$\widetilde{x} = X^{1/2},\ \widetilde{x} \to x,\ Y \to y,\ \frac{1}{2}\tau \to t,$$
$$b = \frac{1-A}{A-3},\ (A \neq 3), \qquad 2\mu_i \to \mu_i. \tag{9}$$

□

Instead of considering system (5), from now on, we will study system (6)$_\varepsilon$ with $A \in (4, +\infty)$. In the right half-plane, the non-Hamiltonian integral system (6)$_0$ has a first integrable of the form

$$H(x,y) = x^{-3}\left(\frac{1}{2}y^2 + Ax^4 + 2(A-2)x^2 - \frac{A-4}{3}\right) = h \tag{10}$$

with integrating factor $M(x,y) = x^{-4}$, and it has a center at $(1,0)$ and a saddle point at $(\sqrt{\frac{A-4}{A}}, 0)$. Let Γ_h, $h \in (h_1, h_2)$ be the compact component of $\{(x,y)|H(x,y) = h\}$, where

$$h_1 = \frac{8}{3}(A-1) > 0,\ \ h_2 = \frac{8}{3}(A-3)\sqrt{\frac{A}{A-4}} > 0,\ \ A > 4.$$

Γ_{h_1} and Γ_{h_2} correspond to the center $(1,0)$ and the homoclinic loop, respectively. The phase portrait of system (6)$_0$ in the right half-plane is shown in Fig. 1.

Denoted by

$$J_i(h) = \oint_{\Gamma_h} x^{i-4} y\mathrm{d}x, \quad h \in (h_1, h_2), \quad i = 0, \pm 1, \pm 2, \cdots. \tag{11}$$

By above discussion, the Melnikov function $M_1(h)$ of system (8)$_\varepsilon$, defined in the proof of Proposition 2.1, is reduced to the bifurcation function of system (6)$_\varepsilon$ in the form

$$\mu_1 J_0(h) + \mu_2 J_2(h) + \mu_3 J_{-2}(h),\ h \in (h_1, h_2). \tag{12}$$

It is well known that the number of limit cycles in (6)$_\varepsilon$ which can be bifurcated from the annulus of (6)$_0$ is equal to the number of zeros of (12). Therefore, our main goal of this chapter is to estimate the number of zeros of Abelian integral (12). To do this, (12) will be reduced to a simpler form. First of all, we give the following proposition.

Proposition 2.2. *If $\mu_3 = 0$, then system (6)$_\varepsilon$ has at most one hyperbolic limit cycles in the right half-plane.*

Proof. If $\mu_3 = 0$, $b \neq -1$, then system $(8)_\varepsilon$ is a quadratic system with an invariant line. It has been proved [24] that the quadratic system with an invariant line has at most one limit cycles, and this limit cycle must be hyperbolic if it exists. Therefore, system $(8)_\varepsilon$ with $\mu_3 = 0, b \neq -1$ has at most one hyperbolic limit cycle. Since we get $(6)_\varepsilon$ by the scaling (9), the desired result for $(6)_\varepsilon$ follows. □

Proposition 2.3. *If $\mu_3 \neq 0$, $A \neq 4$, then the integral (12) can be expressed in the form*

$$I(h) = \alpha J_0(h) + hJ_1(h) + \gamma J_2(h), \tag{13}$$

where α and γ are real constants.

Proof. Rewrite (10) in the form

$$\frac{1}{2}y^2 + Ax^4 + 2(A-2)x^2 - \frac{A-4}{3} = hx^3. \tag{14}$$

Multiplying both sides of (14) by $x^{-6}y$ and then integrating over Γ_h, we get

$$\frac{1}{2}\oint_{\Gamma_h} x^{-6}y^3 \mathrm{d}x + AJ_2(h) + 2(A-2)J_0(h) - \frac{A-4}{3}J_{-2}(h) = hJ_1(h). \tag{15}$$

It follows from (10) that along Γ_h we have

$$\begin{aligned} &x^{-2}y\left(\frac{\partial H}{\partial x}\mathrm{d}x + \frac{\partial H}{\partial y}\mathrm{d}y\right) \\ &\quad = x^{-5}y^2\mathrm{d}y + \left(-\frac{3}{2}x^{-6}y^3 + Ax^{-2}y - 2(A-2)x^{-4}y + (A-4)x^{-6}y\right)\mathrm{d}x = 0, \end{aligned}$$

which implies

$$\oint_{\Gamma_h} x^{-5}y^2\mathrm{d}y - \frac{3}{2}\oint_{\Gamma_h} x^{-6}y^3\mathrm{d}x + AJ_2(h) - 2(A-2)J_0(h) + (A-4)J_{-2}(h) = 0.$$

Since

$$\oint_{\Gamma_h} x^{-5}y^2\mathrm{d}y = \frac{1}{3}\oint_{\Gamma_h} x^{-5}\mathrm{d}y^3 = \frac{5}{3}\oint_{\Gamma_h} x^{-6}y^3\mathrm{d}x,$$

we have

$$\oint_{\Gamma_h} x^{-6}y^3\mathrm{d}x = -6AJ_2(h) + 12(A-2)J_0(h) - 6(A-4)J_{-2}(h). \tag{16}$$

Substituting (16) into (15), we obtain

$$J_{-2}(h) = \frac{3}{10(A-4)}\left(8(A-2)J_0(h) - hJ_1(h) - 2AJ_2(h)\right). \tag{17}$$

Taking (17) into (12), we have

$$\left(\mu_1 + \frac{12(A-2)}{5(A-4)}\mu_3\right)J_0(h) - \frac{3}{10(A-4)}\mu_3 hJ_1(h) + \left(\mu_2 - \frac{3A}{5(A-4)}\mu_3\right)J_2(h). \tag{18}$$

Since $\mu_3 \neq 0$, without loss of generality, we can assume $-3\mu_3/(10(A-4)) = 1$. Let

$$\alpha = \mu_1 + \frac{12(A-2)}{5(A-4)}\mu_3, \ \ \gamma = \mu_2 - \frac{3A}{5(A-4)}\mu_3, \ \ \mu_3 = -\frac{10}{3}(A-4). \tag{19}$$

The expression then (13) follows. □

From now on, we always suppose that $H(x,y)$ is defined as (10) and assume $A \in (4,+\infty)$ unless the opposite is claimed, and we consider the annulus Γ_h has the negative (clockwise) orientation.

3 Picard–Fuchs System and Some Preliminary Results

In this and the next section, we are going to estimate the number of zeros of Abelian integrals $I(h)$ defined in (13). The Abelian integrals $J_i(h)$ with $i = -1,0,1,2$ are all analytic functions in the interval $[h_1,h_2)$, see for instance [22].

Lemma 3.1. *For the Abelian integrals $J_i(h)$ with $i = -1,0,1,2$, we have the following:*

(i) $J_i(h_1) = 0, \ i = -1,0,1,2.$

(ii) $\lim_{h\to h_1} \frac{J_i(h)}{J_0(h)} = 1, \ i = -1,1,2.$ *Hence* $J'_{-1}(h_1) = J'_0(h_1) = J'_1(h_1) = J'_2(h_1)$.

(iii) $J_i(h) > 0, \ J'_i(h) > 0, \ h \in (h_1,h_2).$

Proof. Since the oval Γ_h tends to the center $(1,0)$ as $h \to h_1$, we get (i). Let $(x_1(h),0)$, $(x_2(h),0)$ be the intersection points of Γ_h with x-axis, $x_1(h) < x_2(h)$. By the integral mean value theorem, we have

$$\lim_{h\to h_1}\frac{J_{-1}(h)}{J_0(h)} = \lim_{h\to h_1}\frac{2\int_{x_1(h)}^{x_2(h)} x^{-5}y\mathrm{d}x}{2\int_{x_1(h)}^{x_2(h)} x^{-4}y\mathrm{d}x} = \lim_{h\to h_1}\frac{\int_{x_1(h)}^{x_2(h)} x\cdot x^{-4}y\mathrm{d}x}{\int_{x_1(h)}^{x_2(h)} x^{-4}y\mathrm{d}x}$$

$$= \lim_{h\to h_1}\frac{\zeta\int_{x_1(h)}^{x_2(h)} x^{-4}y\mathrm{d}x}{\int_{x_1(h)}^{x_2(h)} x^{-4}y\mathrm{d}x} = \lim_{h\to h_1}\zeta = 1,$$

where $\zeta \in (x_1(h), x_2(h))$ and $\lim_{h\to h_1} x_i(h) = 1,\ i = 1,2$. Using the same arguments, we know (ii) holds for $i = 1,2$. It follows from (10) that

$$\frac{\partial y}{\partial h} = \frac{x^3}{y},$$

which implies

$$J_i'(h) = \oint_{\Gamma_h} x^{i-4}\frac{\partial y}{\partial h}\mathrm{d}x = \oint_{\Gamma_h} \frac{x^{i-1}}{y}\mathrm{d}x.$$

Since Γ_h lies in the right half-plane and has the negative orientation, we get

$$\begin{aligned} J_i(h) &= \int\int_{int\Gamma_h} x^{i-4}\mathrm{d}x\mathrm{d}y > 0, \\ J_i'(h) &= 2\int_{x_1(h)}^{x_2(h)} \frac{x^{i-1}}{\sqrt{2hx^3 - 2Ax^4 - 4(A-2)x^2 + \frac{2(A-4)}{3}}}\mathrm{d}x > 0. \end{aligned}$$

□

Lemma 3.2. *The Abelian integrals $J_{-1}(h), J_0(h), J_1(h)$, and $J_2(h)$ satisfy the following Picard–Fuchs system:*

$$\begin{pmatrix} 6\mathcal{C}h & 2(\mathcal{B}^2-4A\mathcal{C}) & -\mathcal{B}h & 0 \\ -2\mathcal{B} & 3h & -4A & 0 \\ 0 & -2\mathcal{B} & 3h & -4A \\ -4\mathcal{C} & 0 & -2\mathcal{B} & h \end{pmatrix} \begin{pmatrix} J_{-1}' \\ J_0' \\ J_1' \\ J_2' \end{pmatrix} = \begin{pmatrix} 8\mathcal{C} & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J_{-1} \\ J_0 \\ J_1 \\ J_2 \end{pmatrix}, \tag{20}$$

where $\mathcal{B} = 2(A-2)$, $\mathcal{C} = -(A-4)/3 \neq 0$.

Proof. See [18] for details. □

Differentiating both sides of (20), we get

$$\begin{pmatrix} 6\mathcal{C}h & 2(\mathcal{B}^2-4A\mathcal{C}) & -\mathcal{B}h & 0 \\ -2\mathcal{B} & 3h & -4A & 0 \\ 0 & -2\mathcal{B} & 3h & -4A \\ -4\mathcal{C} & 0 & -2\mathcal{B} & h \end{pmatrix} \begin{pmatrix} J_{-1}'' \\ J_0'' \\ J_1'' \\ J_2'' \end{pmatrix} = \begin{pmatrix} 2\mathcal{C} & 0 & \mathcal{B} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J_{-1}' \\ J_0' \\ J_1' \\ J_2' \end{pmatrix},$$

which implies the following:

Proposition 3.3 ([18]). *If $A \neq 4$, then the following relation holds:*

$$\Delta(h) \begin{pmatrix} J_{-1}'' \\ J_0'' \\ J_1'' \\ J_2'' \end{pmatrix} = \begin{pmatrix} s_{-1,-1}(h) & s_{-1,1}(h) \\ s_{0,-1}(h) & s_{01}(h) \\ s_{1,-1}(h) & s_{11}(h) \\ s_{2,-1}(h) & s_{21}(h) \end{pmatrix} \begin{pmatrix} J_{-1}' \\ J_1' \end{pmatrix}, \tag{21}$$

where

$$\begin{aligned}
\Delta(h) &= -9(A-4)(h-h_1)(h+h_1)(h-h_2)(h+h_2),\\
s_{-1,-1}(h) &= \frac{1}{3}(A-4)h[-9h^2+64A(A-2)],\\
s_{-1,1}(h) &= \frac{2}{3}h[9(A-2)h^2-32A(2A^2-8A+9)],\\
s_{0,-1}(h) &= -\frac{4}{9}(A-4)[9(A-2)h^2-64A(A-1)(A-3)],\\
s_{01}(h) &= -\frac{4}{3}[-3(3A^2-12A+8)h^2+64A(A-1)(A-2)(A-3)],\\
s_{1,-1}(h) &= -\frac{64}{3}(A-4)h,\\
s_{11}(h) &= -s_{-1,-1}(h),\\
s_{2,-1}(h) &= -\frac{4}{9}(A-4)[-9(A-4)h^2+64(A-3)(A-2)(A-1)],\\
s_{21}(h) &= \frac{4}{9}(A-4)[9(A-2)h^2-64A(A-1)(A-3)].
\end{aligned}$$

Lemma 3.4. *Consider the Abelian integrals $J_{-1}(h)$, $J_0(h)$, $J_1(h)$ and $J_2(h)$.*

(i) The derivatives follow the following relations:

$$\begin{aligned}
J''_{-1}(h_1) &= \frac{1}{192}(A-1)(5A+4)J'_0(h_1),\\
J''_0(h_1) &= \frac{1}{192}(5A^2-13A-4)J'_0(h_1),\\
J''_1(h_1) &= \frac{5}{192}(A-4)(A-1)J'_0(h_1),\\
J''_2(h_1) &= \frac{1}{192}(A-4)(5A-17)J'_0(h_1),\\
J'''_{-1}(h_1) &= \frac{1}{73{,}728}(16+440A+2625A^2-2170A^3+385A^4)J'_0(h_1),\\
J'''_0(h_1) &= \frac{35}{73{,}728}(A-1)(A-4)(11A^2-31A-4)J'_0(h_1),\\
J'''_1(h_1) &= \frac{5}{73{,}728}(A-4)(-164+693A-462A^2+77A^3)J'_0(h_1),\\
J'''_2(h_1) &= \frac{35}{73{,}728}(A-4)^2(A-1)(11A-35)J'_0(h_1).
\end{aligned}$$

(ii) The following asymptotic expansions hold near $h = h_2$:

$$J_{-1}(h) = J_{-1}(h_2) - \frac{1}{2\sqrt{2}}\frac{A}{A-4}(h-h_2)\ln|h-h_2| + \cdots,$$

$$J_0(h) = J_0(h_2) - \frac{1}{2\sqrt{2}}\sqrt{\frac{A}{A-4}}(h-h_2)\ln|h-h_2| + \cdots,$$

$$J_1(h) = J_1(h_2) - \frac{1}{2\sqrt{2}}(h-h_2)\ln|h-h_2| + \cdots,$$

$$J_2(h) = J_2(h_2) - \frac{1}{2\sqrt{2}}\sqrt{\frac{A-4}{A}}(h-h_2)\ln|h-h_2| + \cdots.$$

Proof. Since $h = h_1$ corresponds to the center $(1,0)$ of system $(6)_0$, $J_i(h), i = -1,0,1,2$, is an analytic function at $h = h_1$ [22]. We get (i) from (21) and Lemma 3.1.

Consider the analytic Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial y}, \;\; \dot{y} = -\frac{\partial H}{\partial x} + \varepsilon x^{i-4} y.$$

The expansions (ii) follow by using the results in [22], or Theorem A.1 in Appendix of [28] for the above Hamiltonian system. □

Corollary 3.5. *For the Abelian integral $I(h)$ in (13), we have*

$$\begin{aligned}
I(h_1) &= 0,\\
I'(h_1) &= J_0'(h_1)(\alpha + h_1 + \gamma),\\
I''(h_1) &= \frac{J_0'(h_1)}{576}\Big(3(5A^2-13A-4)\alpha\\
&\quad +8(5A^3-30A^2+45A+124)+3(A-4)(5A-17)\gamma\Big),\\
I'''(h_1) &= \frac{5(A-1)(A-4)J_0'(h_1)}{221184}\Big(21(11A^2-31A-4)\alpha\\
&\quad +8(77A^3-462A^2+693A+268)+21(A-4)(11A-35)\gamma\Big).
\end{aligned}$$

Moreover, if $I'(h_1) = I''(h_1) = 0$, then $I'''(h_1) < 0$.

Proof. It follows from straightforward computations; we get $I(h_1), I'(h_1), I''(h_1)$, and $I'''(h_1)$ from Lemma 3.1, 3.4, and (13). Now we consider the equations $I'(h_1) = I''(h_1) = 0$, which gives

$$\alpha = -\frac{4(A^2-5A+16)}{3(A-3)}, \;\; \gamma = -\frac{4(A-5)(A+2)}{3(A-3)}. \tag{22}$$

Substituting (22) into $I'''(h_1)$, we have

$$I'''(h_1) = -\frac{25}{96}(A-4)(A-1) < 0. \tag{23}$$

□

In order to estimate the number of zeros for the Abelian integral $I(h)$, we define

$$\omega(h) = \frac{J'_{-1}(h)}{J'_1(h)}, \quad W(h) = \Delta(h)\frac{I''(h)}{J'_1(h)}. \tag{24}$$

It follows from (13) and (21) that we have

$$\begin{aligned}\Delta(h)I''(h) &= \alpha\Delta(h)J''_0(h) + h\Delta(h)J''_1(h) + \gamma\Delta(h)J''_2(h) + 2\Delta(h)J'_1(h)\\ &= f(h)J'_{-1}(h) + g(h)J'_1(h);\end{aligned}$$

therefore, we have

$$W(h) = f(h)\omega(h) + g(h), \tag{25}$$

where the two auxiliary functions $f(h)$ and $g(h)$ are given by

$$\begin{aligned}f(h) &= \alpha s_{0,-1}(h) + hs_{1,-1}(h) + \gamma s_{2,-1}(h)\\ &= \frac{4}{9}(A-4)\{3(-16-3(A-2)\alpha + 3(A-4)\gamma)h^2\\ &\quad +64(A-3)(A-1)(A\alpha - (A-2)\gamma\},\\ g(h) &= \alpha s_{01}(h) + hs_{11}(h) + \gamma s_{21}(h) + 2\Delta(h).\end{aligned}$$

On the other hand, it follows from the first and the third equations of system (21) that the ratio $\omega(h)$ satisfies the following Riccati equation:

$$\Delta(h)\omega' = -s_{1,-1}(h)\omega^2 + 2s_{-1,-1}(h)\omega + s_{-1,1}(h). \tag{26}$$

Then we can prove the following proposition.

Proposition 3.6. *The ratio $W(h)$ satisfies the following Riccati equation:*

$$\Delta(h)f(h)W' = R_0(h)W^2 + R_1(h)W + R_2(h), \tag{27}$$

where

$$\begin{aligned}R_0(h) &= -s_{1,-1}(h),\\ R_1(h) &= \Delta(h)f'(h) + 2s_{1,-1}(h)g(h) + 2s_{-1,-1}(h)f(h),\\ R_2(h) &= \Delta(h)(f(h)g'(h) - f'(h)g(h)) - s_{1,-1}(h)g^2(h) - 2s_{-1,-1}(h)f(h)g(h)\\ &\quad +s_{-1,1}(h)f^2(h).\end{aligned}$$

Moreover, $f(h)$ and $g(h)$ are even functions and $R_2(h)$ is a odd function, $R_2(\pm h_1) = R_2(\pm h_2) = R(0) = 0$, $\deg f(h) = 2$, $\deg g(h) = 4$, $\deg R_2(h) = 9$.

Proof. Equation (27) follows from (25) and (26). Note $\Delta(h) = \Delta(-h)$ and $s_{0,j}(h) = s_{0,j}(-h)$, $hs_{1,j}(h) = (-h)s_{1,j}(-h)$, $s_{2,j}(h) = s_{2,j}(-h)$ for $j = -1, 1$, then $f(h)$ and $g(h)$ are even functions. The conclusion for $R_2(h)$ follows by straightforward computations. □

It follows from (24), the definition of $W(h)$ that the zeros of $W(h)$ correspond to the inflection points of $I(h)$ when exist. To give a precise result on the number of zeros of Abelian integrals $I(h)$, we need to estimate the upper bound for the number of zeros of $W(h)$.

Lemma 3.7. *The auxiliary functions $f(h)$ and $g(h)$ defined for $W(h)$, $\omega(h)$, and $W(h)$ have the following properties:*

(i) For $f(h)$ and $g(h)$, we have

$$\begin{aligned} f(h_1) &= -\tfrac{512}{9}(A-1)(A-4)(\alpha+h_1+\gamma),\\ g(h_1) &= -f(h_1),\\ f(h_2) &= -\frac{512}{27}(A-3)(3A\alpha+8A(A-3)+3(A-4)\gamma),\\ g(h_2) &= -\frac{A}{A-4}f(h_2). \end{aligned}$$

(ii) For $\omega(h)$, we have

$$\begin{aligned} &\omega(h_1) = 1,\ \omega(h_2) = A/(A-4),\\ &\omega'(h_1) = (A-1)/8,\ \lim_{h\to h_2}\omega(h) = +\infty,\\ &\omega'(h) > 0,\ h \in (h_1, h_2). \end{aligned}$$

(iii) $W(h_1) = W(h_2) = 0$, and

$$W'(h_1) = \frac{4096(A-1)I''(h_1)}{3J_0'(h_1)},\quad \lim_{h\to h_2} W'(h) = +\infty\cdot sgn(f(h_2)). \tag{28}$$

The second equality in (28) holds for $f(h_2) \neq 0$.

Proof. The assertions (i) follow from straightforward computations. Since the right hand of (26) is a quadratic polynomial in ω and $\Delta(h) > 0$, $-s_{1,-1}(h) > 0$,

$$4s^2_{-1,-1}(h) + 4s_{1,-1}(h)s_{-1,1}(h) = 36(A-4)^2h^2(h-h_1)(h+h_1)(h-h_2)(h+h_2) < 0$$

for $h \in (h_1, h_2)$, we have $\omega'(h) > 0$. One gets $\omega(h_1)$ and $\omega'(h_1)$ from (24), Lemmas 3.1 and 3.4(i).

To get the asymptotic behavior near $h = h_2$, we consider the first and the third equations of system (21):

$$\Delta(h)\begin{pmatrix} J''_{-1} \\ J''_1 \end{pmatrix} = \begin{pmatrix} s_{-1,-1}(h) & s_{-1,1}(h) \\ s_{1,-1}(h) & s_{11}(h) \end{pmatrix}\begin{pmatrix} J'_{-1} \\ J'_1 \end{pmatrix}. \tag{29}$$

Using the analytic theory of ordinary differential equations (see [9], or the Appendix of [19]), system (29) has a fundamental solution matrix of the form

$$\mathbf{TP}(h)\begin{pmatrix} 1 & \ln|h-h_2| \\ 0 & 1 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} A & A+24 \\ A-4 & A-4 \end{pmatrix},$$

where $\mathbf{P}(h)$ is analytic at $h = h_2$, $\mathbf{P}(h_2)$ is a unite matrix. Therefore, $J'_{-1}(h)$ and $J'_1(h)$ have the form

$$\begin{aligned} J'_{-1}(h) &= C_2 A \ln|h-h_2| + C_1 A + C_2(A+24) + \cdots, \\ J'_1(h) &= C_2(A-4)\ln|h-h_2| + (C_1+C_2)(A-4) + \cdots, \end{aligned}$$

where C_1 and C_2 are real constants. If $C_2 = 0$, then $J_{-1}(h)$ and $J'_1(h)$ are analytic at $h = h_2$, which contradicts Lemma 3.4(ii). Hence, we have $C_2 \neq 0$. This yields

$$\omega(h) = \frac{J'_{-1}(h)}{J'_1(h)} = \frac{A}{A-4} + \frac{24}{(A-4)\ln|h-h_2|} + \cdots$$

as $h \to h_2$. Therefore

$$\omega(h_2) = \frac{A}{A-4}, \quad \omega'(h) = -\frac{24}{(A-4)(h-h_2)\ln^2|h-h_2|} + \cdots$$

as $h \to h_2$, which implies $\lim_{h\to h_2^-} \omega'(h) = +\infty$.

The conclusions (iii) follows from (i), (ii), and (25). □

Let $\bar{h} > 0$ be the zero of $f(h)$.Then by the definition of $f(h)$ (see (25)), if the root exists, we have

$$\bar{h} = \sqrt{\frac{64(A-3)(A-1)(A\alpha-(A-2)\gamma)}{3(16+3(A-2)\alpha-3(A-4)\gamma)}}. \tag{30}$$

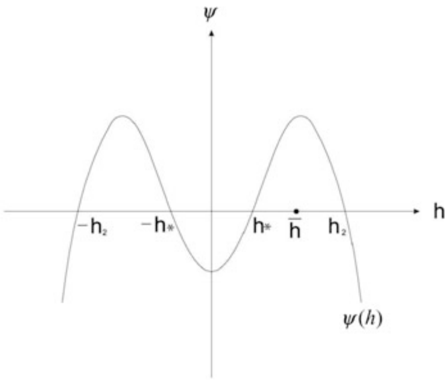
Corollary 3.8. *If $f'(\bar{h}) < 0$, $\bar{h} \in [h_1, h_2)$, then $g(\bar{h}) \geq 0$ and $W(h) > 0$ for $h \in (\bar{h}, h_2)$. Moreover, $g(\bar{h}) = 0$ if and only if $\bar{h} = h_1$.*

Proof. If $f'(\bar{h}) < 0$, then $f(h_2) < 0 \leq f(h_1)$ and the leading part of $f(h)$ is negative, namely,

$$-16 - 3(A-2)\alpha + 3(A-4)\gamma < 0, \tag{31}$$

which implies

Fig. 2 The graph of $\Psi(h)$



$$-32-(A-2)\alpha+(A-4)\gamma < -32+\frac{16}{3} = -\frac{80}{3} < 0. \tag{32}$$

By direct computation,

$$g(\bar{h}) = \frac{81(-32-(A-2)\alpha+(A-4)\gamma)}{256(A-4)(-16-3(A-2)\alpha+3(A-4)\gamma)^2} f(h_1)f(h_2). \tag{33}$$

This yields $g(\bar{h}) \geq 0$ and $g(\bar{h}) = 0$ if and only if $h_1 = \bar{h}$.

Define the following auxiliary function:

$$\Psi(h) = \frac{A}{A-4} f(h) + g(h) = -15(A-4)(h^2-h_2^2)(h^2-h_*^2), \tag{34}$$

where

$$h_*^2 = \frac{8}{15}(A-1)(\alpha+\gamma+16(A-1)) \tag{35}$$

and suppose $h_* > 0$ if the right hand of (35) is positive. It is obvious that $\Psi(h)$ has four real zeros (resp. two real zeros) at $h = \pm h_2, h = \pm h_*$ (resp. $h = \pm h_2$) if $\alpha+\gamma+16(A-1) \geq 0$ (resp. $\alpha+\gamma+16(A-1) < 0$).

We split the rest of this proof into two cases.

Case 1. $\alpha+\gamma+16(A-1) \geq 0$.

We notice that $\Psi(h)$ is an even polynomial with $\deg \Psi(h) = 4$ and the leading coefficient is $-15(A-4) < 0$. The graph of $\Psi(h)$ is drawn in Fig. 2. Since $\Psi(h_2) = 0$, $\Psi(\bar{h}) = g(\bar{h}) \geq 0$, we have $h_* < \bar{h} < h_2$ and $\Psi(h) > 0$ for $h \in (\bar{h}, h_2)$. On the other hand, $f'(\bar{h}) < 0$ implies $f(h) < 0$ for $h \in (\bar{h}, h_2)$. Hence,

$$\frac{\Psi(h)}{f(h)} = \frac{g(h)}{f(h)} + \frac{A}{A-4} < 0,\ h \in (\bar{h}, h_2),$$

i.e., $g(h)/f(h) < -A/(A-4)$. Lemma 3.7(ii) have shown $1 \leq \omega(h) \leq A/(A-4)$, which yields $\omega(h) + g(h)/f(h) < 0$, $h \in (\bar{h}, h_2)$. Using (25) again,

$$W(h) = f(h)\left(\omega(h) + \frac{g(h)}{f(h)}\right) > 0, \quad h \in (\bar{h}, h_2).$$

Case 2. $\alpha + \gamma + 16(A-1) < 0$.

In this case, we obtain from (34) that $\Psi(h) > 0$ for $h \in (h_1, h_2)$. The results follows from a similar argument as above. □

Corollary 3.9. *If* $\bar{h} \in (h_1, h_2]$ *and* $f'(\bar{h}) < 0$*, then* $W(h) > 0, h \in (h_1, \bar{h})$.

Proof. One can verify that $g(\bar{h}) \geq 0$ from (32) and (33).

Define an auxiliary function

$$\Phi(h) = f(h) + g(h).$$

One can verify that

$$\Phi(h) = (h - h_1^2)\Big(-15(A-4)h^2 + 8(A-3)(A\alpha + (A-4)\gamma + 16A(A-3))\Big),$$

from which one immediately yields $\Phi(h_1) = 0$, $\Phi(\bar{h}) = g(\bar{h}) \geq 0$. Since $\Phi(h)$ is an even function with $\deg \Phi(h) = 4$ and its leading coefficient is $-15(A-4) < 0$, we conclude that $\Phi(h) > 0, h \in (h_1, \bar{h})$, by the same arguments as in the proof of Corollary 3.8. Therefore,

$$\frac{\Phi(h)}{f(h)} = 1 + \frac{g(h)}{f(h)} > 0, \; h \in (h_1, \bar{h}),$$

i.e., $g(h)/f(h) > -1$, where we use the fact $f(h) > 0$ for $h \in (h_1, \bar{h})$. It follows from Lemma 3.7(ii) that $\omega(h) > 1$, which gives

$$\omega(h) + \frac{g(h)}{f(h)} > 1 - 1 = 0, \quad h \in (h_1, \bar{h}).$$

So

$$W(h) = f(h)\left(\omega(h) + \frac{g(h)}{f(h)}\right) > 0, \quad h \in (h_1, \bar{h}).$$

□

Proposition 3.10. *If* $\bar{h} \in [h_1, h_2]$ *and* $f'(\bar{h}) < 0$*, then* $I''(h) > 0$ *and* $I(h)$ *has at most one zero in* (h_1, h_2).

Proof. We know from (32) and (33) that $W(\bar{h}) = g(\bar{h}) = 0$ if and only if either $\bar{h} = h_1$ or $\bar{h} = h_2$, which implies $W(\bar{h}) > 0$ if $\bar{h} \neq h_i$, $i = 1, 2$. Therefore, it follows from (24), Corollaries 3.8 and 3.9 that $I''(h) > 0$ for $h \in (h_1, h_2)$, namely, $I(h)$ has no inflection point in (h_1, h_2). Since $I(h_1) = 0$, the claim is proved. □

4 Estimate for the Number of Zeros of $I(h)$

In this section, we estimate the number of zeros of $I(h)$ for $h \in (h_1, h_2)$ (counting the multiplicities), denoted by $\#I(h)$. For convenience, in what follows, we always suppose $h \in (h_1, h_2)$ if the symbol $\#I(h)$ (resp. $\#f(h)$, $\#I'(h)$, etc.) is used.

It follows from Proposition 3.6 that $\#f(h) \le 1$, $\#R_2(h) \le 2$. Since $\#I''(h) = \#W(h)$ and $I(h_1) = 0$, we have

$$\#I(h) \le \#W(h) + 1, \quad h \in (h_1, h_2). \tag{36}$$

Proposition 4.1. *$\#I''(h) \le 2, \#I(h) \le 3$, if one of the following conditions holds:*

(i) $\#f(h) + \#R_2(h) \le 1$
(ii) $\#f(h) + \#R_2(h) \le 2,\ W'(h_1) f(h_2) < 0$
(iii) $W'(h_1) = 0,\ \#f(h) = 0$

Proof. Let $\widetilde{h}_1, \widetilde{h}_2$ be the two consecutive zeros of $W(h)$ in (h_1, h_2). Then the Riccati equation (27) implies

$$\Delta(\widetilde{h}_i) f(\widetilde{h}_i) W'(\widetilde{h}_i) = R_2(\widetilde{h}_i),\ i = 1, 2,$$

which yields $R_2(\widetilde{h}_1) R_2(\widetilde{h}_2) \le 0$ if $f(h) \ne 0$ in $(\widetilde{h}_1 - \varepsilon, \widetilde{h}_2 + \varepsilon)$, $0 < \varepsilon \ll 1$. Therefore, between any two consecutive zeros of $W(h)$, there must exist at least one zero of $R_2(h)$ or $f(h)$. This gives

$$\#I''(h) = \#W(h) \le \#f(h) + \#R_2(h) + 1. \tag{37}$$

The inequality (37) has been used in many papers, see for instance [18]. The desired result for (i) follows from (36) and (37).

If the condition (ii) holds, then by Lemma 3.7, $W'(h_1) \lim_{h \to h_2} W'(h) < 0$, $W(h_1) = W(h_2) = 0$, which implies $\#W(h)$ must be an even number. On the other hand, one gets $\#W(h) \le 3$ from (37). Therefore, either $\#W(h) = 0$ or $\#W(h) = 2$ holds. This yields $\#I''(h) \le 2, \#I(h) \le 3$.

By direct computation,

$$R_2'(h_1) = \frac{4096}{3} (A - 1) f(h_1) W'(h_1) = R_2'(-h_1). \tag{38}$$

If condition (iii) holds, then it follows from (38) that $R_2(h)$ has a zero at h_1 with multiplicity at least two, which implies $\#R_2(h) \le 1$. Using (37) again, one obtains $\#I''(h) \le 2$. This yields $\#I(h) \le 3$. □

Proposition 4.2. *If $f(h) \equiv 0$, then $\#I(h) = 0$.*

Proof. It follows from the definition of $f(h)$ in (25) that $f(h) \equiv 0$ if and only if

$$A\alpha - (A-2)\gamma = 0, \ -16 - 3(A-2)\alpha + 3(A-4)\gamma = 0,$$

which yields

$$\alpha = -\frac{4}{3}(A-2), \ \gamma = -\frac{4}{3}A.$$

This gives

$$W(h) = g(h) = -15(A-4)(h-h_1)(h+h_1)(h-h_2)(h+h_2),$$

which implies $\#W(h) = \#I''(h) = 0$ by (24) and (25). On the other hand, Corollary 3.5 and Lemma 3.7(i) give

$$f(h_1) = -\frac{512}{9}(A-4)(A-1)\frac{I'(h_1)}{J_0'(h_1)}, \tag{39}$$

which implies $I'(h_1) = 0$ if $f(h) \equiv 0$. Suppose that $I(h)$ has a zero at $h = \widetilde{h} \in (h_1, h_2)$, then there must exist $h^* \in (h_1, \widetilde{h})$ such that $I'(h^*) = 0$ by mean value theorem. Since $I'(h_1) = I'(h^*) = 0$, $I''(h)$ have at least one zero in (h_1, h^*), i.e., $\#I''(h) \geq 1$, which yields contradiction. Therefore, one gets $\#I(h) = 0$ by the above discussion. □

By Proposition 3.6, the graph of $W(h), h \in [h_1, h_2]$, is the trajectory of the following system:

$$\begin{cases} \dot{h} &= \Delta(h) f(h), \\ \dot{W} &= R_0(h)W^2 + R_1(h)W + R_2(h). \end{cases} \tag{40}$$

Since $\#I''(h) = \#W(h)$ and the abscissa of the intersection point of the trajectory $W(h)$ with h-axis is a zero of the function $W(h)$, $h \in (h_1, h_2)$, we are going to estimate how many times the trajectory $W(h)$ intersects with the h-axis. First of all, we give the following lemmas:

Lemma 4.3. *Let $\bar{h} \in (h_1, h_2)$ be the zero of $f(h)$, defined in (30). If $f'(\bar{h}) > 0$, then*

(i) The straight lines $h = h_1$, $h = \bar{h}$, and $h = h_2$ are invariant lines of system (40).
(ii) In the right half-plane, system (40) has two saddle nodes at $B_1(h_1, 0)$ and $B_2(h_2, 0)$, a unstable node at $S_1(\bar{h}, W_1)$, and a saddle point at $S_2(\bar{h}, W_2)$, where

$$W_1 = g(\bar{h}), \ \ W_2 = g(\bar{h}) - \frac{\Delta(\bar{h}) f'(\bar{h})}{R_0(\bar{h})}, \ \ W_1 > W_2, \ \ R_0(\bar{h}) > 0.$$

(iii) The graph of $W(h)$, $h \in [h_1, h_2]$, is composed of five trajectories of (40), namely, the critical points B_1, S_1, B_2 and two trajectories $\widehat{S_1B_1}$, $\widehat{S_1B_2}$.

Proof. The first two claims can be obtained by direct computations. The results (iii) follow from Lemma 3.7 and (25). □

We note that, if $f(\bar{h}) = f'(\bar{h}) = 0$, then $\bar{h} = 0 \notin (h_1, h_2)$.

Lemma 4.4. *Suppose $\#f(h) = 0$ in (h_1, h_2), then in the right half-plane, system (40) has two invariant lines $h = h_1$, $h = h_2$ and two saddle nodes at B_1, B_2. The graph of $W(h)$, $h \in [h_1, h_2]$, is composed of the critical points B_1, B_2 and the trajectory connecting B_1 and B_2.*

Proposition 4.5. *If $\bar{h} \in (h_1, h_2)$, $f'(\bar{h}) > 0$, then $\#I(h) \le 3$.*

Proof. For this case, $f(h_1) < 0 < f(h_2)$ and $W(h_1) = W(h_2) = 0$, $\lim_{h \to h_2^-} W'(h) = +\infty$, see Lemma 3.7(iii). This shows that $W'(h) > 0$, $W(h) < 0$ as $h \to h_2^-$.

In Proposition 4.1(i), we have shown that $\#I(h) \le 3$ if $\#R_2(h) = 0$, $\#f(h) = 1$. Hence, in the rest of this proof, we suppose $\#R_2(h) = 1, 2$. We split the proof into the following three cases:

Case 1. $W'(h_1) > 0$.

It follows from (38) that $R_2'(h_1) < 0$, which implies that $R_2(h) < 0$ as $h \to h_1^+$. On the other hand, one gets $I'(h_1) > 0$, $I''(h_1) > 0$ from (39) and (28), which yields that $I(h)$ is increasing and convex as $h \to h_1^+$.

This case is split into the following four subcases:

Subcase 1. $W_2 < W_1 < 0$.

Since $S_1(\bar{h}, W_1)$ is an unstable node, we have $R_2(\bar{h}) > 0$, which implies that $R(h)$ has a zero in the interval $(h_1, \bar{h})$.

Suppose $R_2(h)$ has another zero in the interval $(\bar{h}, h_2)$, then the trajectory $\widehat{S_1B_1}$ must intersects h-axis once and $\widehat{S_1B_2}$ intersects h-axis at most two points, see Fig. 3a. This means $\#W(h) = \#I''(h) \le 3$. Since we have shown that $I(h)$ is increasing and convex as $h \to h_1^+$, $I(h)$ has at most 3 zeros in (h_1, h_2), see Fig. 3b.

If $\#R_2(h) = 1$, then $R_2(h)$ has one zero in the interval $(h_1, \bar{h})$ and no zero in $(\bar{h}, h_2)$. By the same arguments as above, we obtain $\#I(h) \le 1$.

Subcase 2. $W_2 < W_1 = g(\bar{h}) = 0$.

The expression (33) yields

$$\gamma = \frac{32 + (A-2)\alpha}{A-4}.$$

Taking the above value of γ into (25), we obtain

$$W(h) = \frac{320}{3}(A-4)(h-\bar{h})(h+\bar{h})\left(\omega(h) - \frac{9}{64}h^2 + A(A-2)\right).$$

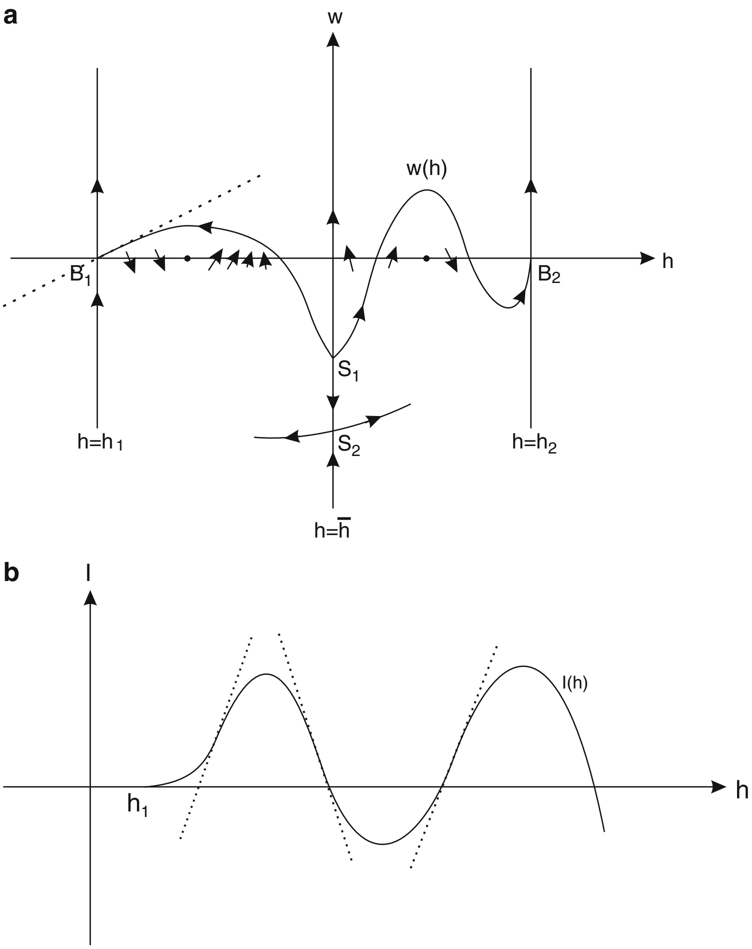


Fig. 3 **(a)** The possible phase portraits of system (40) **(b)** The possible graph of I(h)

Define

$$\widetilde{W}(h) = \omega(h) - \frac{9}{64}h^2 + A(A-2).$$

By the same arguments as in the proof of Proposition 3.6, $\widetilde{W}(h)$ satisfies

$$\Delta(h)\widetilde{W}' = R_0(h)\widetilde{W}^2 + \widetilde{R}_2(h),$$

where

$$\widetilde{R}_2(h) = \frac{135}{64}(A-4)h(h-h_1)(h+h_1)(h-h_2)(h+h_2).$$

Since $\#\widetilde{R}_2(h)=0$, the inequality (37) shows that $\#\widetilde{W}(h)\le 1$. On the other hand, Lemma 3.7 implies $\widetilde{W}(h_1)=\widetilde{W}(h_2)=0$ and

$$\widetilde{W}'(h_1)=\omega'(h_1)-\frac{9}{32}h_1=-\frac{5}{8}(A-1)<0,\ \lim_{h\to h_2^-}\widetilde{W}'(h)=+\infty,$$

which yields $\widetilde{W}(h)<0$, $\widetilde{W}'(h)<0$ as $h\to h_1^+$ and $\widetilde{W}(h)<0$, $\widetilde{W}'(h)>0$ as $h\to h_2^-$. This implies that $\#\widetilde{W}(h)$ must be an even number or zero. We have proved that $\#\widetilde{W}(h)\le 1$. Therefore, $\#\widetilde{W}(h)=0$.

The above discussion shows that $I''(h)$ has only one simple zero at $h=\bar{h}$. Noting $I(h)$ is increasing and convex as $h\to h_1^+$, we have $\#I(h)\le 1$.

Here we list the conclusions for other subcases. All results can be obtained by similar arguments as above.

Subcase 3. $W_2<0<W_1$: $\#I(h)\le 1$ if $\#R_2(h)=1$ and $\#I(h)\le 3$ if $\#R_2(h)=2$.

Subcase 4. $0\le W_2\le 0<W_1$: $\#I(h)\le 1$.

Case 2. $W'(h_1)<0$.

Proposition 4.1(ii) implies that $\#I(h)\le 3$ if $W'(h_1)<0$, $f(h_2)>0$, $\#R_2(h)\le 1$. We only need to consider the case $\#R_2(h)=2$.

It follows from (38) that $R_2'(h_1)>0$, which implies that $R_2(h)>0$ as $h\to h_1^+$. Since $\#R_2(h)=2$, we have $R_2(h)>0$ as or $h\to h_2^-$.

Suppose $W(\bar{h})=W_1=g(\bar{h})>0$. Since $W'(h_1)<0$ implies $W(h)<0$ as $h\to h_1^+$ and $S_1(\bar{h},W_1)$ is an unstable node, the trajectory $\widehat{S_1B_1}$ must intersects h-axis, which means that $R_2(h)$ has at least one zero in $(h_1,\bar{h})$. Due to the same reason, $R_2(h)$ has at least one zero in $(\bar{h},h_2)$. Therefore, $R_2(h)$ has a unique zero in $(h_1,\bar{h})$ (resp. $(\bar{h},h_2)$). We note $S_2(\bar{h},W_2)$ is a saddle and attractive along the invariant line $h=\bar{h}$, hence $W_2<0$ must hold if $W_1=g(\bar{h})>0$. By the same arguments as Case 1, we have $\#I''(h)\le 2$, which gives $\#I(h)\le 3$.

If $g(\bar{h})=0$, then it follows from Case 1 that $\#\widetilde{W}(h)=0$, which shows that $I''(h)$ has a unique zero at $h=\bar{h}$. Therefore, $\#I(h)\le 2$. Please note we do not use the condition $W'(h_1)>0$ in the proof of $\#\widetilde{W}(h)=0$ in Subcase 2 of Case 1.

If $g(\bar{h})<0$, then one gets $\#I(h)\le 3$ by repeating the same arguments as Case 1.

Case 3. $W'(h_1)=0$.

For this case, (38) implies that $R_2(h)$ has a zero at $h=h_1$ with multiplicity at least two, which gives $\#R_2(h)\le 1$. By Proposition 4.1(i), $I(h)$ has at most three zeros if $\#R_2(h)=0$. In the rest of this proof, we suppose $\#R_2(h)=1$.

By Lemma 3.7(iii) and Corollary 3.5, $W'(h_1)=0$ if and only if

$$\gamma=-\frac{3(5A^2-13A-4)\alpha+8(5A^3-30A^2+45A+124)}{3(A-4)(5A-17)}.$$

Taking the above value of γ into (33) and noting $f(h_1)<0<f(h_2)$,

$$g(\bar{h})=-\frac{243(5A-17)^2f^2(h_1)f(h_2)}{10485760(A-4)(4(A+1)(A^2-7A+18)+3(A-3)(A-1)\alpha)^2}<0.$$

The denominator of $g(\bar{h})$ is not equal to zero since the leading part of $f(h)$ is not zero. It follows from the definition of $R_2(h)$ in Proposition 3.6 that

$$R_2(\bar{h}) = g(\bar{h})(-\Delta(\bar{h})f'(\bar{h}) - s_{1,-1}(\bar{h})g(\bar{h})) > 0.$$

If $R_2(h) > 0$ as $h \to h_1^+$, then the above discussion implies that $R_2(h)$ has a unique zero in $(\bar{h}, h_2)$ and no zero in $(h_1, \bar{h})$. This yields $W(h)$ has at most one zero in $(h_1, \bar{h})$ and two zeros in $(\bar{h}, h_2)$ by using (37). Suppose $W(h)$ has a zero in $(h_1, \bar{h})$, then $W(\bar{h}) = g(\bar{h}) < 0$ implies $W(h) > 0$ as $h \to h_1^+$. By (24) and (39), $I(h)$ is monotone and convex as $h \to h_1^+$. Therefore, $\#I(h) = \#I''(h) \leq 3$. If $W(h)$ has no zero in $(h_1, \bar{h})$, then $W(h)$ has at most two zeros in $(\bar{h}, h_2)$, which also gives $\#I(h) \leq 3$.

If $R_2(h) < 0$ as $h \to h_1^+$, then $R_2(h)$ has a zero in $(h_1, \bar{h})$ and no zero in $(\bar{h}, h_2)$. Since $W'(h) \to +\infty$ as $h \to h_2^-$ (see Lemma 3.7), we know that $W(h) < 0$ as $h \to h_2^-$, which means that $W(h)$ has at least two zeros in $(\bar{h}, h_2)$ if there exists. However, if $W(h)$ has two zeros in $(\bar{h}, h_2)$, then $R_2(h)$ has at least one zero between them, see the proof of Proposition 4.1, which contradicts $R_2(h) \neq 0$, $h \in (\bar{h}, h_2)$. In the interval $(h_1, \bar{h})$, one gets from (37) that $\#I''(h) \leq 2$. Therefore, $\#I(h) \leq 3$.

□

Proposition 4.6. *If* $\#f(h) = 0,\ f(h) \not\equiv 0$, *then* $\#I(h) \leq 3$.

Proof. We have proved for this case that $\#I(h) \leq 3$ in Proposition 4.1 if one of the following conditions holds: (1)$\#R_2(h) \leq 1$, (2) $\#R_2(h) = 2, W'(h_1)f(h_2) < 0$ and (3) $W'(h_1) = 0$. It will be proved in Proposition 4.7 and the next section that $\#I(h) \leq 1$ if either $f(h_1) = 0$ or $f(h_2) = 0$. Therefore, in this proof, we only consider the case $\#R_2(h) = 2, W'(h_1)f(h_2) > 0,\ f(h_1)f(h_2) \neq 0$. Since $\#f(h) = 0$ implies $f(h_1)f(h_2) > 0$, it follows from (38) that $R_2'(h_1) > 0$. By using Lemma 4.4 and the same arguments as in the proof of Proposition 4.5, we obtained $\#I(h) \leq 2$. Here the details are omitted. □

Proposition 4.7. $I'(h_1) = 0$ *if and only if* $f(h_1) = 0$.

(i) *If* $I'(h_1) = 0$, *then* $I(h)$ *has at most one zero in* (h_1, h_2), $\#I''(h) \leq 1$.
(ii) *If* $I'(h_1) = I''(h_1) = 0$, *then* $I''(h) < 0, \#I(h) = 0$.

Proof. By (39), we know that $I'(h_1) = 0$ if and only if $f(h_1) = 0$, which yields

$$\alpha = -h_1 - \gamma. \tag{41}$$

(i) In Proposition 3.10, we have proved $\#I(h) \leq 1$ if $\bar{h} = h_1,\ f(h_1) = 0, f'(h_1) < 0$. On the other hand, $f(h_1) = f'(h_1) = 0$ if and only if $f(h) \equiv 0$, which yields $\#I(h) = 0$; see Proposition 4.2. In what follows we suppose $f'(h_1) > 0$, which implies

$$f(h_2)=\frac{2048}{27}(A-3)(4A+3\gamma)>0 \tag{42}$$

if (41) holds.

Substituting (41) into (25), we have

$$W(h)=(h^2-h_1^2)\overline{W}(h)=(h^2-h_1^2)(f_1\omega(h)+g_1(h)), \tag{43}$$

where

$$f_1=\frac{9}{256}(A-4)f(h_2)>0,\ g_1(h)=-15(A-4)h^2+8A(A-3)(-40+12A-\gamma),$$

which implies

$$\overline{W}(h_2)=0,\ \overline{W}(h_1)=\overline{W_1}=-\frac{32}{3}(-40-12A+4A^2+3(A-3)\gamma). \tag{44}$$

It follows from (26) that $\overline{W}(h)$ satisfies

$$\Delta(h)f_1\overline{W}'=h(\overline{R}_0\overline{W}^2+\overline{R}_1(h)\overline{W}+\overline{R}_2(h)), \tag{45}$$

where

$$\begin{aligned}\overline{R}_0&=\frac{64}{3}(A-4),\ \ \overline{R}_1(h)=\frac{3}{2}(A-4)^2\overline{W}_1(h^2-h_2^2),\\ \overline{R}_2(h)&=-\frac{32}{3}(A-4)^2(h^2-h_2^2)\{-45(A-4)(10-12A+4A^2+3(A-3)\gamma)h^2\\ &\quad+8(A-3)(16A(-135+112A-59A^2+10A^3)\\ &\quad+24(-30+42A-29A^2+5A^3)\gamma+9(A-3)\gamma^2)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{R}_1^2(h)-4\overline{R}_0\overline{R}_2(h)&=\frac{27}{8}(A-4)^4(-240-12A+4A^2+3(A-3)\gamma)\\ &\qquad f(h_2)(h^2-h_1^2)(h^2-h_2^2).\end{aligned}$$

Case 1. $-240-12A+4A^2+3(A-3)\gamma\geq 0$

In this case, $\overline{R}_1^2(h)-4\overline{R}_0\overline{R}_2(h)\leq 0$ for $h\in(h_1,h_2)$. Since the right hand of (45) is a quadratic polynomial of $\overline{W}$ and $\overline{R}_0>0$, we have $\overline{R}_0\overline{W}^2+\overline{R}_1(h)\overline{W}+\overline{R}_2(h)\geq 0$, which gives $\overline{W}'(h)>0$. This implies $\#\overline{W}(h)=\#W(h)\leq 1$.

Case 2. $-240-12A+4A^2+3(A-3)\gamma<0,\ \overline{W}_1\neq 0.$

Consider the system

$$\dot{h} = f_1\Delta(h), \;\; \dot{\overline{W}} = h(\overline{R}_0\overline{W}^2 + \overline{R}_1(h)\overline{W} + \overline{R}_2(h)), \tag{46}$$

which has two saddle-node points at $(h_1,\overline{W}_1)$ and $(h_2,0)$. Since $\overline{R}_1^2(h) - 4\overline{R}_0\overline{R}_2(h) > 0$ for $h \in (h_1,h_2)$ in this case, system (46) has three zero isoclines $h = 0$ and $\overline{W}^{\pm}(h)$, defined by

$$\overline{R}_0\overline{W}^2 + \overline{R}_1(h)\overline{W} + \overline{R}_2(h) = 0, \tag{47}$$

where

$$\overline{W}^{\pm}(h) = \frac{-\overline{R}_1(h) \pm \sqrt{\overline{R}_1^2(h) - 4\overline{R}_0\overline{R}_2(h)}}{2\overline{R}_0}, \;\; h \in (h_1,h_2),$$

which gives

$$\overline{W}^{\pm}(h_1) = \overline{W}_1, \; \overline{W}^{\pm}(h_2) = 0, \; \lim_{h\to h_1^+} \frac{\mathrm{d}\overline{W}^{\pm}(h)}{\mathrm{d}h} = \pm\infty, \; \lim_{h\to h_2^-} \frac{\mathrm{d}\overline{W}^{\pm}(h)}{\mathrm{d}h} = \mp\infty. \tag{48}$$

In what follows, we are going to prove that $\overline{W}^{+}(h)$ (resp. $\overline{W}^{-}(h)$) has unique maximum (resp. minimum) in the interval (h_1,h_2). Assume that $\mathrm{d}\overline{W}(h)/\mathrm{d}h = 0$ at $h = \mathcal{H}$, that is to say, $h = \mathcal{H}$ is an extreme point. Differentiating (47) with respect h, we have

$$\begin{aligned}\overline{W}(\mathcal{H}) &= \mathcal{W}(\mathcal{H}) = -\frac{\overline{R}_2'(h)}{\overline{R}_1'(h)}\bigg|_{h=\mathcal{H}} \\ &= -\frac{128}{9\overline{W}_1}\bigg[45(A-4)(10-12A+4A^2+3(A-3)\gamma)\mathcal{H}^2 \\ &\quad -4(A-3)\Big(16A(-210+227A-119A^2+20A^3) \\ &\quad +24(2A-1)(30-27A+5A^2)\gamma+9(A-3)\gamma^2\Big)\bigg],\end{aligned}$$

which shows $\mathcal{W}(\mathcal{H}) = \mathcal{W}(-\mathcal{H})$. Substituting $\overline{W} = \mathcal{W}(\mathcal{H})$ into (47), we get

$$\rho(\mathcal{H}) = \overline{R}_0\mathcal{W}^2(\mathcal{H}) + \overline{R}_1(\mathcal{H})\mathcal{W}(\mathcal{H}) + \overline{R}_2(\mathcal{H}) = 0.$$

Since $\overline{R}_0$ is a constant and $\overline{R}_1(\mathcal{H})$, $\overline{R}_2(\mathcal{H})$, *and* $\mathcal{W}(\mathcal{H})$ are even functions in $\mathcal{H}$, $\rho(\mathcal{H})$ is also an even polynomial function in $\mathcal{H}$ with $\deg \rho(\mathcal{H}) = 4$, which implies $\rho(\mathcal{H}) = 0$ has at most two positive zeros. Therefore, the algebraic curve (47) has at most two extreme values $\mathcal{W}(\mathcal{H}_1)$ and $\mathcal{W}(\mathcal{H}_2)$. On the other

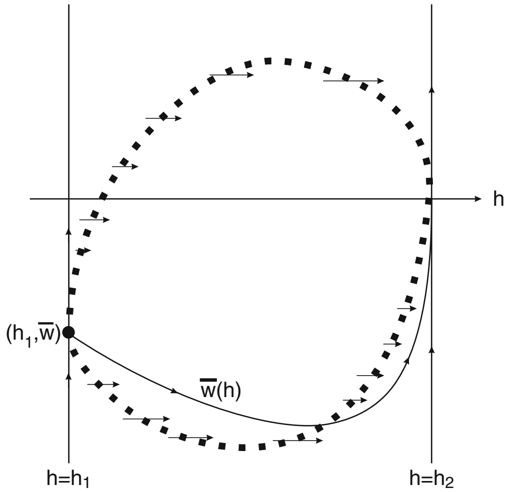


Fig. 4 The phase portraits of system (46) for the case $\overline{W}_1 < 0$

hand, (48) shows that $\overline{W}^+(h)$ and $\overline{W}^-(h)$ have at least a maximum and a minimum in the interval (h_1,h_2), respectively. The desired result follows.

It follows from Lemma 3.7 that

$$\overline{W}'(h_1) = \frac{1}{3}(A-4)(A-1)(-240-12A+4A^2+3(A-3)\gamma) < 0$$

and

$$\lim_{h\to h_2^-} \overline{W}(h) = +\infty, \quad \overline{W}(h_1) = \overline{W}_1, \quad \overline{W}(h_2) = 0,$$

which means that $\overline{W}(h)$ is a trajectory starting from $(h_1,\overline{W}_1)$ to $(h_2,0)$. In the phase plane of system (45), the region $\{(h,\overline{W})|h_1 \le h \le h_2\}$ is divided into three parts by the zero isoclines $\overline{W}^{\pm}(h)$ and the invariant lines $h = h_i$, $i = 1,2$. Since $f_1\Delta(h) > 0$ and the right hand of the second equation of (45) can be rewritten as the form $h\overline{R}_0(\overline{W}-\overline{W}^+(h))(\overline{W}-\overline{W}^-(h))$, the trajectories are decreasing in $\{(h,\overline{W})|\ h_1 < h < h_2,\ \overline{W}^-(h) < \overline{W} < \overline{W}^+(h)\}$ and increasing in either $\{(h,\overline{W})|\ h_1 < h < h_2,\ \overline{W} > \overline{W}^+(h)\}$ or $\{(h,\overline{W})|\ h_1 < h < h_2,\ \overline{W} < \overline{W}^-(h)\}$, respectively. Since $\overline{W}'(h_1) < 0$ and $\lim_{h\to h_2^-} \overline{W}(h) = +\infty$, there must exist $\bar{h}^*$ such that (a) $\overline{W}(h)$ is decreasing in the interval $(h_1,\bar{h}^*)$; (b) $\overline{W}(h)$ intersects $\overline{W}^-(h)$ at $h = \bar{h}^*$ and hence $\overline{W}'(\bar{h}^*) = 0$; and c) $\overline{W}(h)$ is increasing in $(\bar{h}^*,h_2)$. This implies that $\overline{W}(h)$ intersects h-axis at a unique point if $\overline{W}_1 > 0$ and $\overline{W}(h) < 0$ if $\overline{W}_1 \le 0$, see Fig. 4 for the case $\overline{W}_1 < 0$. Finally, one obtains $\#\overline{W}(h) \le 1$.

Case 3. $\overline{W}_1 = 0$.

In this case, $\overline{R}_2(h) = 24000(A-4)^3(h^2-h_1^2)(h^2-h_1^2)$, $\overline{W}_1'(h_1) = -\frac{200}{3}(A-4)(A-1) < 0$, $\lim_{h\to h_2^-} \overline{W}(h) = +\infty$, which shows that $\overline{W}(h) < 0$ as $h \to$

h_i, $i = 1,2$. Therefore, $\#\overline{W}(h)$ must be an even number or equal to zero. Since $\#\overline{R}_2(h) = 0$, one gets $\#\overline{W}(h) \le 1$ by using (37) to $\overline{W}(h), f_1$, and $\overline{R}_2(h)$. Hence, $\#\overline{W}(h) = 0$, $\overline{W}(h) < 0$.

We get from (24) and (43) that $\#I''(h) = \#\overline{W}(h) \le 1$ for the above three cases. Since $I(h_1) = I'(h_1) = 0$, the claim (i) follows.

(ii) If $I'(h_1) = I''(h_1) = 0$, then (22) holds and $I'''(h_1) < 0$. Taking (22) into (44), one gets $\overline{W}_1 = 0$. We have known in Case (3) of (i) that $\overline{W}(h) < 0$, $h \in (h_1, h_2)$, if $\overline{W}_1 = 0$, which yields $I''(h) \ne 0$. Since $I(h_1) = I'(h_1) = I''(h_1) = 0$, $I'''(h_1) < 0$, one obtains $I(h) < 0$, $h \in (h_1, h_2)$.

□

In the next section, we will prove that $\#I(h) \le 1$ if $f(h_2) = 0$ in Corollary 5.3. Summing up the above discussions, we have the following:

Theorem 4.8. *$I(h)$ has at most three zeros (counting their multiplicities) in the interval (h_1, h_2).*

5 Proof of Theorem 1.1

Note that $I(h)$, defined in (13), is equal to (12) with coefficients satisfying (19). Let

$$\begin{aligned} I_2(h) &= J_2(h) - \frac{A-4}{A} J_0(h), \\ I_{-2}(h) &= J_{-2}(h) - \frac{A}{A-4} J_0(h). \end{aligned} \tag{49}$$

We have shown $J_0(h) > 0$ in Lemma 3.1. Then Abelian integral (12) with (19) is reduced to

$$\begin{aligned} I(h) &= \bar{\mu}_1 J_0(h) + \mu_2 I_2(h) + \mu_3 I_{-2}(h) \\ &= J_0(h)(\bar{\mu}_1 + \mu_2 \xi(h) + \mu_3 \eta(h)), \end{aligned} \tag{50}$$

where

$$\begin{aligned} \bar{\mu}_1 &= \mu_1 + \frac{A-4}{A}\mu_2 + \frac{A}{A-4}\mu_3, \\ \xi(h) &= \frac{I_2(h)}{J_0(h)}, \\ \eta(h) &= \frac{I_{-2}(h)}{J_0(h)}. \end{aligned} \tag{51}$$

Lemma 5.1. *Consider I_2 and I_{-2}.*

(i)

$$I_2(h_1) = I_{-2}(h_1) = 0,\ I_2(h) > 0,\ I_{-2}(h) < 0,\ h \in (h_1, h_2].$$

This implies $\xi(h) > 0,\ \eta(h) < 0$ for $h \in (h_1, h_2]$.

(ii) We have

$$\begin{aligned}
I'_{-2}(h_1) &= -\frac{4}{A-4} J'_0(h_1),\\
I''_{-2}(h_1) &= \frac{A^2-5A-20}{48(A-4)} J'_0(h_1),\\
I'''_{-2}(h_1) &= \frac{5(A-1)(7A^2-35A-20)}{18432} J'_0(h_1),\\
I'_2(h_1) &= \frac{4}{A} J'_0(h_1),\\
I''_2(h_1) &= -\frac{(A-1)(A-4)}{48A} J'_0(h_1),\\
I'''_2(h_1) &= -\frac{35(A-4)^2(A-1)^2}{18432A} J'_0(h_1).
\end{aligned}$$

(iii) The following asymptotic expansions hold near $h = h_2$:

$$\begin{aligned}
I_2(h) &= I_2(h_2) + I'_2(h_2)(h-h_2) + \cdots,\\
I_{-2}(h) &= I_{-2}(h_2) + I'_{-2}(h_2)(h-h_2) + \cdots.
\end{aligned}$$

Proof. Since $x \geq \sqrt{\frac{A-4}{A}}$ for $(x,y) \in \Gamma_h$, one gets

$$I_2(h) = \oint_{\Gamma_h} x^{-4}\left(x^2 - \frac{A-4}{A}\right) y\mathrm{d}x = 2\int_{x_1^*(h)}^{x_2^*(h)} x^{-4}\left(x^2 - \frac{A-4}{A}\right) y\mathrm{d}x > 0,$$

and

$$\begin{aligned}
I_{-2}(h) &= \oint_{\Gamma_h} x^{-4}\left(x^{-2} - \frac{A}{A-4}\right) y\mathrm{d}x\\
&= -\frac{2A}{A-4}\int_{x_1^*(h)}^{x_2^*(h)} x^{-6}\left(x^2 - \frac{A-4}{A}\right) y\mathrm{d}x < 0,
\end{aligned}$$

where $y = \sqrt{2\left(-Ax^4 + hx^3 - 2(A-2)x^2 + \frac{A-4}{3}\right)}$, $x_1^*(h)$, and $x_2^*(h)$ are the abscissas of the intersection points of Γ_h with x-axis. The other assertions follow from Lemmas 3.1 and 3.4 and (17). □

Lemma 5.2. $\xi'(h) < 0,\ (I_{-2}(h)/I_2(h))' < 0,\ h \in (h_1, h_2)$.

Proof. If $J_2(h)/J_0(h)$ is not monotone, then there must exist ν_1 and ν_2 such that $\nu_1 J_0(h) + \nu_2 J_2(h)$ has at least two zeros, which implies system $(6)_\varepsilon$ has at least two limit cycles in the right half-plane, provided $\mu_1 = \nu_1$, $\mu_2 = \nu_2$, $\mu_3 = 0$. This contradicts Proposition 2.2. Hence $J_2(h)/J_0(h)$ is a monotonic function. Since

$$\xi(h) = \frac{J_2(h)}{J_0(h)} - \frac{A-4}{A},$$

$\xi(h)$ is also a monotonic function in the interval (h_1, h_2). By Lemma 5.1,

$$\xi(h) = \frac{4}{A} + \frac{3-A}{16}(h-h_1) - \frac{(A-3)(55A^2-311A+292)}{18432}(h-h_1)^2 + \cdots, \quad (52)$$

which gives $\xi'(h_1) < 0$. The assertion $\xi'(h) < 0$ follows.

To prove the monotonicity of $I_{-2}(h)/I_2(h)$, we use a similar argument as in [17]. Rewrite (10) in the form

$$H(x,y) = \frac{1}{2}x^{-3}y^2 + \phi(x),$$

where

$$\phi(x) = Ax + 2(A-2)x^{-1} - \frac{A-4}{3}x^{-3}.$$

As in the proof of Lemma 5.1, denote by $x_1^*(h), x_2^*(h)$ the abscissas of the intersection points of Γ_h with x-axis. Since

$$\phi'(x)(x-1) = Ax^{-4}(x-1)^2(x+1)(x^2 - \frac{A-4}{A}) > 0,\ x \in (x_1^*(h), 1) \cup (1, x_2^*(h)), \quad (53)$$

for any $h \in (h_1, h_2)$, $x \in (x_1^*(h), 1)$, there exists a unique $\widetilde{x} \in (1, x_2^*(h))$ such that $\phi(x) = \phi(\widetilde{x})$. Therefore, we can define the function $\widetilde{x} = \widetilde{x}(x)$ for $x \in (x_1^*(h), 1)$. For the ratio of two Abelian integrals

$$\frac{\oint_{\Gamma_h} \kappa_2(x) y \mathrm{d}x}{\oint_{\Gamma_h} \kappa_1(x) y \mathrm{d}x}, \quad (54)$$

we define a criterion function

$$\zeta(x) = \frac{\kappa_2(x)\sqrt{\frac{1}{2}\widetilde{x}^{-3}}\phi'(\widetilde{x}) - \kappa_2(\widetilde{x})\sqrt{\frac{1}{2}x^{-3}}\phi'(x)}{\kappa_1(x)\sqrt{\frac{1}{2}\widetilde{x}^{-3}}\phi'(\widetilde{x}) - \kappa_1(\widetilde{x})\sqrt{\frac{1}{2}x^{-3}}\phi'(x)}.$$

Let

$$\kappa_1(x) = x^{-4}\left(x^2 - \frac{A-4}{A}\right), \;\; \kappa_2(x) = x^{-6}\left(x^2 - \frac{A-4}{A}\right). \tag{55}$$

Then

$$\zeta(x) = \frac{-1 + x^2 + x\widetilde{x} + \widetilde{x}^2 + (x+\widetilde{x})\sqrt{x\widetilde{x}}}{\sqrt{x\widetilde{x}}(x + \widetilde{x} + \sqrt{x\widetilde{x}} + x\widetilde{x}\sqrt{x\widetilde{x}})},$$

hence

$$\zeta'(x) = \Theta(x,\widetilde{x}) + \Theta(\widetilde{x},x)\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}x},$$

where

$$\Theta(x,\widetilde{x}) = -\frac{(\widetilde{x}-1)(\widetilde{x}+1)(3x + \widetilde{x} + 2\sqrt{x\widetilde{x}} + x^3 + 2x^2\sqrt{x\widetilde{x}} + 3x^2\widetilde{x} + 4x\widetilde{x}\sqrt{x\widetilde{x}})}{2x\sqrt{x\widetilde{x}}(x + \widetilde{x} + \sqrt{x\widetilde{x}} + x\widetilde{x}\sqrt{x\widetilde{x}})^2}.$$

Since $\sqrt{(A-4)/A} < x < 1 < \widetilde{x}$, one gets $\Theta(x,\widetilde{x}) < 0$, $\Theta(\widetilde{x},x) > 0$. By $\phi(x) = \phi(\widetilde{x})$ and (53), we know that

$$\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}x} = \frac{\phi'(x)}{\phi'(\widetilde{x})} < 0,$$

which implies $\zeta'(x) < 0$. It follows from Theorem 2 in [17] that the ratio (54) associated with (55) is an increasing function. Noting

$$\oint_{\Gamma_h} \kappa_1(x) y\mathrm{d}x = I_2(h), \;\; \oint_{\Gamma_h} \kappa_2(x) y\mathrm{d}x = -\frac{A-4}{A} I_{-2}(h),$$

we have $(I_{-2}(h)/I_2(h))' < 0$. □

Corollary 5.3. *If $f(h_2) = 0$, then $I(h)$ has at most one zero in the interval (h_1, h_2).*

Proof. By direct computation, it follows from (51) and (19) that

$$\bar{\mu}_1 = -\frac{9}{512A(A-3)} f(h_2).$$

If $f(h_2) = 0$, then $I(h) = \mu_2 I_2(h) + \mu_3 I_{-2}(h)$. Lemma 5.2 shows that $I(h)$ has at most one zero in this case. □

Since $\xi'(h) < 0$, $\xi = \xi(h)$ has an inverse function $h = h^{-1}(\xi)$. In $\xi\eta$-plane, define the curve

$$\Omega = \{(\xi,\eta) | \xi = \xi(h), \, \eta = \eta(h), \, h \in [h_1, h_2]\}. \tag{56}$$

For $h \in (h_1, h_2)$, the number of zeros of $I(h)$, defined in (50), is the number of the intersection points of the curve Ω with the straight line:

$$L: \quad \bar{\mu}_1 + \mu_2 \xi + \mu_3 \eta = 0. \tag{57}$$

In what follows we are going to study the geometric properties of the curve Ω. Denote by L_C, L_S the tangents to Ω at the endpoints $C(\xi(h_1), \eta(h_1)$ and $S(\xi(h_2), \eta(h_2))$, respectively. Let L_{CS} be the line passing through both C and S. Lemma 5.1 gives

$$\xi(h_1) = \frac{4}{A},\ \eta(h_1) = -\frac{4}{A-4},\ \xi(h_2) = \frac{I_2(h_2)}{J_0(h_2)},\ \eta(h_2) = \frac{I_{-2}(h_2)}{J_0(h_2)}.$$

$I(h)$ has the following asymptotic expansion near the Hamiltonian value $h = h_1$ which corresponds to the center $(1,0)$ of system $(6)_0$:

$$I(h) = I'(h_1)(h-h_1) + \frac{I''(h_1)}{2}(h-h_1)^2 + \frac{I'''(h_1)}{6}(h-h_1)^3 + \cdots. \tag{58}$$

It is well known that the system $(6)_\varepsilon$ at the center $(1,0)$ has a Hopf bifurcation of order 1 if $I'(h_1) = 0, I''(h_1) \neq 0$, of order 2 if $I'(h_1) = I''(h_1) = 0, I'''(h_1) \neq 0$, respectively.

Proposition 5.4. *(i) $I'(h_1) = 0$ if and only if L passes through C.*
(ii) The equation of L_C is

$$A(A+1)(A-4)\xi + A(A-3)(A-4)\eta + 16 = 0. \tag{59}$$

$I'(h_1) = I''(h_1) = 0$ if and only if $L = L_C$.
(iii) For the center $(1,0)$ of system $(6)_\varepsilon$, two is the highest order of Hopf bifurcation.

Proof. (i) By (57), L passes through $C(\xi(h_1), \eta(h_1)) = C(4/A, -4/(A-4))$ if and only if

$$\bar{\mu}_1 + \frac{4}{A}\mu_2 - \frac{4}{A-4}\mu_3 = 0. \tag{60}$$

It follows from (51), (19), and Corollary 3.5 that (60) holds if and only if

$$\alpha + \gamma + h_1 = \frac{I'(h_1)}{J_0'(h_1)} = 0.$$

This proves (i).

(ii) By Lemma 5.1, $\eta(h)$ has the following expansion near $h = h_1$:

$$\eta(h) = -\frac{4}{A-4} + \frac{A+1}{16}(h-h_1) + \frac{55A^3 - 256A^2 + 221A + 52}{18432}(h-h_1)^2 + \cdots. \tag{61}$$

Using (52) and (61), we have

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right|_C = \frac{\eta'(h_1)}{\xi'(h_1)} = -\frac{A+1}{A-3},$$

which yields the equation of L_C is (59).

On the other hand, $I'(h_1) = I''(h_1) = 0$ if and only if (22) holds. Taking (22) into (57) and using (51) and (19) again, we have that $I'(h_1) = I''(h_1) = 0$ if and only $L = L_C$.

(iii) In Corollary 3.5, we have shown that $I'''(h_1) < 0$ if $I'(h_1) = I''(h_1) = 0$. This proves the assertion (iii).

□

Near the value h_2 corresponding to a saddle-loop Γ_{h_2}, it follows from Lemmas 5.1 and 3.4 that $I(h)$, defined in (50), has the expansion:

$$I(h) = c_0 + c_1(h-h_2)\ln|h-h_2| + c_2(h-h_2) + \cdots, \tag{62}$$

where

$$\begin{aligned} c_0 &= \bar{\mu}_1 J_0(h_2) + \mu_2 I_2(h_2) + \mu_3 I_{-2}(h_2), \\ c_1 &= -\frac{1}{2\sqrt{2}}\sqrt{\frac{A}{A-4}}\bar{\mu}_1, \\ c_2 &= \mu_2 I_2'(h_2) + \mu_3 I_{-2}'(h_2), \quad \text{if} \ \ c_0 = c_1 = 0. \end{aligned}$$

If $c_0 = 0, c_1 \neq 0$ (resp. $c_0 = c_1 = 0, c_2 \neq 0$), system $(6)_\varepsilon$ has at most one (resp. two) limit cycles which tends to Γ_{h_2} as $\varepsilon \to 0$ [22].

Proposition 5.5. *(i) L passes through S if and only if* $c_0 = 0$*;*

(ii) The equation L_S *is*

$$\eta = \frac{I_{-2}(h_2)}{I_2(h_2)}\xi. \tag{63}$$

And $c_0 = c_1 = 0$ *if and only if* $L = L_S$*;*

(iii) If $c_0 = c_1 = 0$*, then* $c_2 \neq 0$.

Proof. By (57), L passes through S if and only if

$$\bar{\mu}_1 + \mu_2\frac{I_2(h_2)}{J_0(h_2)} + \mu_3\frac{I_{-2}(h_2)}{J_0(h_2)} = 0.$$

This holds if and only if

$$\bar{\mu}_1 J_0(h_2) + \mu_2 I_2(h_2) + \mu_3 I_{-2}(h_2) = c_0 = 0.$$

(ii) By Lemma 3.4, $\lim_{h\to h_2} J_0'(h) = +\infty$. Therefore, it follows from Lemmas 5.1 and 3.4 that

$$\frac{d\eta}{d\xi}\bigg|_{h=h_2} = \lim_{h\to h_2} \frac{\eta'(h)}{\xi'(h)} = \lim_{h\to h_2} \frac{I'_{-2}(h)J_0(h) - I_{-2}(h)J'_0(h)}{I'_2(h)J_0(h) - I_2(h)J'_0(h)}$$

$$= \lim_{h\to h_2} \frac{\frac{I'_{-2}(h)J_0(h)}{J'_0(h)} - I_{-2}(h)}{\frac{I'_2(h)J_0(h)}{J'_0(h)} - I_2(h)} = \frac{I_{-2}(h_2)}{I_2(h_2)},$$

which gives (63).

(iii) In the paper [11], the authors proved that at most two limit cycles can appear in a homoclinic loop bifurcation from any quadratic integrable system under quadratic nonconservative perturbation except for one case. System (5) belongs to the cases which has been solved in [11]. Therefore, if $c_0 = c_1 = 0$, then $c_2 \neq 0$.

□

Lemma 5.6. *L_{CS} does not intersect Ω except C and S.*

Proof. By Propositions 5.4 and 5.5, we know that $L = L_{CS}$ if and only if $I'(h_1) = I(h_2) = 0$. It has been shown in Proposition 3.10 and Proposition 4.7 that either $\#I''(h) = 0$ or $\#I''(h) \leq 1$ if $I'(h_1) = 0$. Since $I(h_1) = I'(h_1) = I(h_2) = 0$, $I(h)$ has no zero in (h_1, h_2), which means that L_{CS} has no common point with Ω. □

Lemma 5.7. *(i) If L passes through C and $L \neq L_C$, $L \neq L_{CS}$, then L intersects Ω in at most one point except C.*
(ii) L_C does not intersect Ω except C.

Proof. The assertions follow from Propositions 4.7 and 5.4. □

Lemma 5.8. *L_S does not intersect Ω except S.*

Proof. Proposition 5.5 and (62) yields that $L = L_S$ if and only if $\bar{\mu}_1 J_0(h_2) + \mu_2 I_2(h_2) + \mu_3 I_{-2}(h_2) = \bar{\mu}_1 = 0$, which gives

$$\bar{\mu}_1 = 0,\ \mu_2 = -\frac{I_{-2}(h_2)}{I_2(h_2)}\mu_3.$$

Therefore, it follows from (50) and Lemma 5.2 that

$$I(h) = \mu_2 I_2(h) + \mu_3 I_{-2}(h) = \mu_3 I_2(h)\left(-\frac{I_{-2}(h_2)}{I_2(h_2)} + \frac{I_{-2}(h)}{I_2(h)}\right) > 0,\ h \in (h_1, h_2).$$

It is easy to see that $I_{-2}(h_2)/I_2(h_2) \neq I_{-2}(h_1)/I_2(h_1)$, otherwise $I_{-2}(h)/I_2(h)$ is not a monotonic function, which contradicts Lemma 5.2. Hence,

$$I'(h_1) = \mu_3 I'_2(h_1)\left(-\frac{I_{-2}(h_2)}{I_2(h_2)} + \frac{I_{-2}(h_1)}{I_2(h_1)}\right) \neq 0.$$

By Proposition 5.4, L_S does not pass through C. The lemma is proved. □

Lemma 5.9. *The curve Ω is convex near the endpoints C and S.*

Proof. It follows from (52) and (61) that

$$\left.\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2}\right|_C = \left.\frac{\eta''(h)\xi'(h)-\eta'(h)\xi''(h)}{(\xi'(h))^3}\right|_{h=h_1} = \frac{10(A-1)}{3(A-3)^2} > 0,$$

which shows that Ω is convex near C.

To get the convexity of Ω near S, we now study the relative position between L_S and Ω. Let $(\xi(h),\eta(h)) \in \Omega$ and $(\xi(h),\eta) \in L_S$, then by (63) and Lemma 5.2,

$$\eta(h)-\eta = \eta(h) - \frac{I_{-2}(h_2)}{I_2(h_2)}\xi(h) = \xi(h)\left(\frac{I_{-2}(h)}{I_2(h)} - \frac{I_{-2}(h_2)}{I_2(h_2)}\right) > 0,$$

i.e., $\eta(h) > \eta$ for same $\xi = \xi(h)$. Therefore, L_S stays below Ω, which implies that Ω is convex near S. □

It follows from Lemma 5.6 to 5.9 that:

Proposition 5.10. *The curve Ω is entirely placed in the triangle formed by L_C, L_S, and L_{CS}.*

Lemma 5.11. *(i) Let $h_0 \in (h_1,h_2)$. The equality $I(h_0)=0$ holds if and only if L passes through $(\xi(h_0),\eta(h_0))$; the equalities $I(h_0)=I'(h_0)=0$ hold if and only if L is tangent to the curve Ω at the point $(\xi(h_0),\eta(h_0))$; $I(h_0)=I'(h_0)=I''(h_0)=0$, $I'''(h_0)\neq 0$ hold if and only if $(\xi(h_0),\eta(h_0))$ is the inflection point of Ω.*

(ii) If L does not pass through C and S, then L intersects Ω in at most three points (counted with their multiplicities).

Proof. The assertion (i) follows by the same arguments as in the proof of Proposition 5.4. Using (i) and Theorem 4.8, one gets (ii). □

Proof of Theorem 1.1. By Proposition 2.1, we consider the number of limit cycles in $(6)_\varepsilon$ in the right half-plane, instead of system (5). The theorem will be proved by the arguments used in [10] and [27].

If $\mu_3 = 0$, then the assertion of this theorem follows from Proposition 2.2. In the rest of this proof, we suppose $\mu_3 \neq 0$ without loss of generality.

By the results in [1, 6] the limit cycles of system $(6)_\varepsilon$ are completely related to the zeros of the corresponding Abelian integral $I(h)$. Hence, Theorem 1.1 would be proved if we show that any line L intersects Ω in at most two points (counting the multiplicity) in $\xi\eta$-plane.

In what follows, we are going to prove what the following two assertions hold.

Assertion 1. *Each line L, passing through C or S, intersects Ω in at most two points.* If L passes through C or $L=L_S$, then Assertion 1 follows from Lemmas 5.6–5.8. Suppose that L is the line through S, $L\neq L_S$, $L\neq L_{CS}$, which intersects Ω at another point $\bar{M}$. Then by Lemma 5.9 and Proposition 5.10, the points of Ω near C

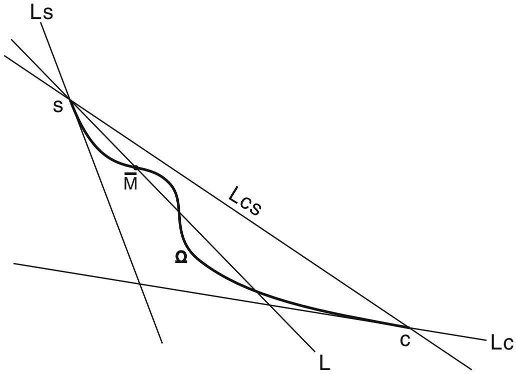


Fig. 5 If L intersects Ω one point except S and $\overline{M}$, then the total number of the intersection points is at least four

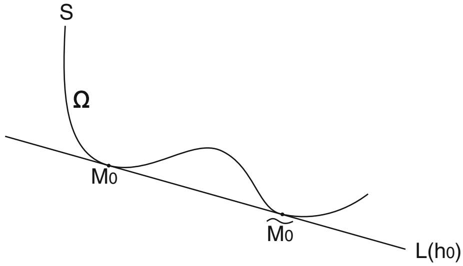


Fig. 6 $L(h_0)$ is tangent to Ω at M_0 and $\widetilde{M}_0$

and S lie on different sides of L, which implies either L does not intersect Ω in any other point except S and $\bar{M}$ or the total number of intersection points is at least three except S; see Fig. 5.

Assume that the latter case is true. It follows from Proposition 5.5 and Lemma 5.11 that $I(h_2) = 0$ and $\#I(h) \geq 3$, $h \in (h_1, h_2)$. Since $I(h_1) = I(h_2) = 0$, one gets $\#I(h) \leq 2$ by repeating the arguments as in the proof of Propositions 3.10, 4.1–4.2, and 4.5–4.6. This yields contradiction.

Assertion 2. *Each tangent* $L(h)$, $h \in (h_1, h_2)$, *to* Ω *at the point* $(\xi(h), \eta(h))$, *has exact one common double point with* Ω.

Starting from S, suppose $M_0(\xi(h_0), \eta(h_0))$, $h_0 \in (h_1, h_2)$, is the first point for which $L(h_0)$ has another common point $\widetilde{M}_0(\xi(\widetilde{h}_0), \eta(\widetilde{h}_0))$ with Ω, $h_0 \neq \widetilde{h}_0$. By Lemma 5.6–5.8, $\widetilde{M}_0 \neq C$, $\widetilde{M}_0 \neq S$. The choice of $\widetilde{M}_0$ being the first such point implies that $L(h_0)$ is tangent to Ω also at $\widetilde{M}_0$, see Fig. 6. This contradicts Lemma 5.11(ii). Consequently, there is no $h_0 \in (h_1, h_2)$ for which $L(h_0)$ has another common point with Ω except the tangency point.

To prove that is a double intersection point, assume the contrary. By Lemma 5.11, M_0 is a triple point and $I(h_0) = I'(h_0) = I''(h_0) = 0$, $I'''(h_0) \neq 0$. Slightly moving the tangent $L(h_0)$, there must exist $\bar{\mu}_1^*$, μ_2^* such that $I(h_1^*) = I'(h_1^*) = 0$, $I(h_2^*) = 0$, where h_1^*, h_2^* depend on $\bar{\mu}_1^*$, μ_2^*, μ_3 and $h_i^* \in (h_1, h_2)$, $i = 1, 2$. Hence, $L(h_1^*)$ is tangent to Ω at $(\xi(h_1^*), \eta(h_1^*))$ and intersects another point $(\xi(h_2^*), \eta(h_2^*))$. This contradicts the fact we have proved above.

The above two assertions and Lemma 5.9 imply that Ω is strictly convex for $h \in [h_1, h_2]$, which yields that any line L intersects Ω in at most two points (counting the multiplicities). The proof is finished. □

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Anomalous Diffusion in Polymers: Long-Time Behaviour

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Abstract We study the Dirichlet boundary value problem for viscoelastic diffusion in polymers. We show that its weak solutions generate a dissipative semiflow. We construct the minimal trajectory attractor and the global attractor for this problem.

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1 Introduction

The concentration behaviour for diffusion of penetrant liquids in polymers cannot always be described by the Fickian diffusion equation

$$\frac{\partial u}{\partial t} = div(D(u)\nabla u), \tag{1}$$

where $u = u(t,x)$ is the concentration, which depends on time t and the spatial point x, and $D(u)$ is the diffusion coefficient. The phenomena running counter to (1) include *case II diffusion*, *sorption overshoot*, *literal skinning, trapping skinning* and *desorption overshoot* [8, 10–12, 23, 24, 26, 30]. There is a number of approaches which explain these non-Fickian properties of polymeric diffusion. They have much in common: they are usually based on taking into account the viscoelastic nature of polymers (cf. [17] and references therein) and on the possibility of glass-rubber phase transition (see e.g. [30] with some review). We are going to study the model

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which is due to Cohen et al. [7, 8, 13]. The Fickian diffusion equation is replaced by the system

$$\frac{\partial u}{\partial t} = D\Delta u + E\Delta\sigma, \tag{2}$$

$$\frac{\partial \sigma}{\partial t} + \beta(u,\sigma)\sigma = \mu u + \nu\frac{\partial u}{\partial t}. \tag{3}$$

Here the second variable $\sigma(t,x)$ is introduced (it is called *stress*), D and E are the diffusion and stress-diffusion coefficients, resp., μ and ν are non-negative constants, and the scalar function β is the inverse of the relaxation time, for instance, β can be [8] taken in the following form:

$$\beta = \beta(u) = \frac{1}{2}(\beta_R + \beta_G) + \frac{1}{2}(\beta_R - \beta_G)\tanh\left(\frac{u-u_{RG}}{\delta}\right) \tag{4}$$

where $\beta_R, \beta_G, \delta, u_{RG}$ are positive constants, $\beta_R > \beta_G$.[1]

Well-posedness issues for initial-boundary value problems for systems of viscoelastic diffusion equations have been studied in [1, 2, 15, 25, 26, 28], see [26, 27] for brief reviews. These results include investigation of system (2) and (3) and more general settings (diffusion with variable coefficients). Let us only recall the main results on global (in time) solvability: strong solutions exist globally for $\nu = 0$ and $D = E$ in the one-dimensional case [1], and for suitable non-constant stress-diffusion coefficient, but not for all initial and boundary data [15]; global existence of weak solutions for the Dirichlet and Neumann problems in the general setting with variable coefficients in the multidimensional case is shown in [26] and [28], resp., without restrictions on the initial and boundary data.

Let us also mention here a study of a system obtained from (2)–(3) by some simplification, in [19], and paper [14], which touches upon some long-time behaviour issues for a free boundary problem for a polymeric diffusion model based on Fick's law. A result on long-time behavior (not in the "attractor framework") of the general second boundary value problem can be found in [28].

In this work we are interested in the long-time behaviour of the solutions to Dirichlet initial-boundary value problem for system (2) and (3). We show that the weak solutions generate a semiflow on a suitable phase space with L_2-topology (this means that there is a unique solution for any data from the phase space, and the solution semigroup is continuous in t and x). However, it is not clear whether this semigroup is asymptotically compact, and the phase space is not complete, so this impedes proving of existence of the usual global attractor for this semigroup in

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[1] Formula (4) describes the following peculiarities of the processes under consideration. The polymer network in the glassy state (low concentration area) is severely entangled, so β is approximately equal to some small β_G. In the high concentration areas the system is in the rubbery state: the network disentangles, so the relaxation time is small, and its inverse is close to $\beta_R > \beta_G$. The glass-rubber phase transition occurs near a certain concentration u_{RG}. However, we assume that β also depends on stress, cf. [2, 11, 26].

this phase space. A possible way out is to use the concept of minimal trajectory attractor. Thus, we construct a minimal trajectory attractor, which generates some generalized global attractor for the weak solutions of the problem in the (completed) phase space.

The theory of trajectory attractors was created by G. Sell, M. Vishik and V. Chepyzhov [5, 6, 20], in order to construct an attractor to weak solutions of the 3D Navier-Stokes equation. A generalization of this approach with a related notion of minimal trajectory attractor may be found in [29, 31]; it is applicable when the system lacks continuity properties or invariance of the trajectory space with respect to time shifts. In both theories, the trajectory attractor generates some generalized global attractor in the phase space. This global attractor has many usual properties of attractors, but its invariance may be shown only under additional conditions (see [31, Sect. 4.2.7]). The idea of trajectory attractor was slightly criticized by Ball [4], for the evolution of the original system is not explicitly involved in its definition. However, an example [31, Remark 4.2.13] shows that the minimal trajectory attractor (and the corresponding global attractor) can well characterize the long-time behaviour of the system, even when the usual global attractor does not exist. That example also illustrates that such situations may appear, in particular, for problems with uniqueness of solutions (i.e. when there exists a solution semigroup). It is a rather unexpected fact because the original theory of trajectory attractors was developed for the problems where the uniqueness is not proved or is absent.[2] Similarly, in this paper we apply the theory of minimal trajectory attractors to a problem with uniqueness. However, in our case, we cannot insist that there is no attractor of the semigroup (semiflow), since this is unknown.

Our paper is organized in the following way. In Sect. 2, we introduce the required function spaces. In Sect. 3, we give a weak formulation of the initial-boundary value problem for system (2)–(3) with existence, uniqueness and regularity results (Theorem 3.2, Remark 3.4). In Sect. 4, we recall the basic issues of the classical and minimal trajectory attractor theories, and construct a dissipative semiflow generated by weak solutions of (2)–(3) (Theorem 4.10). In Sect. 5, we show that the *trajectory space* generated by the weak solutions possesses a minimal trajectory attractor and a global attractor (Theorem 5.2).

2 Function Spaces and Related Notations

$L_p(\Omega)$, $W_p^m(\Omega)$, $H^m(\Omega) = W_2^m(\Omega)$ $(m \in \mathbb{Z}, 1 \le p \le \infty)$, $H_0^m(\Omega) = \overset{\circ}{W}{}_2^m(\Omega)$ $(m \in \mathbb{N})$ are, as usual, Lebesgue and Sobolev spaces of functions defined on a bounded open set (domain) $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. The scalar product and the Euclidian norm in $L_2(\Omega)^k = L_2(\Omega, \mathbb{R}^k)$ are denoted by (u, v) and $\|u\|$, respectively (k is equal to 1 or n). In $H_0^1(\Omega)$,

[2] Trajectory attractors for problems with uniqueness were investigated in [6] only as an intermediate step on the way to usual global attractors of semigroups.

we use the following scalar product and norm: $(u,v)_1 = (\nabla u, \nabla v)$, $\|u\|_1 = \|\nabla u\|$. We recall Friedrichs' inequality

$$\|u\| \le K_\Omega \|u\|_1. \tag{5}$$

Let $L_2^1(\Omega)$ denote the topological subspace of $L_2(\Omega)$ consisting of functions from $H_0^1(\Omega)$.

The Laplace operator $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism. Therefore

$$\Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega) \tag{6}$$

is also an isomorphism. Set $X = X(\Omega) = \Delta^{-1}(H_0^1(\Omega))$. The scalar product and norm in X are $(u,v)_X = (\Delta u, \Delta v)_1$, $\|u\|_X = \|\Delta u\|_1$.

As usual, we identify the space $H^{-1}(\Omega)$ with the space of linear continuous functionals on $H_0^1(\Omega)$ (the dual space). The value of a functional from $H^{-1}(\Omega)$ on an element from $H_0^1(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$ (the "bra-ket" notation). The scalar product and norm in $H^{-1}(\Omega)$ are $(u,v)_{-1} = (\Delta^{-1}u, \Delta^{-1}v)_1$, $\|u\|_{-1} = \|\Delta^{-1}u\|_1$. Note that

$$(u, \Delta v)_{-1} = -\langle u, v \rangle,\ u \in H^{-1}(\Omega), v \in H_0^1(\Omega). \tag{7}$$

The symbols $C(\mathcal{J};E)$, $L_2(\mathcal{J};E)$ etc. denote the spaces of continuous, quadratically integrable etc. functions on an interval $\mathcal{J} \subset \mathbb{R}$ with values in a Banach space E.

Let us remind that a pre-norm in the Frechet space $C([0,+\infty);E)$ may be defined by the formula

$$\|v\|_{C([0,+\infty);E)} = \sum_{i=1}^{+\infty} 2^{-i} \frac{\|v\|_{C([0,i];E)}}{1 + \|v\|_{C([0,i];E)}}.$$

If E is a function space ($L_2(\Omega), H^m(\Omega)$ etc.), then we identify the elements of $C(\mathcal{J};E)$, $L_2(\mathcal{J};E)$ etc. with scalar functions defined on $\mathcal{J} \times \Omega$ according to the formula

$$u(t)(x) = u(t,x), t \in \mathcal{J}, x \in \Omega.$$

We shall also use the function space (T is a positive number):

$$W = W(\Omega,T) = \{u \in L_2(0,T;H_0^1(\Omega)),\ u' \in L_2(0,T;H^{-1}(\Omega))\},$$
$$\|u\|_W = \|u\|_{L_2(0,T;H_0^1(\Omega))} + \|u'\|_{L_2(0,T;H^{-1}(\Omega))};$$

[31, Corollary 2.2.3] implies continuous embedding $W \subset C([0,T];L_2(\Omega))$. Moreover,

$$\left\langle v', v \right\rangle = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|v\|^2,\ v \in W. \tag{8}$$

We also denote

$$W_{loc}(\Omega,+\infty)=\{u\in C([0,+\infty);L_2(\Omega)),\ u|_{[0,T]}\in W(\Omega,T)\ \forall T>0\},$$

$$H^1_{loc}(0,+\infty;H^1_0(\Omega))=\{u\in C([0,+\infty);H^1_0(\Omega)),\ u|_{[0,T]}\in H^1(0,T;H^1_0(\Omega))\ \forall T>0\}.$$

We use the notation $|\cdot|$ for the absolute value of a number and for the Euclidean norm in $\mathbb{R}^n$.

The symbol C stands for various positive constants.

Note that

$$\langle u,v\rangle=(u,v),\ u\in L_2(\Omega),v\in H^1_0(\Omega) \tag{9}$$

and, since $H^{-1}(\Omega)\subset L_2(\Omega)$, one has

$$\|u\|_{-1}\le C\|u\|. \tag{10}$$

3 Basic Properties of the Boundary Value Problem

We study the diffusion of a penetrant in a polymer filling a bounded domain $\Omega\subset\mathbb{R}^n$, $n\in\mathbb{N}$, which is described by the following boundary value problem:

$$\frac{\partial u}{\partial t}=D\Delta u+E\Delta\sigma,\ (t,x)\in[0,\infty)\times\Omega, \tag{11}$$

$$\frac{\partial\sigma}{\partial t}+\beta_0(u,\sigma)\sigma=\mu u+\nu\frac{\partial u}{\partial t},\ (t,x)\in[0,\infty)\times\Omega, \tag{12}$$

$$u(t,x)=\varphi(x),\ (t,x)\in[0,\infty)\times\partial\Omega. \tag{13}$$

Here $u=u(t,x):[0,\infty)\times\overline{\Omega}\to\mathbb{R}$ is the unknown concentration of the penetrant (at the spatial point x at the moment of time t), $\sigma=\sigma(t,x):[0,\infty)\times\overline{\Omega}\to\mathbb{R}$ is the unknown stress, $\varphi:\partial\Omega\to\mathbb{R}$ is a given boundary condition, μ,ν, D and E are positive constants,[3] $\beta_0:\mathbb{R}^2\to\mathbb{R}$ is a given function, $\beta_R\ge\beta_0(u,\sigma)\ge\beta_G>0$; $\Omega\subset\mathbb{R}^n$ is supposed to be a bounded open set such that $X(\Omega)\subset W^1_{p_0}(\Omega)$ for some $p_0>2$ (cf. [26]). For definiteness, we assume that the boundary condition for the stress is also prescribed:

$$\sigma(t,x)=\phi(x),\ (t,x)\in[0,\infty)\times\partial\Omega. \tag{14}$$

Equation (12) yields the following relation between the boundary conditions ϕ and φ:

$$\beta_0(\varphi,\phi)\phi=\mu\varphi,\ x\in\partial\Omega. \tag{15}$$

W.l.o.g. the functions φ and ϕ are defined and satisfy (15) on $\overline{\Omega}$.

[3]The case $\mu=0$ (“the Maxwell model” [9, 30]) is admissible as well.

We assume that the functions β_0, ϕ and φ are C^2- smooth. Moreover, for simplicity, let $\beta_0(u,\sigma) \equiv \beta_\infty \geq \beta_G$ for large $|u|+|\sigma|$ (this assumption is admissible in the considered model, see Remark 3.6 below).

Set

$$v(t,x) = u(t,x) - \varphi(x), \varpi(t,x) = \sigma(t,x) - \phi(x),$$
$$\beta(x,v,\varpi) = \beta_0(v+\varphi(x), \varpi+\phi(x)),$$
$$h(x) = D\Delta\varphi(x) + E\Delta\phi(x),$$
$$g(x,v,\varpi) = \mu\varphi(x) - \beta_0(v+\varphi(x),\varpi+\phi(x))\phi(x).$$

Then we can rewrite (11)–(14) in the following form:

$$\frac{\partial v}{\partial t} = D\Delta v + E\Delta\varpi + h, \tag{16}$$

$$\frac{\partial \varpi}{\partial t} + \beta(x,v,\varpi)\varpi = g(x,v,\varpi) + \mu v + \nu\frac{\partial v}{\partial t}, \tag{17}$$

$$v|_{\partial\Omega} = \varpi|_{\partial\Omega} = 0. \tag{18}$$

This form of the studied problem will be useful below. However, in order to set the problem finally, we need the variable $\tau(t,x) = \varpi(t,x) - \nu v(t,x)$. We denote

$$d = D + \nu E,$$
$$\gamma(x,v,\tau) = \mu v - \beta(x,v,\tau+\nu v)\tau - \nu\beta(x,v,\tau+\nu v)v + g(x,v,\tau+\nu v).$$

Then problem (16)–(18) becomes

$$\frac{\partial v}{\partial t} = d\Delta v + E\Delta\tau + h, \tag{19}$$

$$\frac{\partial \tau}{\partial t} = \gamma(x,v,\tau), \tag{20}$$

$$v|_{\partial\Omega} = \tau|_{\partial\Omega} = 0. \tag{21}$$

It can be completed with the initial condition:

$$v(0,x) = v_0(x),\ \tau(0,x) = \tau_0(x),\ x \in \Omega. \tag{22}$$

Definition 3.1. A pair of functions (v,τ) from the class

$$v \in W_{loc}(\Omega,+\infty), \tau \in H^1_{loc}(0,+\infty; H^1_0(\Omega)) \tag{23}$$

is a *weak* solution to problem (19)–(21) if equality (19) holds in the space $H^{-1}(\Omega)$ for a.a. $t \in (0,+\infty)$, and (20) holds in $H^1(\Omega)$ a.e. on $(0,+\infty)$.

Theorem 3.2. *Given $v_0 \in L_2(\Omega)$ and $\tau_0 \in H_0^1(\Omega)$, there exists a unique weak solution to problem (19)–(21) which belongs to (23) and satisfies (22).*

Note that (22) makes sense due to the embeddings $W(\Omega,T) \subset C([0,T];L_2(\Omega))$, $H^1(0,T;H_0^1(\Omega)) \subset C([0,T];H_0^1(\Omega)), T > 0$.

Proof. Let us calculate the gradient of $\gamma(x,v,\tau)$:

$$\begin{aligned}\nabla\gamma(x,v,\tau) &= \mu\nabla v - \beta(x,v,\tau+vv)\nabla\tau - \frac{\partial\beta}{\partial x}(x,v,\tau+vv)\tau \\ &-\frac{\partial\beta}{\partial v}(x,v,\tau+vv)\tau\nabla v \\ &-v\frac{\partial\beta}{\partial\varpi}(x,v,\tau+vv)\tau\nabla v - \frac{\partial\beta}{\partial\varpi}(x,v,\tau+vv)\tau\nabla\tau \\ &-v\beta(x,v,\tau+vv)\nabla v - v\frac{\partial\beta}{\partial x}(x,v,\tau+vv)v \\ &-v\frac{\partial\beta}{\partial v}(x,v,\tau+vv)v\nabla v - v^2\frac{\partial\beta}{\partial\varpi}(x,v,\tau+vv)v\nabla v - v\frac{\partial\beta}{\partial\varpi}(x,v,\tau+vv)v\nabla\tau \\ &+\frac{\partial g}{\partial x}(x,v,\tau+vv) + \frac{\partial g}{\partial v}(x,v,\tau+vv)\nabla v + v\frac{\partial g}{\partial\varpi}(x,v,\tau+vv)\nabla v \\ &+\frac{\partial g}{\partial\varpi}(x,v,\tau+vv)\nabla\tau.\end{aligned}$$

Compactness of Ω gives boundedness of ϕ and φ. Hence, the function $\beta(x,v,\varpi)$ is equal to β_∞ for large $|v|+|\varpi|$. This implies that the gradient $\nabla\gamma(x,v,\tau)$ can be expressed in the form

$$h_1(x,v,\tau)\nabla v + h_2(x,v,\tau)\nabla\tau + \theta(x,v,\tau),$$

where

$$\begin{aligned}|h_1(x,v,\tau)| + |h_2(x,v,\tau)| &\le K_h, \\ |\theta(x,v,\tau)| &\le K_\theta(|v|+|\tau|+1)\end{aligned}$$

with some constants K_h, K_θ.

Therefore, by [26, Theorem 3.1], there is a pair of functions (v,τ) from (23) which satisfies (19), (22) and

$$\Delta\frac{\partial\tau}{\partial t} = \Delta[\gamma(x,v,\tau)], \tag{24}$$

in $H^{-1}(\Omega)$ a.e. on $(0,+\infty)$ (more precisely, that theorem gives existence of solutions on finite time intervals, but then the solution on the positive semi-axis can be constructed step by step in a standard way, see [27]).

Let us show that (v,τ) satisfies (20). Since $\frac{\partial\tau}{\partial t}\in H_0^1(\Omega)$ for a.a. $t>0$, it suffices to prove that $\gamma(x,v,\tau)\in H_0^1(\Omega)$ for a.a. $t>0$. Observe that $\gamma(x,0,0)\equiv 0$ due to (15). The above representations of $\gamma(x,v,\tau)$ and $\nabla\gamma(x,v,\tau)$ imply $\gamma(x,v(t,x),\tau(t,x))$ $\in H^1(\Omega)$ (for a.a. $t>0$). Let v_m,τ_m be sequences of smooth functions with compact supports in Ω, $v_m\to v(t),\tau_m\to\tau(t)$ in $H^1(\Omega)$. Then $\gamma(\cdot,v_m,\tau_m)\in H_0^1(\Omega)$. It remains to observe that $\gamma(\cdot,v_m,\tau_m)\to\gamma(\cdot,v(t),\tau(t))$ weakly in $H^1(\Omega)$. Really, by Krasnoselskii's theorem [16, 22] on continuity of Nemytskii operators we have

$$\gamma(\cdot,v_m,\tau_m)\to\gamma(\cdot,v(t),\tau(t)),h_1(\cdot,v_m,\tau_m)\to h_1(\cdot,v(t),\tau(t)),$$
$$h_2(\cdot,v_m,\tau_m)\to h_2(\cdot,v(t),\tau(t)),\theta(\cdot,v_m,\tau_m)\to\theta(\cdot,v(t),\tau(t))$$

strongly in $L_2(\Omega)$. Moreover, due to boundedness of h_1 and h_2, w.l.o.g. we may assume that $h_1(\cdot,v_m,\tau_m)\to h_1(\cdot,v(t),\tau(t))$, $h_2(\cdot,v_m,\tau_m)\to h_2(\cdot,v(t),\tau(t))$ *-weakly in $L_\infty(\Omega)$. But $\nabla v_m\to\nabla v(t)$, $\nabla\tau_m\to\nabla\tau(t)$ strongly in $L_2(\Omega)$. Therefore,

$$h_1(\cdot,v_m,\tau_m)\nabla v_m\to h_1(\cdot,v(t),\tau(t))\nabla v(t),h_2(\cdot,v_m,\tau_m)\nabla\tau_m\to h_2(\cdot,v(t),\tau(t))\nabla\tau(t)$$

weakly in $L_2(\Omega)$.

Remark 3.3. It seems that a more profound reasoning of this kind proves that for any Nemytskii operator $\mathcal{N}:H^1(\Omega)\to H^1(\Omega)$ one has $\mathcal{N}(\xi)-\mathcal{N}(0)\in H_0^1(\Omega)$ for all $\xi\in H_0^1(\Omega)$.

It remains to prove uniqueness. If $\varpi=\tau+\nu v$, then $\varpi\in W_{loc}(\Omega,+\infty)$, and the pair (v,ϖ) satisfies (16), (17). It suffices to show uniqueness of the pair (v,ϖ). Let (v_1,ϖ_1), (v_2,ϖ_2) be solutions of (16), (17) in the class $W_{loc}(\Omega,+\infty)\times W_{loc}(\Omega,+\infty)$ with the same initial conditions. Let us denote $w=v_1-v_2$, $\xi=\varpi_1-\varpi_2$. Then

$$w'=D\Delta w+E\Delta\xi, \tag{25}$$

$$\xi'+\beta(x,v_1,\varpi_1)\varpi_1-\beta(x,v_2,\varpi_2)\varpi_2=g(x,v_1,\varpi_1)-g(x,v_2,\varpi_2)+\mu w+\nu w'. \tag{26}$$

Calculate the $H^{-1}(\Omega)$-scalar product of (25) and $\mu w+\nu w'$ for a.a. $t\in(0,\infty)$, and take (7) and (9) into account:

$$\mu(w',w)_{-1}+\nu(w',w')_{-1}=-\mu D(w,w)-\mu E(\xi,w)-\nu D\left\langle w',w\right\rangle-\nu E\left\langle w',\xi\right\rangle. \tag{27}$$

Take the "bra-ket" of (26) and $E\xi$ for a.a. $t\in(0,\infty)$, and remember (9):

$$\begin{aligned}&E\left\langle\xi',\xi\right\rangle+E(\beta(x,v_1,\varpi_1)\varpi_1-\beta(x,v_2,\varpi_2)\varpi_2,\xi)\\&\quad=E(g(x,v_1,\varpi_1)-g(x,v_2,\varpi_2),\xi)+\mu E(w,\xi)+\nu E\left\langle w',\xi\right\rangle.\end{aligned} \tag{28}$$

Adding (27) and (28), and omitting the second term, which is positive, we conclude:

$$\begin{aligned}&\frac{\mu}{2}\frac{\mathrm{d}\|w\|_{-1}^2}{\mathrm{d}t}+\mu D\|w\|^2+\frac{\nu D}{2}\frac{\mathrm{d}\|w\|^2}{\mathrm{d}t}+\frac{E}{2}\frac{\mathrm{d}\|\xi\|^2}{\mathrm{d}t}\\ &\quad\le E(\beta_0(v_2+\varphi,\varpi_2+\phi)(\varpi_2+\phi)-\beta_0(v_1+\varphi,\varpi_1+\phi)(\varpi_1+\phi),\xi)\\ &\quad\le C(\|\xi\|+\|w\|)\|\xi\|. \end{aligned} \tag{29}$$

The last inequality follows from boundedness of $\frac{\partial[\beta_0(u,\sigma)\sigma]}{\partial u}$ and $\frac{\partial[\beta_0(u,\sigma)\sigma]}{\partial\sigma}$, Lagrange's theorem and the Cauchy-Buniakowski inequality.

Integration from 0 to t yields

$$\frac{\mu}{2}\|w\|_{-1}^2+\mu D\int_0^t\|w(s)\|^2\,\mathrm{d}s+\frac{\nu D}{2}\|w\|^2+\frac{E}{2}\|\xi\|^2\le C\int_0^t(\|\xi(s)\|^2+\|w(s)\|^2)\,\mathrm{d}s.$$

Hence,

$$\|w(t)\|^2+\|\xi(t)\|^2\le C\int_0^t(\|\xi(s)\|^2+\|w(s)\|^2)\,\mathrm{d}s,\ t\ge 0. \tag{30}$$

Thus, by the Gronwall lemma, $\xi\equiv w\equiv 0$. □

Remark 3.4. Weak solutions $(v(t),\tau(t))$ to (19)–(21) belong to $H_0^1(\Omega)^2$ for all $t>0$. It follows from a simple regularity result for reaction-diffusion equations. Consider the problem

$$\frac{\partial\upsilon}{\partial t}-a\Delta\upsilon=f, \tag{31}$$

$$\upsilon|_{\partial\Omega}=0, \tag{32}$$

$$\upsilon(0,x)=\upsilon_0(x),\ x\in\Omega. \tag{33}$$

Given $a>0$, $\upsilon_0\in L_2(\Omega)$ and $f\in L_2(0,T;H^{-1}(\Omega))$, there exists a unique weak solution $\upsilon\in W(\Omega,T)$, $T>0$. Since $W(\Omega,T)\subset C([0,T],L_2(\Omega))$, the operator

$$\Upsilon:L_2(\Omega)\times L_2(0,T;H^{-1}(\Omega))\to L_2(\Omega),\ \Upsilon(\upsilon_0,f)=\upsilon(t),0<t\le T,$$

is well-defined. Since $\tau\in H^1(0,T;H_0^1(\Omega))$, it suffices to apply the following lemma to (19).

Lemma 3.5. *The operator Υ transforms $L_2(\Omega)\times H^1(0,T;H^{-1}(\Omega))$ into $H_0^1(\Omega)$.*

Proof. W.l.o.g. $a = 1$. The solution υ can be considered as the sum $\upsilon_1 + \upsilon_2$ of the weak solutions to the following problems:

$$\frac{\partial \upsilon_1}{\partial t} - \Delta\upsilon_1 = f, \tag{34}$$

$$\upsilon_1|_{\partial\Omega} = 0, \tag{35}$$

$$\upsilon_1(0) = -\Delta^{-1} f(0), \tag{36}$$

$$\frac{\partial \upsilon_2}{\partial t} - \Delta\upsilon_2 = 0, \tag{37}$$

$$\upsilon_2|_{\partial\Omega} = 0, \tag{38}$$

$$\upsilon_2(0) = \upsilon_0 + \Delta^{-1} f(0). \tag{39}$$

It is easy to see that $\upsilon_1(t) = -\Delta^{-1} f(0) + \int_0^t \upsilon_3(s)\,\mathrm{d}s$, where $\upsilon_3 \in W(\Omega,T)$ is determined by the problem

$$\upsilon_3' - \Delta\upsilon_3 = f', \tag{40}$$

$$\upsilon_3|_{\partial\Omega} = 0, \tag{41}$$

$$\upsilon_3(0) = 0. \tag{42}$$

Obviously, $\upsilon_3 \in L_2(0,T;H_0^1(\Omega))$, so $\upsilon_1 \in H^1(0,T;H_0^1(\Omega)) \subset C([0,T];H_0^1(\Omega))$. Moreover, see e.g. [6, Proposition XV.3.5], $\upsilon_2(t) \in H_0^1(\Omega)$. □

Remark 3.6. There is no essential loss in generality of the model in assuming that $\beta_0(u,\sigma)$ is equal to some constant β_∞ for large $|u| + |\sigma|$. In fact, physically, $|u(t,x)| \le 1$ (u is the concentration, so it cannot exceed 100%). Let $\varsigma = \sigma - \nu u$. Then (12) yields (cf. [26, 27])

$$\begin{aligned}\varsigma(t,x) = {} & \varsigma(0,x)\exp\left(-\int_0^t \beta_0(u(\xi,x),\nu u(\xi,x)+\varsigma(\xi,x))\,\mathrm{d}\xi\right)\\ & + \int_0^t \exp\left(\int_t^s \beta_0(u(\xi,x),\nu u(\xi,x)+\varsigma(\xi,x))\,\mathrm{d}\xi\right)\\ & \times[\mu - \nu\beta_0(u(s,x),\nu u(s,x)+\varsigma(s,x))]\,u(s,x)\,\mathrm{d}s.\end{aligned}$$

Hence,

$$|\varsigma(t,x)| \le \mathrm{e}^{-t\beta_G}|\varsigma(0,x)| + (\mu+\nu\beta_R)\int_0^t \mathrm{e}^{(s-t)\beta_G}\,\mathrm{d}s \le \mathrm{e}^{-t\beta_G}|\varsigma(0,x)| + \frac{\mu+\nu\beta_R}{\beta_G}. \tag{43}$$

Thus, if $|\varsigma(0,x)|$ is uniformly bounded, ς is also bounded:

$$|\varsigma(t,x)| \leq C, \forall t > 0, x \in \Omega. \tag{44}$$

Moreover, (43) implies that long-time behaviour of $\varsigma(t,x)$ is bounded for any $|\varsigma(0,x)|$. Hence, ς (and, therefore, σ) is bounded for typical regimes which may be observed in reality. Thus, the relaxation time (and its inverse β_0) can be experimentally determined[4] only for bounded u and σ, whereas "at infinity" we can choose it at discretion, for instance, we can let $\beta_0(u,\sigma) \equiv \beta_\infty$ for large $|u| + |\sigma|$.

4 Semigroups, Semiflows, Trajectory Attractors and Global Attractors

Let us recall some basics of the attractor theory. Let E be a metric space.

Definition 4.1. A family of mappings $\mathcal{S}_t : E \to E$, $t \geq 0$, is called a *semigroup* if $\mathcal{S}_0$ is the identity map I and

$$\mathcal{S}_t \circ \mathcal{S}_s = \mathcal{S}_{t+s} \tag{45}$$

for any $t, s \geq 0$.

Definition 4.2. A set $P \subset E$ is called *attracting* (for $\mathcal{S}_t$) if for any bounded set $B \subset E$ and any open neighborhood W of P there exists $h \geq 0$ such that $\mathcal{S}_t B \subset W$ for all $t \geq h$.

Definition 4.3. A set $P \subset E$ is called *absorbing* (for $\mathcal{S}_t$) if for any bounded set $B \subset E$ there is $h \geq 0$ such that for all $t \geq h$ one has $\mathcal{S}_t B \subset P$.

Definition 4.4. A set $A \subset E$ is called *invariant* (for $\mathcal{S}_t$) if

$$\mathcal{S}_t A = A$$

for any $t \geq 0$.

Definition 4.5. A set $\mathcal{A} \subset E$ is called a *global attractor* (of $\mathcal{S}_t$) if

(i) $\mathcal{A}$ is compact;
(ii) $\mathcal{A}$ is invariant for $\mathcal{S}_t$;
(iii) $\mathcal{A}$ is attracting for $\mathcal{S}_t$.

If there exists a global attractor of $\mathcal{S}_t$, then it is unique (see e.g. [31, Corollary 4.1.1]).

Definition 4.6. A semigroup $\mathcal{S}_t : E \to E$ is called *dissipative* if there is a bounded absorbing set.

[4] e.g. in form (4).

Definition 4.7. A semigroup $\mathcal{S}_t : E \to E$ is called *asymptotically compact* if, for any bounded sequence $y_m \in E$ and any sequence of numbers $t_m \to +\infty$, the sequence $\mathcal{S}_{t_m}(y_m)$ contains a converging subsequence.

Definition 4.8. A semigroup $\mathcal{S}_t : E \to E$ is called a *semiflow* if the map

$$(t,y) \mapsto \mathcal{S}_t(y)$$

is continuous from $[0,+\infty) \times E$ to E.

The next result follows e.g. from [4, Theorem 3.3, Corollary 4.3]:

Theorem 4.9. *A semiflow $\mathcal{S}_t : E \to E$ has a global attractor $\mathcal{A} \subset E$ if and only if it is dissipative and asymptotically compact. If E is connected, then $\mathcal{A}$ is a connected set.*

In order to describe the dynamics of weak solutions for problem (19)–(21), one may put $E = L_2(\Omega) \times L_2^1(\Omega)$, and define the semigroup $S_t : E \to E$ in the standard way: if $y = (v_0, \tau_0)$, and (v,τ) is the corresponding weak solution of (19)–(22), then we set $S_t(y) = (v(t), \tau(t))$. Since weak solutions belong to $C([0,T]; L_2(\Omega)) \times C([0,T]; H^1(\Omega))$ for all $T > 0$, the map $t \mapsto S_t(y)$ is continuous for each $y \in E$. On the other hand, a reasoning similar to the proof of uniqueness in Theorem 3.2 shows that the map $S_t : E \to E$ is continuous uniformly with respect to $t \in [0,T]$ for all $T > 0$. Then the map $(t,y) \mapsto S_t(y)$ is continuous, and, taking into account Lemma 5.1 (see below), we arrive at

Theorem 4.10. *$S_t : E \to E$ is a dissipative semiflow.*

However, it seems to be very hard or impossible to establish asymptotic compactness of this semiflow since the space E is not complete (and this is usually important, cf. [18]). If we try to change the phase space E and to find so-called (E,F)-attractors [3, 31], i.e. attractors which attract in the topology of some space F, which is weaker than the one of E, then we need continuity of S_t in this weaker topology, which is not clear. Moreover, it seems (due to the properties of the generalized attractor which we construct below) that the attractor can contain non-differentiable elements of $L_2(\Omega)$, and then Remark 3.4 implies that it cannot be invariant. Thus, we are going to use the theory of trajectory attractors, so let us briefly describe the required notions.

Let E and E_0 be Banach spaces, $E \subset E_0$, E is reflexive. Fix some set

$$\mathcal{H}^+ \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$$

of solutions (strong, weak, etc.) for any given autonomous differential equation or boundary value problem. Hereafter, the set $\mathcal{H}^+$ will be called the *trajectory space* and its elements will be called *trajectories*. Generally speaking, the nature of $\mathcal{H}^+$ may be different from the just described one.

Definition 4.11. A set $P \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is called *attracting* (for the trajectory space $\mathcal{H}^+$) if for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty;E)$, one has

$$\sup_{u \in B} \inf_{v \in P} \|T(h)u - v\|_{C([0,+\infty);E_0)} \underset{h\to\infty}{\to} 0.$$

Here $T(h)$ stands for the translation (shift) operators,

$$T(h)(u)(t) = u(t+h).$$

Note that $T(h)$ is a semiflow on $C([0,+\infty);E_0)$.

Definition 4.12. A set $P \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is called *absorbing* (for the trajectory space $\mathcal{H}^+$) if for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0,+\infty;E)$, there is $h \geq 0$ such that for all $t \geq h$:

$$T(t)B \subset P.$$

Definition 4.13. A set $\mathcal{U} \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is called the *minimal trajectory attractor* (for the trajectory space $\mathcal{H}^+$) if

(i) $\mathcal{U}$ is compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$;
(ii) $T(t)\mathcal{U} = \mathcal{U}$ for any $t \geq 0$;
(iii) $\mathcal{U}$ is attracting in the sense of Definition 4.11;
(iv) $\mathcal{U}$ is contained in any other set satisfying conditions (i), (ii), (iii).

Definition 4.14. A set $\mathcal{A} \subset E$ is called the *global attractor* (in E_0) for the trajectory space $\mathcal{H}^+$ if

(i) $\mathcal{A}$ is compact in E_0 and bounded in E;
(ii) for any bounded in $L_\infty(0,+\infty;E)$ set $B \subset \mathcal{H}^+$ the attraction property is fulfilled:

$$\sup_{u \in B} \inf_{v \in \mathcal{A}} \|u(t) - v\|_{E_0} \underset{t\to\infty}{\to} 0$$

(iii) $\mathcal{A}$ is the minimal set satisfying conditions (i) and (ii) (that is, $\mathcal{A}$ is contained in every set satisfying conditions (i) and (ii)).

Theorem 4.15 (see [31, Corollary 4.2.1, Lemma 4.2.9]). *Assume that there exists an absorbing set P for the trajectory space $\mathcal{H}^+$, which is relatively compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$. Then there exists a minimal trajectory attractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$.*

The structure of minimal trajectory attractors is discussed in [31, Chap. 4] (see also [6]). In particular, the minimal trajectory attractor contains the set of those solutions to the considered problem that can be continued to the whole real axis being uniformly bounded in E and continuous with values in E_0; however, some extra elements may appear. The structure of global attractors is determined by the structure of minimal trajectory attractors:

Theorem 4.16 (see [31, Theorem 4.2.2]). *If there exists a minimal trajectory attractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$, then there is a global attractor $\mathcal{A}$ for the trajectory space $\mathcal{H}^+$, and for all $t \geq 0$ one has $\mathcal{A} = \{y(t) | y \in \mathcal{U}\}$.*

5 Attractors for the Polymeric Diffusion Problem

We are going to construct the minimal trajectory attractor and the global attractor for problem (19)–(21). Let us choose $L_2(\Omega)^2$ as the space E and the space $H^{-\delta}(\Omega)^2$ as the space E_0, where $\delta \in (0,1]$ is a fixed number. The trajectory space $\mathcal{H}^+$ is the set of all weak solutions to (19)–(21) from class (23). It is contained in $L_\infty(0,+\infty;E)$ due to Lemma 5.1 (below) and in $C([0,+\infty);E_0)$ (clearly).

Dissipativity of problem (19)–(21) is stated by the following result:

Lemma 5.1. *Let (v,τ) be a weak solution to (19)–(21). Then*

$$\frac{\nu D}{2}\|v(t)\|^2 + \frac{E}{2}\|\tau(t)+\nu v(t)\|^2 + \frac{\mu}{2}\|v(t)\|_{-1}^2$$
$$\leq \mathrm{e}^{-\gamma t}\left(\frac{\nu D}{2}\|v(0)\|^2 + \frac{E}{2}\|\tau(0)+\nu v(0)\|^2 + \frac{\mu}{2}\|v(0)\|_{-1}^2\right) + \Gamma \tag{46}$$

for all $t>0$, where Γ and γ are some fixed positive numbers, which are independent of v,τ and t.

Proof. The pair $(v, \varpi = \tau + \nu v)$ satisfies (16), (17). Take the $H^{-1}(\Omega)$-scalar product of (16) and $\mu v + \nu v'$ and the "bra-ket" of (17) and $E\varpi$ for a.a. $t \in (0,\infty)$, and add the results (cf. proof of Theorem 3.2):

$$\frac{\mu}{2}\frac{\mathrm{d}\|v\|_{-1}^2}{\mathrm{d}t} + \nu\|v'\|_{-1}^2 + \mu D\|v\|^2 + \frac{\nu D}{2}\frac{\mathrm{d}\|v\|^2}{\mathrm{d}t} + \frac{E}{2}\frac{\mathrm{d}\|\varpi\|^2}{\mathrm{d}t}$$
$$+E(\beta(x,v,\varpi)\varpi,\varpi) = (h, \mu v + \nu v')_{-1} + E(g(x,v,\varpi),\varpi). \tag{47}$$

Hence,

$$\frac{\mu}{2}\frac{\mathrm{d}\|v\|_{-1}^2}{\mathrm{d}t} + \nu\|v'\|_{-1}^2 + \mu D\|v\|^2 + \frac{\nu D}{2}\frac{\mathrm{d}\|v\|^2}{\mathrm{d}t} + \frac{E}{2}\frac{\mathrm{d}\|\varpi\|^2}{\mathrm{d}t}$$
$$+E\beta_G\|\varpi\|^2 \leq C(\|v\|_{-1} + \|v'\|_{-1} + \|\varpi\|) \leq C(\|v\| + \|v'\|_{-1} + \|\varpi\|). \tag{48}$$

The Cauchy inequality for scalars can be written in the form $C\eta \leq \varepsilon\eta^2 + \frac{C^2}{4\varepsilon}$. Thus, (48) implies

$$\frac{\mu}{2}\frac{\mathrm{d}\|v\|_{-1}^2}{\mathrm{d}t} + \frac{\mu D}{2}\|v\|^2 + \frac{\nu D}{2}\frac{\mathrm{d}\|v\|^2}{\mathrm{d}t} + \frac{E}{2}\frac{\mathrm{d}\|\varpi\|^2}{\mathrm{d}t} + \frac{E\beta_G}{2}\|\varpi\|^2 \leq C. \tag{49}$$

Let $\chi(t) = \frac{\mu}{2}\|v(t)\|_{-1}^2 + \frac{\nu D}{2}\|v(t)\|^2 + \frac{E}{2}\|\varpi(t)\|^2$. Obviously,

$$\frac{\mu D}{2}\|v(t)\|^2 + \frac{E\beta_G}{2}\|\varpi(t)\|^2 \geq \gamma\chi(t)$$

for some $\gamma > 0$. Thus, $\chi'(t) + \gamma\chi(t) \leq C$, so $\chi(t) \leq \mathrm{e}^{-\gamma t}\chi(0) + (1+\gamma^{-1})C$ by [6, Lemma II.1.3], and (46) follows. □

The main result of this section is

Theorem 5.2. *The trajectory space $\mathcal{H}^+$ possesses a minimal trajectory attractor and a global attractor.*

Proof. Due to Theorems 4.15 and 4.16, it suffices to find an absorbing set for the trajectory space $\mathcal{H}^+$, which is relatively compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$. Consider the set P of weak solutions to (19)–(21) which satisfy the estimate

$$\frac{\nu D}{2}\|v(t)\|^2 + \frac{E}{2}\|\tau(t) + \nu v(t)\|^2 + \frac{\mu}{2}\|v(t)\|_{-1}^2 \leq 2\Gamma, \forall t \geq 0. \tag{50}$$

It is an absorbing set for the trajectory space $\mathcal{H}^+$ and is bounded in $L_\infty(0,+\infty;E)$. By (19), the set $\{v',\ (v,\tau) \in P\}$ is bounded in $L_\infty(0,+\infty;H^{-2}(\Omega))$. Moreover, (20) yields that $\{\tau',\ (v,\tau) \in P\}$ is bounded in $L_\infty(0,+\infty;L_2(\Omega))$. The embedding $E \subset E_0$ is compact. By [21, Corollary 4], the set $\{y|_{[0,M]}, y \in P\}$ is relatively compact in $C([0,M];E_0)$ for any $M > 0$. This implies (cf. [31, p. 183]) that P is relatively compact in $C([0,+\infty);E_0)$. □

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